Level Crossing Probabilities of the Ornstein – Uhlenbeck Process

Dr. József Dénes

Institute of Informatics and Mathematics, Faculty of Light Industry, Budapes Tech Department of Science, University of West Hungary
denes.jozsef@nik.bmf.hu

Abstract: The Ornstein Uhlenbeck process is a Gaussian process $X_t$ with independent increments and autocorrelation $E(X_t X_{t+s}) = e^{-|t-s|}/2$. First the Laplace transform of the probability density $P(X_t = x | X_0 = p)$ is computed. Using this, the Laplace transform of $X_t$ first time reaching a given value $x$ is derived. It is proved that these results agree with the special case derived earlier by Bellman and Harris (Pacific J. Math. 1, 1951).

1 Definitions

The Ornstein Uhlenbeck process is a stationary Gaussian-Markov process $X_t$ such that the joint distribution of $X_{t_1}, X_{t_2}, \ldots, X_{t_m}$ is a gaussian and is dependent only on the differences $t_j - t_i$ where $i < j$ and the autocorelation function is given by

$$E(X_s \cdot X_{s+t}) = \frac{1}{2} e^{-|t|}$$ (1.1)

$$E X_t = 0 \text{ and } EX_t^2 = \frac{1}{2}.$$ (1.2)

Let $X$ be a random vector with normal distribution, then the density of its probability distribution is:

$$\frac{1}{2\pi|\Sigma|} e^{-\frac{1}{2} x^T \Sigma^{-1} x}$$
where \( X = \begin{pmatrix} X \\ Y \end{pmatrix} \) and \( \Sigma \) is the correlation matrix:

\[
\begin{pmatrix}
\rho_1 & \sigma \\
\sigma & \rho_2
\end{pmatrix}
\]

with \( \rho_1 = EX^2, \rho_2 = EY^2, \sigma_1 = EXY \) and \( |\Sigma| = \rho_1 \rho_2 - \sigma^2 \). Clearly

\[
\Sigma^{-1} = \begin{pmatrix}
\rho_2 & -\sigma \\
-\sigma & \rho_1
\end{pmatrix}
\]

Hence the joint probability density

\[
P(X = x, Y = y) = \frac{1}{2\pi\sqrt{\rho_1 \rho_2 - \sigma^2}} \exp\left(-\frac{\rho_2 x^2 - 2\sigma xy + \rho_1 y^2}{2(\rho_1 \rho_2 - \sigma^2)}\right).
\]

It follows from here that

\[
P(Y = y | X = x) = \frac{1}{2\pi\sqrt{\rho_1 \rho_2 - \sigma^2}} \exp\left(-\frac{\rho_2 x^2 - 2\sigma xy + \rho_1 y^2}{2(\rho_1 \rho_2 - \sigma^2)}\right)
\]

\[
= \frac{1}{\sqrt{2\pi \rho_1}} \exp\left(-\frac{\rho_1 \rho_2 - \sigma^2}{\rho_1} \left(\frac{y - x}{\rho_1}\right)^2\right).
\]

Applying this to what concerns us, the Ornstein-Uhlenbeck process, we can determine the probability density \( P(X_t = x | X_0 = p) \).

Clearly
\[ \rho_1 = \rho_2 = \frac{1}{2}, \sigma = \frac{e^{-t}}{2} \text{ so } \frac{2(\rho_1 \rho_2 - \sigma^2)}{\rho_1} = \frac{2}{2} \left( \frac{1}{2} - \frac{e^{-2t}}{4} \right) = 1 - e^{-2t}. \frac{\sigma}{\rho_1} = e^{-t} \]

Hence:

\[ P(X_t = x \mid X_0 = p) = \frac{e^{-(x-ep^{-1})^2/(1-e^{-2t})}}{\sqrt{\pi(1-e^{-2t})}}. \quad (1.3) \]

We shall denote this with \( P(t,p,x) \) or \( P(p,x) \) and call it the fundamental function. The special cases \( p = 0 \) and \( x = 0 \) are important also:

\[ P(X_t = x \mid X_0 = 0) = \frac{e^{-x^2/(1-e^{-2t})}}{\sqrt{\pi(1-e^{-2t})}}. \quad (1.4) \]

\[ P(X_t = 0 \mid X_0 = p) = \frac{e^{-p^2 e^{-2t}/(1-e^{-2t})}}{\sqrt{\pi(1-e^{-2t})}}. \quad (1.5) \]

By simple substitution it is easy to prove that (1.3) satisfies the forward equation:

\[ \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial t} + u \]

and the backward equation:

\[ \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial p^2} - p \frac{\partial u}{\partial p}. \]

This also implies that (1.4) satisfies the forward equation and (1.5) satisfies the backward equation.
2 The Laplace Transforms of the Fundamental Functions

Since both \( \frac{e^{-x^2/(1-e^{-2t})}}{\sqrt{\pi(1-e^{-2t})}} \) and \( \frac{e^{-x^2/(1-e^{-2t})}}{\sqrt{\pi(1-e^{-2t})}} \) satisfy the forward equation their Laplace transforms must satisfy the second order ordinary differential equation, that is the equation

\[
U'' + 2xU' + 2(1-s)U = 0
\]  

(2.1)

To find the solutions of (2.1) let us consider the confluent hypergeometric equation

\[
xy'' = +(c - x)y'0 - ay = 0
\]  

(2.2)

The two solutions of this are the:

\[
_1F_1(a, c; x) = 1 + \frac{a}{c!} + \frac{a(a + 1)x^2}{c(c + 1)2!} \ldots
\]

and \( x^{1-c}_1F_1(a + 1 - c, 2 - c; x) \) Kummer functions. Let us consider the following transformation of (2.2) \( u = y(kx^2) \) where \( k \) is an arbitrary nonzero constant.

Clearly:

\[
u = y(kx^2)
\]

\[
u' = 2kxy'
\]

\[
u'' = 2ky' + 4k^2x^2y''.
\]

Hence:
\[ y = u \\
\frac{y'}{2} = \frac{u'}{2} \\
y'' = \frac{u''}{x} \\
y''' = \frac{x}{4k^2x^2} 
\]
Substituting these into (2.2) gives:

\[
kx^2\left(\frac{u'' - \frac{u'}{x}}{x^2}\right) + \left(c - \frac{kx^2}{2}\right)\frac{u'}{2kx} - au = 0
\]

which in turn, after some simplification becomes:

\[
u'' + \left(\frac{2c - 1}{x} - \frac{2kx}{x^2}\right)u' - 4kau = 0.
\]

Putting \(c = \frac{1}{2}\) gives: \(u'' - \frac{2kxu'}{x} - 4kau = 0\).

Let us compare this with (2.1)

\[
U'' + 2xU' + 2(1-s)U = 0 \\
- 2k = 2 \\
- 4ka = 2(1-s).
\]

Hence we get for \(k\) and for \(ak = -1\) and \(a = \frac{1-s}{2}\). Therefore the solutions of (2.1) are \(F_1 = F\left(\frac{1-s}{2}, \frac{1}{2}; -x^2\right)\) and \(F_2 = xF\left(\frac{1-s}{2}, \frac{3}{2}; -x^2\right)\).

Now we are in the position to determine the Laplace transform of \(\frac{e^{-x^2}}{\sqrt{\pi(1-e^{-2t})}}\).

Clearly it must be of the form \(AF_1 + xBF_2\) where \(A\) and \(B\) some constans. To this end Laplace transform will be evaluated for some special cases. The Laplace transform of \(\frac{e^{-x^2}}{\sqrt{\pi(1-e^{-2t})}}\) is clearly:
Writing $t$ instead of $e^{-t}$ transforms it into a Mellin type integral:

$$
\int_{0}^{\infty} \frac{x^2}{\sqrt{\pi(1-e^{-2t})}} e^{-st} dt.
$$

Substituing $\sqrt{t}$ instead of $t$ yields

$$
\int_{0}^{1} \frac{x^2}{\sqrt{\pi(1-t^2)}} t^{s-1} dt.
$$

For $x = 0$ this becomes the beta function type integral:

$$
\frac{1}{2\sqrt{\pi}} \int_{0}^{1} \frac{t^{\frac{s-1}{2}}}{\sqrt{1-t}} dt = \frac{1}{2\sqrt{\pi}} B\left(\frac{1}{2}, \frac{s}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{s}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{1+s}{2}\right)}.
$$

Hence

$$
A = \frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)}.
$$

Clearly $A$ is the Laplace transform of $\frac{x^2}{\sqrt{\pi(1-e^{-2t})}}$. To determine the value of $B$ let us consider the $x$ derivative of the Laplace transform, which is:

$$
-\int_{0}^{1} x \frac{x^2}{\sqrt{\pi(1-t^2)}} t^{s-1} dt.
$$
In the present case we cannot take the $x \to 0$ limit by simply substituting $x^2$
for $x$ because $\frac{x e^t}{t^{3/2}}$ does not converge uniformly to 0 as $x \to 0$. In
fact it is a “delta function type function”, its integral being
\[
\int_0^1 \frac{x^2}{\sqrt{\pi} t^{3/2}} \, dt = 1.
\]
For it is know from theory of the heat equation that, for an arbitrary continuous
function $f(t)$
\[
\lim_{x \to 0} \int_0^t \frac{x}{\sqrt{\pi} (t-r)^{3/2}} f(r) \, dr = \lim_{x \to 0} \int_0^t \frac{x}{\sqrt{\pi} r^{3/2}} f(t-r) \, dr = f(t).
\]
Hence in the present case:
\[
- \lim_{x \to 0} \int_0^1 \frac{x e^t}{\sqrt{\pi} (1-t)^{3/2}} t^{s-1} \, dt = t^{s-1} \bigg|_{t=1} = -1,
\]
thus $B = -1$. Therefore the Laplace transform of $\frac{e^{(1-x^2)}}{\sqrt{\pi (1-e^{-2t})}}$ is
\[
AF_1 - x F_2 = \frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)} F\left(\frac{1-s}{2}, \frac{1}{2}; -x^2\right) - x F\left(\frac{1-s}{2}, \frac{3}{2}; -x^2\right).
\]
Now we compute the The Laplace transform of \( \frac{e^{-\frac{x^2}{2}}}{\sqrt{\pi(1-e^{-2t})}} \). It has been shown that it statisfies the backward equation \( u_t = -pu_p + \frac{u_{pp}}{2} \). Therefore its Laplace transform is the solution of the second order linear differential equation
\[ sU = -pU_p + \frac{U_{pp}}{2} \]
that is of the equation
\[ U'' + 2pU' + 2sU = 0 \]
Now the solution of \( u'' - 2kxu' - 4kuu = 0 \) are
\[ F\left(a, \frac{1}{2}; kx^2\right) \text{ and } \]
\[ xF\left(a + \frac{1}{2}, \frac{3}{2}; kx^2\right). \]
Comparing the two equations we get for \( k \)
\[ 2k = 2 \quad \text{and} \quad 4ka = 2s \]
that is \( k = 1 \) and \( a = \frac{s}{2} \). Thus the Laplace transform must be the linear combination of \( G_1 = F\left(\frac{s}{2}, \frac{1}{2}; p^2\right) \) and \( pG_2 = pF\left(\frac{1+s}{2}, \frac{3}{2}; p^2\right) \). To find the coefficients of \( G_1 \) and \( pG_2 \) let us inspect the Laplace transform itself.
\[ \int_0^\infty \frac{e^{-\frac{x^2}{2}}}{\sqrt{\pi(1-e^{-2t})}} e^{-st} \, dt. \]
Writing \( t \) instead of \( e^{-t} \) it transforms again into the Mellin type integral:
\[ \int_0^1 \frac{e^{-\frac{p^2t^2}{2}}}{\sqrt{\pi(1-t^2)}} t^{s-1} \, dt. \]
Substituing \( \sqrt{t} \) instead of \( t \) yields
\[
\frac{1}{2\sqrt{\pi}} \int_0^1 e^{\frac{s}{1-t}} t^{\frac{1}{2}} dt.
\]

Again putting \( p = 0 \) this becomes:

\[
\frac{1}{2\sqrt{\pi}} \int_0^1 \frac{s}{t} dt = A.
\]

The coefficient of \( pG_2 \) can be evaluated the same way as was done for

\[
\frac{x^2}{\sqrt{\pi}(1-e^{-2t})} \quad \text{and it is found to be again } -1.
\]

Thus the Laplace transform of

\[
\frac{x^2}{\sqrt{\pi}(1-e^{-2t})}
\]

is

\[
AG_1 - pG_2 = A \left\{ \text{F} \left( s \frac{1}{2}, \frac{3}{2}; p^2 \right) - p \text{F} \left( 1 - \frac{1}{2}, \frac{3}{2}; p^2 \right) \right\}.
\]

The above result can be arrived at directly from the Laplace transform of

\[
\frac{e^{\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})}.
\]

To this end let us inspect

\[
\int_0^\infty e^{\frac{p^2}{1-e^{-2t}} - st} dt
\]

using

\[
\frac{p^2 e^{-2t}}{1-e^{-2t}} = \frac{p^2}{1-e^{-2t}} - p^2.
\]
This becomes \( e^{p^2} \int_0^\infty \frac{e^{-\frac{p^2}{2}}}{\sqrt{\pi (1 - e^{-2t})}} e^{-st} dt \) and the integrand here is of the same form as of the Laplace transform of \( \frac{e^{-\frac{x^2}{(1 - e^{-2t})}}}{\sqrt{\pi (1 - e^{-2t})}} \) except we have \( p \) instead of \( x \).

Therefore the Laplace transform of \( \frac{e^{-\frac{x^2}{(1 - e^{-2t})}}}{\sqrt{\pi (1 - e^{-2t})}} \) is

\[
e^{p^2} \left\{ \frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)} \right\} F\left(\frac{1-s}{2}, \frac{1}{2}; -p^2\right) - \text{pF}\left(1 - \frac{s}{2}, \frac{3}{2}; -p^2\right) \]

Applying Kummer’s formula \( F(a, c; x) = e^x F(c - a, c; x) \) we get for the Laplace transform of \( \frac{e^{-\frac{p^2}{2}e^{-2t}}}{\sqrt{\pi (1 - e^{-2t})}} \)

\[
\frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)} F\left(\frac{s}{2}, \frac{1}{2}; p^2\right) - \text{pF}\left(1 - \frac{1+s}{2}, \frac{3}{2}; p^2\right). 
\]
3 The Laplace Transforms of \( \frac{e^{-(xpe^{-1})^2}}{\sqrt{\pi(1-e^{-2t})}} \)

We have seen that the \( \frac{e^{-(xpe^{-1})^2}}{\sqrt{\pi(1-e^{-2t})}} \) fundamental function satisfies both the forward and backward equations, therefore its Laplace transform must satisfy both of the ordinary differential equations:

\[
U'' + 2pU' + 2(1-s)U = 0 \quad (3.1)
\]

\[
U'' - 2pU' - 2sU = 0. \quad (3.2)
\]

Because of (3.1) must be of the form: \( HF_1 + KxF_2 \), where \( H \) and \( K \) must be some linear combinations of \( G_1 \) and \( pG_2 \) since it satisfies (3.2) as well. Let us observe that \( \frac{e^{-(xpe^{-1})^2}}{\sqrt{\pi(1-e^{-2t})}} \) is analytic in \( x \) for all values \( p \) and \( t \) except when \( t = 0 \) and \( x = p \), in the latter case it is undefined. Therefore its Laplace transform is analytic in the \( x \leq p \) domain as well. Putting \( x = 0 \) in \( \frac{e^{-(xpe^{-1})^2}}{\sqrt{\pi(1-e^{-2t})}} \) gives

\[
\frac{e^{-p^2e^{-t^2}}}{\sqrt{\pi(1-e^{-2t})}} \text{ and we have seen that its Laplace transform is } AG_1 - pG_2, \text{ so } H = AG_1 - pG_2 \text{ (when } x \leq p \text{). The determination of } K \text{ is more involved. Differentiating the fundamental function by } x \text{ gives:}
\]

\[
2pe^{-t}e^{P^2e^{-t^2}} \sqrt{\pi\left(1-e^{-2t}\right)^2} = e^{p^2t} \frac{2pe^{-t}e^{P^2e^{-t^2}}}{\sqrt{\pi\left(1-e^{-2t}\right)^2}}.
\]

Clearly the coefficient \( K \) is the Laplace transform of (3.3). To evaluate it let us compute the following convolution integral:
\[
\begin{align*}
\text{e}^{p^2} & \frac{2p^2 \text{e}^{-\frac{p^2}{2(1-e^{-2t})}}}{\sqrt{\pi} \left(1 - e^{-2t}\right)^{\frac{3}{2}}} \ast \frac{1}{\sqrt{\pi} \left(1 - e^{-2t}\right)^{\frac{3}{2}}}.
\end{align*}
\]

(3.4)

It has been shown that the Laplace transform of the second factor in (3.4) is \( A \), so the Laplace transform of (3.3) is the Laplace transform of (3.4) divided into \( A \).

Next we evaluate (3.4):

\[
\int_0^t \text{e}^{p^2} \frac{2p^2 \text{e}^{-\frac{p^2}{2(1-e^{-2t})}}}{\sqrt{\pi} \left(1 - e^{-2t}\right)^{\frac{3}{2}}} \frac{1}{\sqrt{\pi} \left(1 - e^{-2(t-r)}\right)^{\frac{3}{2}}} \text{d}r =
\]

putting \( r \) for \( e^{-r} \) yields:

\[
\int_T^1 \text{e}^{p^2} \frac{2p^2 \text{e}^{-\frac{p^2}{2(1-r^2)}}}{\sqrt{\pi} \left(1 - r^2\right)^{\frac{3}{2}}} \frac{1}{\sqrt{\pi} \left(r^2 - T^2\right)} \text{d}r =
\]

where \( T = e^{-r} \). Substituting \( \sqrt{r} \) for \( r \) gives:

\[
\text{e}^{p^2} \int_T^1 \frac{p^2}{\sqrt{\pi} \left(1 - r^2\right)^{\frac{3}{2}}} \frac{1}{\sqrt{\pi} \left(r^2 - T^2\right)} \text{d}r =
\]

\[
= \text{e}^{p^2} \int_0^{1-T^2} \frac{p^2}{\sqrt{\pi} \left(1 - T^2 - r\right)^{\frac{3}{2}}} \frac{1}{\sqrt{\pi} r} \text{d}r,
\]

\[
= \text{e}^{-p^2} \cdot \frac{p^2}{\sqrt{\pi} t^3} \ast \frac{1}{\sqrt{\pi} t} \bigg|_{t=1-T^2} = \frac{e^{-p^2} e^{-2t}}{\sqrt{\pi} (1 - 2^{-2t})}.
\]

Thus we have for the coefficient \( K = \frac{\mathcal{A}G_1 - \mathcal{P}G_2}{\mathcal{A}} \). Hence the Laplace transform

\[
\frac{(x-p^{-1})^2}{\sqrt{\pi} (1 - e^{-2t})}
\]

of \( \text{e}^{\frac{(x-p^{-1})^2}{\sqrt{\pi} (1 - e^{-2t})}} \) is for \( x \leq p \):
\[
(AG_1 - pG_2)F_1 + xF_2 \frac{AG_1 - pG_2}{A} = \frac{(AF_1 + xF_2)(AG_1 - pG_2)}{A}.
\]

Next let us consider the case \( p \leq x \). If the same computation is repeated but instead of \( x = 0 \) we look at \( p = 0 \), that is we compute the coefficients of \( G_1 \) and \( pG_2 \). Putting \( p = 0 \) in \( \frac{x}{\sqrt{\pi(1 - e^{-2t})}} \) gives \( \frac{x}{\sqrt{\pi(1 - e^{-2t})}} \). Its Laplace transform is \( AF_1 - xF_2 \), carrying through similar computation as was done for the coefficient of \( xF_2 \) we get for the coefficient for \( pG_2 \frac{AF_1 - xF_2}{A} \). Thus the Laplace transform of \( \frac{x}{\sqrt{\pi(1 - e^{-2t})}} \) when \( p \leq x \) is:

\[
\frac{(x - p)^{-1}}{\sqrt{\pi(1 - e^{-2t})}}.
\]

Hence the Laplace transform of \( \frac{(x - p)^{-1}}{\sqrt{\pi(1 - e^{-2t})}} \) is:

\[
\begin{cases}
\frac{(x - p)^{-1}}{\sqrt{\pi(1 - e^{-2t})}} & \text{if } p \leq x \\
\frac{(x - p)^{-1}}{\sqrt{\pi(1 - e^{-2t})}} & \text{if } x \leq p.
\end{cases}
\]

(3.5)

4 Level Crossing Probabilites

Let the random variable \( FC \) or \( FC(x) \) (first crossing) be the smallest possible value of \( t \) such that \( X_t = x \) given \( X_0 = p \). Let \( \phi(t, p, x) \) be the distribution of \( FC \), clearly: \( \phi(t, p, x) \ast P(t, x, x) = P(t, p, x) \).

That is: \( \phi(t, p, x) \ast \frac{e^{-(x - p)^2}}{\sqrt{\pi(1 - e^{-2t})}} = \frac{e^{-(x - p)^2}}{\sqrt{\pi(1 - e^{-2t})}} \).
Now the probability that $X_t$ stays below $x$ is: 

$$P\left(\sup_{0 \leq r \leq t} X_r \leq x\right) = 1 - \int_0^t \varphi(r) \, dr.$$ 

Let us denote the Laplace transform of $\varphi$ by $\Psi$, then $\Psi$ for $0 \leq p \leq x$ using (3.5) can be expressed as

$$\psi = \frac{(AG_1(p) + pG_2(p))(AF_1(x) + xF_2(x))}{(AG_1(x) + xG_2(x))(AF_1(x) + xF_2(x))}$$ (4.1)

$$= \frac{AG_1 + pG_2}{AG_1 + xG_2}$$ (4.2)

For the special case when $p = 0$, that is when $X_t$ reaches level $x$ subject to the initial condition $X_0 = 0$ is

$$\psi = \frac{A}{AG_1 + xG_2}.$$ (4.3)

For this case Bellman and Harris [1] found the following expression:

$$\psi = \int_0^\infty e^{-y^2 - 2sy + s^2} \, dy.$$ (4.4)

For the case $p \geq x \geq 0$ :

$$\psi(p, x) = \frac{(AF_1 + xF_2)(AG_1 - pG_2)}{(AF_1 + xF_2)(AG_1 - xG_2)} \frac{A}{A}$$ (4.5)

$$= \frac{AG_1 - pG_2}{AG_1 - xG_2}$$ (4.6)

For the special case $p > 0$, $x = 0$ we have:

$$\psi(p, 0) = \frac{AG_1 - pG_2}{A}.$$ (4.7)

Using (4.7) it is not difficult to show that (4.2) holds for $p \leq x$ and holds for $p \geq x$ as well. Formula (4.7) easily invertable, for
\[
\frac{1}{A} = \frac{2\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = \frac{s\Gamma\left(\frac{s+1}{2}\right)}{\frac{s}{2}\Gamma\left(\frac{s}{2}\right)} = \frac{s}{2}\Gamma\left(\frac{s}{2} + 1\right).
\]

Clearly \(\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2} + 1\right)}\) is the Laplace transform of \(2 \cdot \frac{e^{-t}}{\sqrt{\pi(1-e^{-2t})}}\). Hence (4.7) is the Laplace transform of:

\[
2 \cdot \frac{d}{dt} \left(\frac{e^{-t}}{\sqrt{\pi(1-e^{-2t})}} \cdot e^{-t}\right) = 2 \cdot \frac{d}{dt} \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z^2} dz = \frac{2pe^{-t}}{\sqrt{\pi}} \frac{e^{- \frac{p^2}{1-e^{-2t}}}}{(1-e^{-2t})^{1/2}}.
\]

### 5 The Equivalence of Bellman-Haris’ and our Result

To show that formulas (4.3) and (4.4) are the same, we have to evaluate the integral \(\int_{0}^{\infty} e^{-y^2 + 2xy} = e^{x^2} \cdot e^{-(x-y)^2} = e^{x^2} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} e^{-y^2} \frac{d^n}{dy^n} \).

Substituting this into the integral we get:

\[
\int_{0}^{\infty} e^{-y^2 + 2xy} y^{s-1} dy = e^{x^2} \cdot \int_{0}^{\infty} e^{-(x-y)^2} y^{s-1} dy = e^{x^2} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \int_{0}^{\infty} \frac{d^n}{dy^n} e^{-y^2} y^{s-1} dy.
\]

Let us observe that the integrals on the right hand side are the Mellin transforms of the functions \(\frac{d^n}{dy^n} e^{-y^2}\). First we compute the Mellin transform of \(e^{-y^2}\) which is:

\[
\int_{0}^{\infty} e^{-y^2} y^{s-1} dy = \frac{1}{2} \int_{0}^{\infty} e^{-y^2} y^{s-1} dy = \frac{1}{2} \Gamma\left(\frac{s}{2}\right).
\]
Let us denote the Mellin transform of a function $f$ by $M$ or $F$. It is not difficult to see that:

$$M(f') = -(s-1)F(s-1)$$
$$M(f'') = -(s-1)(s-2)F(s-2)$$

...  

$$M(f^n) = (-1)^n(s-1)(s-2)\cdots(s-n)F(s-n).$$

Hence the Mellin transforms of $e^{-y^2}$, $\frac{de^{-y^2}}{dy}$, $\frac{d^2e^{-y^2}}{dy^2}$, $\frac{d^3e^{-y^2}}{dy^3}$ ... are

$$\frac{1}{2} \Gamma\left(\frac{s}{2}\right) - \frac{(s-1)}{2} \Gamma\left(\frac{s}{2}\right) - \frac{(s-2)}{2} \Gamma\left(\frac{s}{2}\right) - \frac{(s-3)}{2} \Gamma\left(\frac{s}{2}\right),$$

$$\frac{(s-4)(s-3)(s-2)(s-1)}{2} \Gamma\left(\frac{s}{2}\right),$$

Substituting these into (5.1) gives:

$$e^{-x^2} \left\{ \frac{1}{2} \Gamma\left(\frac{s}{2}\right) + \frac{x}{2!} \Gamma\left(\frac{s}{2}\right) + \frac{x^2}{2!} \Gamma\left(\frac{s}{2}\right) + \frac{x^3}{3!} \Gamma\left(\frac{s}{2}\right) + \frac{x^4}{4!} \Gamma\left(\frac{s}{2}\right) + \cdots \right\}$$

$$= e^{-x^2} \left\{ \frac{1}{2} \Gamma\left(\frac{s}{2}\right) + \frac{x^2}{2!} \Gamma\left(\frac{s}{2}\right) + \frac{x^4}{4!} \Gamma\left(\frac{s}{2}\right) + \cdots \right\}$$

$$+ e^{-x^2} \left\{ \frac{x}{2!} \Gamma\left(\frac{s}{2}\right) + \frac{x^3}{3!} \Gamma\left(\frac{s}{2}\right) + \frac{x^5}{5!} \Gamma\left(\frac{s}{2}\right) + \cdots \right\}$$

$$= e^{-x^2} \left\{ \frac{1}{2} \Gamma\left(\frac{s}{2}\right) + \frac{x^2}{2!} \Gamma\left(\frac{s}{2}\right) + \frac{x^4}{4!} \Gamma\left(\frac{s}{2}\right) + \cdots \right\}$$
Hence Bellman and Harrises formula becomes:

\[
\frac{1}{2} \Gamma\left(\frac{s}{2}\right) \int_0^\infty e^{-y^2+2xy} y^{s-1} \, dy = \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \frac{1}{2} \Gamma\left(\frac{s + 1}{2}\right) + \Gamma\left(\frac{s + 1}{2}\right) xF\left(\frac{1 + s}{2}, \frac{3}{2}; x^2\right).
\]

Diving both the numerator and the denominator of the right hand side into \( \Gamma\left(\frac{s + 1}{2}\right) \) gives \( \frac{A}{AG_1 + xG_2} \) and this completes the proof.

Reference