

# On the Application of the Fuzzified Choquet Integral to Multiple Criteria Evaluation

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*Abstract: The Choquet integral plays an important role in models of multiple criteria evaluation. Compared to the weighted average, the Choquet integral can be used to aggregate partial evaluations of an object even if the criteria are redundant or synergic. In the paper, we employ the Choquet integral for the generalization of the Partial Goals Method. We also discuss the application of the fuzzified Choquet integral to multiple criteria evaluation and propose a new technique for fuzzy measure construction. The theory is demonstrated on an example implemented in a new software capable of dealing with multiple criteria evaluation via the fuzzified Choquet integral.*

*Keywords: Choquet integral; multiple criteria evaluation; fuzzification; software*

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## 1 Introduction

In the paper, we present a way of employing the fuzzified Choquet integral [1] in multiple criteria evaluation. The first part of the paper deals with the Choquet integral [2, 4, 5, 6, 7, 8, 13] and its applications to multiple criteria evaluation. For the Partial Goals Method [11] we compare a weighted average, which is an aggregation operator based on an additive measure, with aggregation operators employing a generalised monotonous measure – a fuzzy measure. We specify the general conditions for the application of the Choquet integral to the aggregation of partial evaluations. The second part of the paper deals with the fuzzified Choquet integral. We focus on the second level fuzzified Choquet integral [1], i.e. we consider partial evaluations and the values of the fuzzy measure (weights of the sets of partial goals) to be modelled by fuzzy numbers. We explain the use of the fuzzy number valued (FNV) fuzzy measure [1] in evaluation models (for example, for the weights of the set of criteria, which are determined expertly). The major contribution of this paper is a proposal of a feasible new way of setting the FNV-fuzzy measure on a set of partial goals. We present the method in a clear algorithmic form and compare it to the general approach to multiple criteria

evaluation – the base of rules method. The last part of the paper is devoted to an example implemented in a new software created for the application of the fuzzified Choquet integral in multiple criteria evaluation. The software employs the theory presented in this paper and the Partial Goals Method.

## 2 Choquet Integral in Multiple Criteria Evaluation

A multiple criteria evaluation problem can be described as follows: Evaluators have a list of criteria and object  $x$ . The object  $x$  is evaluated with respect to each criterion individually and as a consequence, several partial evaluations of  $x$  are formed. Now, the question is how to merge these partial evaluations into a single overall evaluation of object  $x$ ?

There are several possible ways to address this issue. The most general approach is the base of rules method [9]. However, to create the complete basis is usually prohibitively demanding in practice, therefore we seek to employ other, more feasible methods. One of them is the Partial Goals Method (PGM) [11], which is based on the following assumption: Object  $x$  should achieve the overall goal  $G_0$ , which can be replaced by  $n$  partial goals  $G_1, \dots, G_n$ , covering the overall goal without overlapping. The partial goals form a partition of  $G_0$  and each partial goal is associated with a single criterion. The weight of the criterion corresponds to the proportion of the partial goal in the overall goal. The overall evaluation of  $x$  is calculated by weighted average of the partial evaluations, which represent the degrees of fulfillment of the particular partial goals. The overall evaluation of  $x$  then represents the degree of achievement of the overall goal.

The problem becomes more complex when the partial goals overlap. The overlapping goals are associated with redundant criteria, which monitor different aspects of a single feature of the object. As an example we can use the evaluation of students' aptitude for the study of science. Let the students' evaluations be based on their results of Mathematics, Chemistry and Physics tests. Although Mathematics differs from Physics, they have a lot in common. Therefore, the criteria monitor a similar feature and are partly redundant.

Another problem occurs when criteria are synergic. If object  $x$  fully achieves partial goals which are associated to synergic criteria, the evaluation of the object is higher than the sum of the weights of the particular criteria. Analogically, if the object achieves all but one partial goal, its evaluation is diminished more than by the weight of the unfulfilled goal. The synergy among the criteria can be demonstrated by the following example. Let us evaluate the career perspective of young mathematicians. The criteria are: Knowledge of mathematics, Language skills and Communication skills. The knowledge of mathematics is essential for math scientists. Nevertheless, the mathematicians need to possess all the mentioned skills if they want to be internationally competitive. Therefore, a

mathematician who knows languages should be evaluated higher than the one who does not. On the other hand, communicative people who know plenty of languages but nothing about mathematics will never be math scientists, and their evaluation as promising mathematicians should be zero.

The interactions among the criteria make the application of PGM impossible, because the partial evaluations with respect to the interacting criteria cannot be aggregated by weighted average. Nevertheless, if the weighted average is replaced by a more general aggregation operator which can handle redundant or synergic criteria, we can still use the basic idea of PGM. The suitable operator for this task is the Choquet integral, which is an integral with respect to a non-additive measure, called a fuzzy measure [2, 4, 5, 6, 7, 8, 13].

**Definition 1** A fuzzy measure on a finite nonempty set  $\Omega$ ,  $\Omega = \{\omega_1, \dots, \omega_n\}$ , is a set function  $\mu: \wp(\Omega) \rightarrow [0,1]$ , where  $\wp(\Omega)$  is a power set of  $\Omega$ , satisfying the following conditions

$$\mu(\emptyset) = 0, \mu(\Omega) = 1 \text{ (boundary conditions),}$$

$$C \subseteq D \text{ implies } \mu(C) \leq \mu(D) \text{ for any } C, D \in \wp(\Omega) \text{ (monotonicity).}$$

**Definition 2** Let  $\Omega = \{\omega_1, \dots, \omega_n\}$  be a finite nonempty set,  $\mu$  be a fuzzy measure on  $\Omega$ , and  $f: \Omega \rightarrow [0,1]$ , then the discrete Choquet integral of  $f$  is defined as follows

$$(C) \int_{\Omega} f d\mu = \sum_{i=1}^n [f(\omega_i) - f(\omega_{i-1})] \mu(B_{(i)}), \quad (1)$$

where  $(1), \dots, (n)$  is a permutation of indices  $1, \dots, n$ , such that  $f(\omega_{(1)}) \leq f(\omega_{(2)}) \leq \dots \leq f(\omega_{(n)})$ ,  $B_{(i)} = \{\omega_{(i)}, \omega_{(i+1)}, \dots, \omega_{(n)}\}$ , and for  $f(\omega_{(0)}) = 0$  by convention.

Here,  $\Omega$  represents the set of partial goals,  $\mu(A)$ ,  $A \in \wp(\Omega)$ , is interpreted as the proportion of the set of partial goals  $A$  in the overall goal, and the values  $f(\omega_i)$ ,  $i = \{1, \dots, n\}$ , represent partial evaluations of the object  $x$  with respect to criteria associated with partial goals  $\omega_i$ .

The Choquet integral is capable of dealing with synergic or redundant criteria. Moreover, for a special fuzzy measure [12], the Choquet integral can become the weighted average. Apparently, the evaluation method based on PGM employing the Choquet integral instead of weighted average is a natural generalization of PGM.

Note, however, that Generalized PGM (GPGM) cannot handle the problems when the interactions among the criteria are not valid within the whole domain of the criteria, but depend on the combinations of particular values of particular criteria. The typical example of such interactions arises in the evaluation of cuisine. Let us suppose we want to evaluate a meal with respect to criteria “Meat” and “Side-dish”. Here, the evaluation of the meal would depend on the specific combination of the meat and the side-dish. We might like the meat, we might like the side-dish, but if they do not go well together, the meal is not satisfactory.

### 3 A Role of Fuzzy Sets in Multiple Criteria Evaluation

In multiple criteria evaluation, we consider two types of criteria: quantitative and qualitative. Partial evaluations with respect to quantitative criteria are results of measurements transformed by expertly set evaluative functions into the interval  $[0,1]$ , while partial evaluations with respect to qualitative criteria are set directly by the expert. Both kinds of partial evaluations can be burdened by uncertainty: the measurements can be inaccurate, and the expertly set evaluations are subjective. Also the weights of the criteria are usually estimated expertly and are therefore vague. To deal with the uncertainty, we can model the partial evaluations by fuzzy numbers [3].

Let  $U$  be a nonempty set. A fuzzy set  $C$  on  $U$  is defined by the mapping  $C:U \rightarrow [0,1]$ . For each  $x \in U$  the value  $C(x)$  is called a membership degree of the element  $x$  in the fuzzy set  $C$ .  $\mathcal{F}(U)$  denotes the system of all fuzzy sets on the set  $U$ . A fuzzy number  $C$  is a fuzzy set on the set of real numbers  $\mathbb{R}$  with following properties: The kernel of  $C$ ,  $\text{Ker } C = \{x \in \mathbb{R} \mid C(x) = 1\}$ , is nonempty, the  $\alpha$ -cuts of  $C$ ,  $C_\alpha = \{x \in \mathbb{R} \mid C(x) \geq \alpha\}$ , are closed intervals for all  $\alpha \in (0,1]$ , and the support of  $C$ ,  $\text{Supp } C = \{x \in \mathbb{R} \mid C(x) > 0\}$ , is bounded. The family of all fuzzy numbers will be denoted by  $\mathcal{F}_N(\mathbb{R})$ . If  $C$  is a fuzzy number and  $\text{Supp } C \subseteq [a,b]$ , then  $C$  is referred to as a fuzzy number on  $[a,b]$  and the set of all fuzzy numbers on  $[a,b]$  is denoted by  $\mathcal{F}_N([a,b])$ .

Fuzzy number  $C$  with a membership function  $C(\cdot)$  can be alternatively described by a couple of functions  $\underline{c}: [0,1] \rightarrow \mathbb{R}$ ,  $\bar{c}: [0,1] \rightarrow \mathbb{R}$ , such that  $[\underline{c}(\alpha), \bar{c}(\alpha)] = C_\alpha$ ,  $\alpha \in (0,1]$ ,  $[\underline{c}(0), \bar{c}(0)] = \text{Cl}(\text{Supp}(C))$ , where  $\text{Cl}(\cdot)$  stands for the closure of a set. In this paper we will use the interval representation of fuzzy numbers and the notation  $C = \left\{ [\underline{c}(\alpha), \bar{c}(\alpha)] \mid \alpha \in [0,1] \right\}$ .

We say that a fuzzy number  $A = \left\{ \left[ \underline{a}(\alpha), \bar{a}(\alpha) \right] \mid \alpha \in [0,1] \right\}$  is less than or equal to a fuzzy number  $B = \left\{ \left[ \underline{b}(\alpha), \bar{b}(\alpha) \right] \mid \alpha \in [0,1] \right\}$ , denoted by  $A \leq B$ , if  $\underline{a}(\alpha) \leq \underline{b}(\alpha)$  and  $\bar{a}(\alpha) \leq \bar{b}(\alpha)$  for any  $\alpha \in [0,1]$ .

We will call  $F$  a fuzzy number-valued function (FNV-function) if  $F : \Omega \rightarrow \mathcal{F}_N(\mathbb{R})$ , where  $\Omega$  is a nonempty set.

The aggregation of partial fuzzy evaluations with respect to non-interacting criteria with fuzzy weights can be handled by fuzzy weighted average [10]. Analogically, the partial fuzzy evaluations with respect to synergic or redundant criteria related to fuzzy weights are aggregated with help of the fuzzified Choquet integral and FNV-fuzzy measure [1].

**Definition 3** A FNV-fuzzy measure on a finite set  $\Omega$ ,  $\Omega = \{\omega_1, \dots, \omega_n\}$ , is a set function  $\mu_F : \wp(\Omega) \rightarrow \mathcal{F}_N([0,1])$  satisfying the following conditions:

- $\mu_F(\emptyset) = 0, \mu_F(\Omega) = 1$  (boundary conditions),
- $C \subseteq D$  implies  $\mu_F(C) \leq \mu_F(D)$  for any  $C, D \in \wp(\Omega)$  (monotonicity).

The fuzzified Choquet integral, the integral of the FNV-function with respect to FNV-fuzzy measure, is then defined with the help of the extension principle.

**Definition 4** Let  $\Omega = \{\omega_1, \dots, \omega_n\}$  be a nonempty finite set,  $B_1, \dots, B_{2^n-1}$  be all its nonempty subsets,  $\mu_F$  be a FNV-fuzzy measure on  $\Omega$ , and  $F : \Omega \rightarrow \mathcal{F}_N([0,1])$ ,  $F(\omega_i) = H_i$ ,  $i = 1, \dots, n$ , be a FNV-function. The Choquet integral of  $F$  with respect to FNV-fuzzy measure  $\mu_F$  is defined as a fuzzy number  $Y$  with a membership function given for any  $y \in [0,1]$  by

$$\begin{aligned}
 Y(y) = & \max \left\{ \min \left\{ H_1(h_1), \dots, H_n(h_n), \mu_F(B_1)(\mu_1), \dots, \mu_F(B_{2^n-1})(\mu_{2^n-1}) \right\} \mid \right. \\
 & \left. h_i \in [0,1], i = 1, \dots, n, \mu_j \in [0,1], j = 1, \dots, 2^n - 1, \right. \\
 & \left. y = h_{(n)}\beta_n + \sum_{i=1}^{n-1} h_{(i)}[\beta_i - \beta_{i+1}], \beta_k - \beta_{k+1} \geq 0, k = 1, \dots, n-1, \right. \quad (2) \\
 & \left. \text{and for } i = 1, \dots, n, \text{ it holds that } \beta_i = \mu_j, \right. \\
 & \left. \text{where } j \in \{1, \dots, 2^n - 1\} \text{ such that } B_j = \{\omega_{(i)}, \dots, \omega_{(n)}\} \right\}
 \end{aligned}$$

where  $(1), \dots, (n)$  is a permutation of indices  $1, \dots, n$  such that  $h_{(1)} \leq h_{(2)} \leq \dots \leq h_{(n)}$ .

Note that in Definition 4 we are looking for the minimum among  $(n+2^n-1)$  membership degrees, even though there are only  $2n$  variables ( $n$  partial evaluations and  $n$  weights) in the formula for the Choquet integral. The seemingly large number  $(n+2^n-1)$  is a consequence of the need to consider all possible subsets of the set of partial goals, because for each  $n$ -tuple  $h_1, \dots, h_n$  a different  $n$ -tuple of subsets  $B_i$  of the set of partial goals is relevant. The relevant  $n$ -tuple of subsets  $B_i$  is given by the condition in Definition 4. The remaining membership degrees, among which we are looking for the minimum but which are irrelevant for given  $n$ -tuple  $h_1, \dots, h_n$ , can be considered equal to 1, and therefore do not influence the minimum value.

It is difficult to calculate the integral using the definition. The method for simpler calculation of the fuzzified Choquet integral was presented in [1].

## 4 Fuzzy Measure and FNV-Fuzzy Measure

The application of GPGM to a multiple criteria evaluation problem requires the correct construction of the fuzzy measure or FNV-fuzzy measure. Given  $n$  criteria, the evaluator needs to set the weight of each subset of the set of criteria,  $2^n - 2$  values in total (the weights of the empty set and the whole set of criteria are given by the definitions of the fuzzy measure or FNV-fuzzy measure). Moreover, the fuzzy measure should maintain monotonicity arising due to inclusion, and the evaluator should keep that in mind. Since a direct construction is not a trivial task, sometimes it is beneficial to consider more complex methods, which may require more parameters to be entered, but which are easier on the evaluator. Various methods of constructing the fuzzy measure have already been described in an overview work by Grabisch and Labreuche [6]. In the following text, we would like to propose another approach to setting the fuzzy measure and FNV-fuzzy measure.

Let us suppose that object  $x$  has to achieve  $n$  partial goals  $G_1, \dots, G_n$  describing the overall goal  $G_0$ . Let object  $x$  fully achieve partial goal  $G_1$  and totally fail with respect to the rest of the partial goals. Object  $x$  then can be denoted as  $x = (1, 0, \dots, 0)$  and for its overall evaluation  $h_{(1,0,\dots,0)}$  obtained by the Choquet integral it holds

$$h_{(1,0,\dots,0)} = 0 \cdot \mu(\{G_1, \dots, G_n\}) + \dots + 0 \cdot \mu(\{G_1, G_2\}) + 1 \cdot \mu(G_1) = \mu(G_1). \quad (3)$$

Analogically, we can see that the evaluations of objects  $(1, 1, 0, \dots, 0)$  and  $(1, 1, 1, 0, \dots, 0)$  are equal to  $\mu(\{G_1, G_2\})$  and  $\mu(\{G_1, G_2, G_3\})$ , respectively, i.e.

$$h_{(1,1,0,\dots,0)} = 0 \cdot \mu(\{G_1, \dots, G_n\}) + \dots + 1 \cdot \mu(\{G_1, G_2\}) + 0 \cdot \mu(G_1), \quad (5)$$

$$h_{(1,1,1,\dots,0)} = 0 \cdot \mu(\{G_1, \dots, G_n\}) + \dots + 1 \cdot \mu(\{G_1, G_2, G_3\}) + \dots + 0 \cdot \mu(G_1). \quad (6)$$

All in all, the values of fuzzy measure of the sets of partial goals are equal to the evaluations of the corresponding objects. As a consequence, the construction of the fuzzy measure can be simplified by evaluation of  $2^n$  imaginary objects  $x^1 = (0, 0, \dots, 0)$ ,  $x^2 = (1, 0, \dots, 0)$ ,  $x^3 = (0, 1, 0, \dots, 0)$ , ...,  $x^{2^n} = (1, 1, \dots, 1)$ .

The evaluation can be done in two steps. In the first step, we order the objects decreasingly by comparing all pairs of the given alternatives. During the process, we create a matrix  $A = \{a_{ij}\}_{i,j=1}^{2^n}$ , where  $a_{ij} = 1$  if the object from the  $i$ -th row achieves the overall goal better than the object from the  $j$ -th column,  $a_{ij} = 0.5$  if the object from the  $i$ -th row is as good as the object from the  $j$ -th column, and  $a_{ij} = 0$  otherwise. To each object  $x^i$  there is assigned value  $\sum_{j=1}^{2^n} a_{ij}$ ,  $i = 1, \dots, 2^n$ , and all the imaginary objects are arranged into several groups  $\Pi_1 \succ \Pi_2 \succ \dots \succ \Pi_k$ ,  $k \leq 2^n$ , such that objects in each group are assigned the same value and the objects in group  $\Pi_j$  are assigned higher value than objects in group  $\Pi_{j+1}$ ,  $j = 1, \dots, k-1$ .

Using this approach, we need to set only the upper triangle of the matrix, because  $a_{ij} = 1 - a_{ji}$  for any  $i, j \in \{1, \dots, 2^n\}$ . Moreover, some values are not optional (they are fixed from the definition of fuzzy measure); therefore the evaluator needs to set at most  $2^{2^n-1} - 2^{n-1} - 1$  values. During the process of creating the matrix  $A$ , we can also employ the inclusion and transitivity properties to reduce the number of needed elements even further. For example, for  $n = 3$  the 27 required parameters can be reduced up to only 9. The real number of reduced parameters will depend on the actual choice of values and the nature of the problem, and without that knowledge it can be only bounded from above and from below.

In the second step, we use the ordering of the groups  $\Pi_1 \succ \Pi_2 \succ \dots \succ \Pi_k$ , and add information about the intensity of the relation between the groups  $\Pi_j, \Pi_{j+1}$ ,  $j = 1, \dots, k-2$ . The intensity of the relation  $r_j$ ,  $j = 1, \dots, k-2$ , can be described linguistically, e.g. by terms "as good as", "slightly better", "quite better", "strongly better" or "extremely better", with numbers  $1, t_1, t_2, t_3, t_4$  quantifying the relation. For example, we may consider numbers 1, 2, 3, 4, 5. If  $r_j = t_j$ , it means that the objects in the group  $\Pi_j$  are  $t_j$ -times better than the objects in the group  $\Pi_{j+1}$ . The evaluation  $h^j$  of the objects from  $\Pi_j$  is then given as follows,  $h^1 = 1$ ,  $h^j = \frac{1}{r_1 \cdots r_{j-1}}$ ,  $j = 2, \dots, k-1$ ,  $h^k = 0$ . For example, let us suppose

$(1, 1, \dots, 1) \in \Pi_1 \succ (1, 1, \dots, 1, 0) \in \Pi_2$ . If we describe the relation between the first and the second object as “slightly better”, then the evaluation of the object  $(1, 1, \dots, 1, 0)$  is  $\frac{1}{t_1}$  and  $\mu(\{G_1, G_2, \dots, G_{n-1}\}) = \frac{1}{t_1}$ .

It is also possible to have matrix  $A$  created only from ones and zeros, when  $a_{ij} = 1$  if object from the  $i$ -th row is better or as good as the object from the  $j$ -th column, and  $a_{ij} = 0$  otherwise. However, creating this matrix requires more parameters to be set by the evaluator, for  $n$  criteria it is  $2^{2n} - 3^n$  parameters.

The FNV-fuzzy measure can be set in a similar way. The imaginary objects are ordered into groups  $\Pi_1 \succ \Pi_2 \succ \dots \succ \Pi_k$ ,  $k \leq 2^n$ , with the help of matrix  $A$  and the evaluator then compares the successive groups and describes the intensity of the relation between them. This time the linguistic terms “as good as”, “slightly better”, “quite better”, “strongly better” or “extremely better” are associated with fuzzy numbers  $1, T_1, T_2, T_3, T_4$ . After assigning the value  $R_j \in \{1, T_1, T_2, T_3, T_4\}$ ,

$R_j = \left\{ \left[ \underline{r}_j(\alpha), \overline{r}_j(\alpha) \right] \mid \alpha \in [0, 1] \right\}$  to any pair of successive groups  $\Pi_j, \Pi_{j+1}$ ,  $j = 1, \dots, k-2$ , we can evaluate the objects from  $\Pi_j$ ,  $j = 2, \dots, k-1$ , by fuzzy

number  $H^j = \left\{ \left[ \underline{h}^j(\alpha), \overline{h}^j(\alpha) \right] \mid \alpha \in [0, 1] \right\}$ , where for all  $\alpha \in [0, 1]$  holds

$$\underline{h}^j(\alpha) = \frac{1}{\underline{r}_1(\alpha) \cdots \underline{r}_{j-1}(\alpha)}, \quad (7)$$

$$\overline{h}^j(\alpha) = \frac{1}{\overline{r}_1(\alpha) \cdots \overline{r}_{j-1}(\alpha)}. \quad (8)$$

For the evaluation of the objects from the group  $\Pi_1$  and  $\Pi_k$  there holds  $H^1 = 1$ ,  $H^k = 0$ , respectively. The evaluations of the objects are equal to the weights of the corresponding sets of the partial goals; thus the FNV-fuzzy measure is set.

The construction of the fuzzy measure can be described also in the form of an algorithm:

- Step 1 Generate imaginary objects  $x^i$ ,  $i=1, \dots, 2^n$ , such that each object corresponds to a different subset of the set of partial goals  $\{G_1, G_2, \dots, G_n\}$ , and it absolutely satisfies the goals in the subset and totally fails in achieving all the others.



- Step 2 Create matrix  $A = \{a_{ij}\}_{i,j=1}^{2^n}$  : For each  $i=1, \dots, 2^n$  and for each  $j=1, \dots, 2^n$  check if  $a_{ij}$  has already been set. If yes, then  $a_{ji}=1- a_{ij}$ . Otherwise check the following:
- Inclusion: if the set of partial goals corresponding to the object  $x^i$  is a proper superset (equal to, proper subset) of the set of partial goals corresponding to the object  $x^j$ , then  $a_{ij}=1$  ( $a_{ij}=1/2, a_{ij}=0$ ). Proceed to the next element of matrix  $A$ .
  - Transitivity: if there exists an index  $z$  such that  $a_{iz}= a_{zj}=1$  ( $a_{iz}= a_{zj}=0, a_{iz}= a_{zj}=1/2$ ), then  $a_{ij}=1$  ( $a_{ij}=0, a_{ij}=1/2$ ). Proceed to the next element of matrix  $A$ .
  - Goal fulfillment: compare object  $x^i$  with object  $x^j$ . If the object  $x^i$  fulfils the overall goal  $G_0$  better than the object  $x^j$ , then  $a_{ij}=1$ ; if both objects achieve the overall goal equally, then  $a_{ij}=1/2$ ; otherwise  $a_{ij}=0$ . Proceed to the next element of matrix  $A$ .
- Step 3 For each  $i=1, \dots, 2^n$  calculate preference index  $\sum_{j=1}^{2^n} a_{ij}$ .
- Step 4 Order the objects decreasingly according to their preference indexes and bunch them into groups  $\Pi_q, q=1, \dots, k, k \leq 2^n$  such that all the members of the group have the same preference index.
- Step 5 For  $q=1, \dots, k-1$  compare groups  $\Pi_q, \Pi_{q+1}$  and quantify their relationship by number  $r_q$ , which can attain one of the five values  $\{1, t_1, t_2, t_3, t_4\}$ , where each value corresponds to one linguistic term of “as good as”, “slightly better”, “quite better”, “strongly better” or “extremely better”, respectively.
- Step 6 For  $q=2, \dots, k-1$  calculate the evaluation  $h^q$  of the objects from group  $\Pi_q$  :  $h^q = \frac{1}{r_1 \cdots r_{q-1}}$ . The remaining two evaluations are  $h^1=1$  and  $h^k=0$ .
- Step 7 Fuzzy measure of each subset of the set of partial goals is then equal to the evaluation  $h^q$  of the corresponding object  $x^i \in \Pi_q$ .

The FNV-fuzzy measure can be constructed in a similar way, only with steps 5, 6 and 7 replaced by steps 5b, 6b and 7b:

- Step 5b) For  $q=1, \dots, k-1$  compare groups  $\Pi_q, \Pi_{q+1}$  and quantify their relationship by fuzzy number  $R_q = \left\{ \left[ \underline{r}_q(\alpha), \overline{r}_q(\alpha) \right] \mid \alpha \in [0, 1] \right\}$ ,

which can attain one of the five values  $\{1, T_1, T_2, T_3, T_4\}$ , where each value corresponds to one linguistic term of “as good as”, “slightly better”, “quite better”, “strongly better” or “extremely better”, respectively.

Step 6b) For  $q = 2, \dots, k-1$  calculate the evaluation

$$H^q = \left\{ \left[ \underline{h}^q(\alpha), \overline{h}^q(\alpha) \right] \mid \alpha \in [0, 1] \right\}$$

of the objects from group  $\Pi_q$  in the following way: For all  $\alpha \in [0, 1]$  calculate

$$\underline{h}^q(\alpha) = \frac{1}{r_1(\alpha) \cdots r_{q-1}(\alpha)} \quad \text{and} \quad \overline{h}^q(\alpha) = \frac{1}{\underline{r}_1(\alpha) \cdots \underline{r}_{q-1}(\alpha)}.$$

The remaining two evaluations are  $H^1=1$  and  $H^k=0$ .

Step 7b) The FNV-fuzzy measure of each subset of the set of partial goals is then equal to the evaluation  $H^q$  of the corresponding object  $x^i \in \Pi_q$ .

Because of the demanding construction of the FNV- fuzzy measure, it may seem that the implementation of the fuzzified Choquet integral to a multiple criteria evaluation problem is too complicated for practical applications. But let us not step ahead of ourselves and compare it to other methods. Any multiple criteria evaluation problem can be described by fuzzy rules base [9], which can deal with any kind of interactions among the criteria, even if the partial evaluations are uncertain and modelled by fuzzy numbers. However, the formulation of fuzzy rules requires an experienced and patient expert. Considering  $n$  criteria, each of which can attain  $m$  different values, the expert has to formulate  $m^n$  fuzzy rules to describe the problem properly. If we compare it to the number of parameters the expert has to set during the direct construction of the FNV-fuzzy measure, i.e.  $2^n$  fuzzy numbers, we can see that for  $m > 2$  the construction of the fuzzy rule base is more demanding than the direct construction of the FNV-fuzzy measure. As was mentioned before, our technique for FNV-fuzzy measure construction requires setting at most  $2^{2n-1} - 2^{n-1} - 1$  parameters in the first and  $2^n - 2$  parameters in the second step, the exact number depending on the problem we are solving. In total, that is at most  $2^{2n-1} + 2^{n-1} - 3$  parameters, where  $2^{2n-1} - 2^{n-1} - 1$  of them are only zeroes or ones and, consequently, fairly easy to set. It can be seen that for  $m > 3$  the application of the fuzzified Choquet integral with our technique for FNV-fuzzy measure construction requires less parameters than the construction of the complete fuzzy rules base does. The same conclusion can be made for the application of the fuzzy measure and the original Choquet integral.

## 5 Example

To demonstrate the application of GPGM, let us go back to the example of the evaluation of students' aptitudes for mathematical science, where our overall goal is to find out how good any given student is. The goal can be replaced by three redundant partial goals: find out how good the student is at mathematics, physics and chemistry. Partial evaluations of the students are their test results. Because all three tests are evaluated expertly, the partial evaluations are subjective, and therefore modelled by triangular fuzzy numbers. Let us consider a single student with evaluations  $H_M=(0.8 \ 0.9 \ 1)$ ,  $H_{Ph}=(0.4 \ 0.5 \ 0.6)$  and  $H_{Ch}=(0.1 \ 0.2 \ 0.3)$ .

Since the partial goals partly overlap, we aggregate the partial evaluations by fuzzified Choquet integral. At first, we construct the fuzzy measure with help of the technique described above. We create the imaginary objects  $x^1=(0,0,0)$ ,  $x^2=(1,0,0)$ ,  $x^3=(0,1,0)$ ,  $x^4=(0,0,1)$ ,  $x^5=(1,1,0)$ ,  $x^6=(0,1,1)$ ,  $x^7=(1,0,1)$ ,  $x^8=(1,1,1)$  and order them decreasingly  $x^8 \succ x^5 \succ x^7 \succ x^6 \succ x^2 \succ x^3 \succ x^4 \succ x^1$ . Afterwards, we use numbers  $t_1=1.1$ ,  $t_2=1.2$ ,  $t_3=1.3$  and  $t_4=1.4$  to describe the intensity of the relationship between each two consecutive objects and construct the fuzzy measure:  $\mu(M, Ph, Ch)=1$ ,  $\mu(M, Ph)=0.83$ ,  $\mu(M, Ch)=0.76$ ,  $\mu(Ph, Ch)=0.69$ ,  $\mu(M)=0.49$ ,  $\mu(Ph)=0.45$ ,  $\mu(Ch)=0.41$ . The overall evaluation of the student, as calculated by the fuzzified Choquet integral with respect to our fuzzy measure, is then equal to triangular fuzzy number (5.45, 6.45, 7.45). Based on the result, we can linguistically describe the student as "above average".

Let us now calculate the overall evaluation using the FNV-fuzzy measure. The construction of the FNV-fuzzy measure is analogical to the construction of the fuzzy measure in the previous paragraph, with a single difference. To describe the intensity of the relationship between each two successive objects we use triangular fuzzy numbers  $T_1=(1,1,1,1,2)$ ,  $T_2=(1,1,1,2,1,3)$ ,  $T_3=(1,2,1,3,1,4)$  and  $T_4=(1,3,1,4,1,5)$ . Graphical representation of the resulting FNV-fuzzy measure can be seen in Fig. 1.

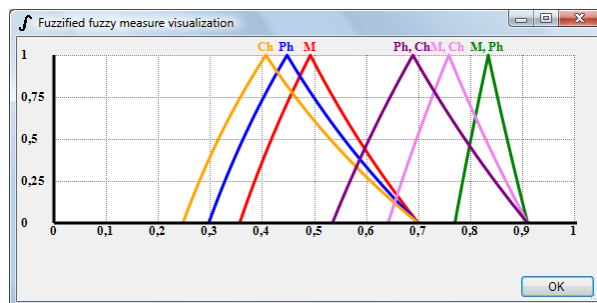


Figure 1  
FNV-fuzzy measure

The overall evaluation of the student together with the partial evaluations is depicted in Fig. 2. As we can see, the evaluation is in the form of a fuzzy number. Its centre of gravity 0.65 is the same as the result from the second paragraph of Section 5. Nevertheless, the shape of this fuzzy number indicates a little possibility that the student is only “below average”.

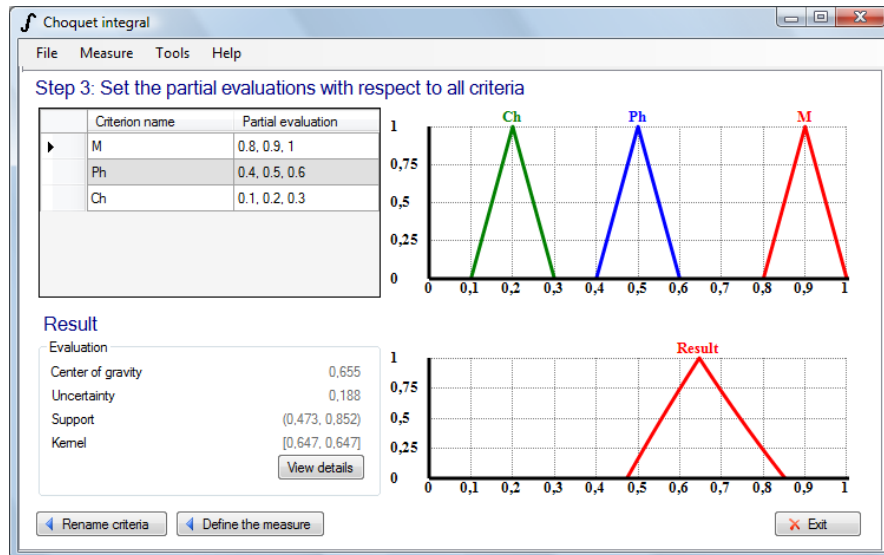


Figure 2

Partial evaluations and the overall evaluation

The calculations used in the example were implemented in a new software that we have created for this purpose. The software is capable of modelling both the fuzzy measures and the partial evaluations by fuzzy numbers, and employs the method presented in [1] for the actual calculations. A demo version of the software can be found on `fuzzme.wz.cz`.

## Conclusions

In the paper we have discussed the application of the Choquet integral to multiple criteria evaluation. We presented the Generalized Partial Goals Method (GPGM), where the aggregation with weighted average was replaced by aggregation with the Choquet integral. We also proposed the modification of GPGM for the multiple criteria evaluation problem with fuzzy partial evaluations and fuzzy weights of criteria. To deal with the problem of the proper construction of the fuzzy/FNV-fuzzy measure, we have proposed an approach based on the comparison of imaginary alternatives. In the last section we presented an example of an application of GPGM using fuzzy measure and FNV-fuzzy measure constructed by the techniques we proposed in the paper.

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### References

- [1] I. Bebčáková, J. Talašová, O. Pavlačka: Fuzzification of Choquet Integral and its Application in Multiple Criteria Decision Making, Neural Network World, Vol. 20, 2010, pp. 125-137
- [2] G. Choquet, Theory of Capacities, Ann. Inst. Fourier. V (1953-54), pp. 131-295
- [3] D. Dubois, H. Prade (Eds.): Fundamentals of Fuzzy Sets, Kluwer Academic Publishers, Dordercht, 2000
- [4] M. Grabisch: Fuzzy Integral in Multicriteria Decision Making, Fuzzy Sets and Systems 69, 1995, pp. 279-298
- [5] M. Grabisch: The Application of Fuzzy Integrals in Multicriteria Decision Making, European Journal of Operational Research 89, 1996, pp. 445-456
- [6] M. Grabisch, C. Labreuche: A Decade of Application of the Choquet and Sugeno Integrals in Multicriteria Decision Making, Annals of Operations Research 175, 2010, pp. 247-286
- [7] M. Grabisch, T. Murofushi, M. Sugeno (Eds.): Fuzzy Measures and Integrals: Theory and Applications, Physica Verlag, Heidelberg, New York, 2000
- [8] M. Grabisch, M. Roubens: Application of the Choquet Integral in Multicriteria Decision Making, Fuzzy Measures and Integrals-Theory and Applications, Physica Verlag, Göttingen, 2000, pp. 348-374
- [9] Mamdani, E. H., Assilian, S. (1975) An Experiment in Linguistic Synthesis with a Fuzzy Logic Controller, Int. J. Man-machine Studies, Vol. 7, 1-13, 1975
- [10] O. Pavlačka, J. Talašová: Application of the fuzzy Weighted Average of Fuzzy Numbers in Decision Making Models, New Dimensions in Fuzzy Logic and Related Technologies, Vol. II, Proceedings of the 5<sup>th</sup> EUSFLAT Conference, Ostrava, 2007, pp. 455-462
- [11] J. Talašová: Fuzzy metody vícekritériálního hodnocení a rozhodování (In Czech), VUP, Olomouc, 2003
- [12] J. Talašová, I. Bebčáková: Fuzzification of Aggregation Operators Based on Choquet Integral, Aplimat – Journal of Applied Mathematics, Vol. 1, 2008, pp. 463-474

- [13] V. Torra, Y. Narukawa: *Modeling Decisions: Information Fusion and Aggregation Operators*, Springer-Verlag, Berlin Heidelberg, 2007
- [14] Michel Grabisch, Jean-Luc Marichal, Radko Mesiar, Endre Pap: *Monograph: Aggregation Functions*, in *Acta Polytechnica Hungarica*, Vol. 6, No. 1, 2009, pp. 79-94