Asymptotic stability of an evolutionary nonlinear Boltzmann-type equation

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In the paper a sufficient condition for the asymptotic stability with respect to total variation norm of semigroup generated by an abstract evolutionary non-linear Boltzmann-type equation in the space of signed measures with the right-hand side being a collision operator is presented. For this purpose a sufficient condition for the asymptotic stability of Markov semigroups acting on the space of signed measures for any distance ([4]), adapted to the total variation norm, joined with the maximum principle for this norm is used. The paper generalizes the result in [4] related to the same type of non-linear Boltzmann-type equation, where the asymptotic stability in the weaker norm, Kantorovich-Wasserstein, was investigated.

Keywords: Asymptotic stability, Markov operators, maximum principle for the total variation metric, nonlinear Boltzmann-type equation

1 Introduction

We are interested in the problem of the stability of solutions u of the following version of the Boltzmann equation

$$\frac{\partial u(t,x)}{\partial t} + u(t,x) = \int_x^\infty \frac{dy}{y} \int_0^y u(t,y-z)u(t,z)dz \qquad t \geq 0, \qquad x \geq 0, \tag{1}$$

with the additional conditions for $t \geq 0$

$$\int_0^\infty u(t,x)\,dx = \int_0^\infty xu(t,x)\,dx = 1, \tag{2}$$

which describes the law of conservation of mass and energy. Equation (1) was presented in the space $L^p(\mathbb{R}_+)$ with $p = 1,2$ and different weights (see [1], [3], [7]). Equation (1) was derived by J. A. Tjon and T. T. Wu from the Boltzmann equation using the Abel transformation (see [14]) and was later called by Barnsley and Cornille (see [1]) the *Tjon–Wu equation*.

Equation (1) governs the evolution of the density distribution function of the energy of particles imbedded in an ideal gas in the equilibrium stage (see [7], [8], [14]). The solution $u(t,\cdot)$ of the problem has an interpretation as a probability distribution function of the energy of particles in an ideal gas. In the time interval $(t,t+\Delta t)$ a particle changes its energy with the probability $\Delta t + o(\Delta t)$ and this change is described by the operator

$$(Pu)(x) = \int_x^\infty \frac{dy}{y} \int_0^y u(y-z)\,u(z)dz. \tag{3}$$

Hence, the change is equal to $[-u(t,x)+P(u(t,x))]\triangle t + o(\triangle t)$.

In order to understand the action of P consider three independent random variables ξ_1, ξ_2 and η, such that ξ_1, ξ_2 have the same density distribution function u and η is uniformly distributed on the interval $[0,1]$. Here we obtain that Pu is the density distribution function of the random variable

$$\eta(\xi_1+\xi_2). \tag{4}$$

This corresponds to the physical assumption that the energies of the particles before a collision are independent quantities and that a particle after collision takes η part of the sum of the energies of the colliding particles.

The assumption that η has the density distribution function of the form $\mathbf{1}_{[0,1]}$ is quite restrictive. In general, if η has the density distribution h, then the random variable (4) has the density distribution function

$$(Pv)(x) = \int_x^\infty h(\frac{x}{y})\frac{dy}{y} \int_0^y u(y-z)\,u(z)dz. \tag{5}$$

The problem of the asymptotic behaviour of solutions of the equation:

$$\frac{\partial u(t,x)}{\partial t} + u(t,x) = \int_x^\infty h(\frac{x}{y})\frac{dy}{y} \int_0^y u(y-z)\,u(z)dz \tag{6}$$

was investigated by A. Lasota and J. Traple in 1999 ([10], Theorem 1.1).

This version is more general than (1). In both versions there are no physical reasons which will allow us to assume that the distribution of energy of particles can be described only by density (so by the absolutely continuous measure).

Following this physical interpretation, Gacki in 2007 (see [4]) considered the evolutionary Boltzmann-type equation

$$\frac{d\psi}{dt} + \psi = P\psi \qquad \text{for} \qquad t \geq 0 \tag{7}$$

with the initial condition

$$\psi(0) = \psi_0, \tag{8}$$

where $\psi_0 \in \mathscr{M}_1(\mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to \mathscr{M}_{sig}(\mathbb{R}_+)$ is an unknown function. Moreover $P : \mathscr{M}_1(\mathbb{R}_+) \to \mathscr{M}_1(\mathbb{R}_+)$ is analogous to (5), but in this case P is an operator acting on the space of probability measures. The operator P will be described precisely in Section 3. By $\mathscr{M}_1(\mathbb{R}_+)$ and $\mathscr{M}_{sig}(\mathbb{R}_+)$ we denote the space of probability measures and the space of finite signed measures respectively. More precisely an operator P is acting on the subset $D \subset \mathscr{M}_1(\mathbb{R}_+)$ given by formula

$$D := \{\mu \in \mathscr{M}_1 : m_1(\mu) = 1\}, \quad \text{where} \quad m_1(\mu) = \int_0^\infty x\mu(dx). \tag{9}$$

Equation (7) was studied in the space $\mathscr{M}_1(\mathbb{R}_+)$. The operator P describes the collision of two particles in general situation.

In [4], the problem of the stability of solutions of a nonlinear Boltzmann-type equation (7) with the initial condition (8) was studied in Kantorovich-Wasserstein norm (see [4], [13]). The proof of the asymptotic stability is based on a property of the Kantorovich-Rubinstein norm in the space of probabilistic measures, which the author called *the maximum principle* (see [5]).

The purpose of our paper is to prove that the semigroup generated by the equation (7) with the initial condition (8) is asymptotically stable with respect to the total variation norm. The basic idea of our method is to apply technique related with the maximum principle for the total variation norm (see [2]).

The maximum principle method in studying the asymptotic stability of Markov semigroup with respect to various metrics was used in the papers [2], [4], [6], [9] and [10].

In order to make the paper self-contained all necessary definitions from the theory of Markov operators, dynamical systems and differential equations in Banach spaces are recalled at the beginning of Sections 2 and 3 respectively.

2 Preliminaries

Let (X,ρ) be a Polish space and let $\mathscr{B}_X$ be σ–algebra of its Borel. We denote by $\mathscr{M}$ the family of all finite (nonnegative) Borel measures on X. and by $\mathscr{M}_1$ we the subset of $\mathscr{M}$ such that $\mu(X) = 1$ for $\mu \in \mathscr{M}_1$. Now let

$$\mathscr{M}_{sig} = \{\mu_1 - \mu_2 : \ \mu_1, \mu_2 \in \mathscr{M}\},$$

be the space of *finite signed measures* endowed with the total variation norm $\|\cdot\|_T$ (under which it is a Banach space).

Fix an element c of X and for every real number $\alpha \geq 1$ we define sets $\mathscr{M}_{1,\alpha}$ and $\mathscr{M}_{sig,\alpha}$

$$\mathscr{M}_{1,\alpha} = \{\mu \in \mathscr{M}_1 : \ m_\alpha(\mu) < \infty\} \qquad \text{and} \qquad \mathscr{M}_{sig,\alpha} = \{\mu \in \mathscr{M}_{sig} : \ m_\alpha(\mu) < \infty\}$$

where

$$m_\alpha(\mu) = \int_X (\rho(x,c))^\alpha |\mu|(dx).$$

It is easy to verify that these spaces do not depend on the choice of c.

Denote by $B(x,r)$ a closed ball in X with center $x \in X$ and radius r. For $\mu \in \mathscr{M}_1$ define *the support of a measure* μ by

$$\operatorname{supp} \mu = \{x \in X : \ \mu(B(x,\varepsilon)) > 0 \ \text{ for every } \varepsilon > 0\}.$$

The support of a measure being a stationary solution will play an important role in the proof of the asymptotic stability of the equation (7). Every set $\mathscr{M}_{1,\alpha}$, for $\alpha \geq 1$ contains the subset of all measures $\mu \in \mathscr{M}_1$ with a compact support.

In the proof of the main result of this paper an important role is played by some property of the total variation norm, directly connected with the strong contractivity, which is called the maximum principle. The relation between contractivity and the maximum principle will be described below in Theorem 2.1.

The Maximum principle for total variation norm formulated as follows: Let $\mu_1, \mu_2 \in \mathscr{M}$. Then

$$\|\mu_1 - \mu_2\|_T = \|\mu_1\|_T + \|\mu_2\|_T \tag{10}$$

if and only if μ_1 and μ_2 are mutually singular (i.e. if there are two sets $A, B \in \mathscr{B}$ such that $A \cap B = \emptyset$, $A \cup B = X$ and $\mu_1(B) = \mu_2(A) = 0$). (For details see [2], p. 325).

We start with a definition of Markov operator

Definition 2.1. An operator $P : \mathscr{M} \to \mathscr{M}$ is called a *Markov operator* if it satisfies the following conditions:

(i) P is positively linear

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$$

for $\lambda_1, \lambda_2 \geq 0$ and $\mu_1, \mu_2 \in \mathscr{M}$,

(ii) P preserves the measure of the space

$$P\mu(X) = \mu(X) \qquad \text{for} \qquad \mu \in \mathscr{M}. \tag{11}$$

Note that every Markov operator P can be uniquely extended as an operator to the space of signed measures.

In what follows we will understand by d the distance generated by the total variation norm on $\mathscr{M}_{sig}$. A Markov operator $P : \mathscr{M}_{sig} \to \mathscr{M}_{sig}$ is called *contracting* or *nonexpansive* with respect to d if

$$d(P\mu_1, P\mu_2) \leq d(\mu_1, \mu_2) \qquad \text{for} \qquad \mu_1, \mu_2 \in \mathscr{M}_{sig}. \tag{12}$$

A Markov operator $P : \mathscr{M}_{sig} \to \mathscr{M}_{sig}$ is called *strongly contracting* or *contractive in the class* $\widetilde{\mathscr{M}} \subset \mathscr{M}_{sig}$ with respect to d if

$$d(P\mu_1, P\mu_2) < d(\mu_1, \mu_2) \qquad \text{for} \qquad \mu_1, \mu_2 \in \widetilde{\mathscr{M}}. \tag{13}$$

Definition 2.2. We say that the measures $\mu, \nu \in \mathscr{M}$ *overlap supports* if there is no set $A \in \mathscr{B}$ such that

$\mu(A) = 0$ and $\nu(A^c) = 0$

Contractivity of Markov operators in total variation plays an important role in investigation of asymptotics of solutions of equation (1). We have

Theorem 2.1. *Let P be a Markov operator. Assume that $P\mu_+, P\mu_-$ overlap supports for every nontrivial measure $\mu \in \mathscr{M}_{sig}$. Then Markov operator P is strongly contracting with respect to the distance d generated by the total variation norm.*

In the proof of this theorem, the crucial role is played by the inequality:

$$d(P\mu_+, P\mu_-) \leq ||P\mu_+||_T + ||P\mu_-||_T.$$

Applying the maximum principle to $P\mu_+$ and $P\mu_-$, we obtain the strong inequality. But we have

$$||P\mu_+|_T = ||\mu_+||_T \text{ and } ||P\mu_-||_T = ||\mu_-||_T,$$

so using the maximum principle once more (for μ_+ and μ_-), we directly obtain that P is strongly contracting. For details see [2], p. 326.

Now we recall few facts from the theory of dynamical systems.

Let T be a *nontrivial semigroup* of nonnegative real numbers i.e. $\{0\} \subsetneq T \subset \mathbb{R}_+$ and $t_1 + t_2 \in T$, $t_1 - t_2 \in T$ for $t_1, t_2 \in T$, $t_1 \geq t_2$.

A family of Markov operators $(P^t)_{t \in T}$ is called a *semigroup* if

$P^{t+s} = P^t P^s \qquad$ for $\qquad t, s \in T$

and $P^0 = I$ where I is the identity operator.

A semigroup $(P^t)_{t \in T}$ is called a *semidynamical system* if the transformation $\mathscr{M}_{sig} \ni \mu \to P^t \mu \in \mathscr{M}_{sig}$ is continuous for every $t \in T$.

Remark 2.1. Every Markov operator $P : \mathscr{M}_{sig} \to \mathscr{M}_{sig}$ is continuous with respect to the total variation norm. Consequently, every semigroup $(P^t)_{t \in T}$ of Markov operators is a semidynamical system.

If a semidynamical system $(P^t)_{t \in T}$ is given, then for every fixed $\mu \in \mathscr{M}_{sig}$ the function $T \ni t \to P^t \mu \in \mathscr{M}_{sig}$ will be called a *trajectory* starting from μ and denoted by $(P^t\mu)$. A point $\nu \in \mathscr{M}_{sig}$ is called a *limiting point* of a trajectory $(P^t\mu)$ if there exists a sequence (t_n), $t_n \in T$, such that $t_n \to \infty$ and

$$\lim_{n\to\infty} P^{t_n}\mu = \nu.$$

The set of all limiting points of the trajectory $(P^t\mu)$ will be denoted by $\Omega(\mu)$.

We say that a trajectory $(P^t\mu)$ is *sequentially compact* if for every sequence (t_n), $t_n \in T$, $t_n \to \infty$, there exists a subsequence (t_{k_n}) such that the sequence $(P^{t_{k_n}}\mu)$ is convergent to a point $\nu \in \mathscr{M}_{sig}$.

Remark 2.2. If the trajectory $(P^t\mu)$ is sequentially compact, then $\Omega(\mu)$ is a nonempty, sequentially compact set.

A point $\mu_* \in \mathscr{M}_{sig}$ is called *stationary* (or *invariant*) with respect to a semidynamical system $(P^t)_{t\in T}$ if

$$P^t\mu_* = \mu_* \qquad \text{for} \qquad t \in T. \tag{14}$$

A semidynamical system $(P^t)_{t\in T}$ is called *asymptotically stable* if there exists a stationary point $x_* \in X$ such that

$$\lim_{t\to\infty} P^t\mu = \mu_* \qquad \text{for} \qquad \mu \in \mathscr{M}_{sig}. \tag{15}$$

Remark 2.3. Since $(\mathscr{M}_{sig}, \|\cdot\|_T)$ is a Hausdorff space, an asymptotically stable dynamical system has exactly one stationary point.

We say that a Markov semigroup $(P^t)_{t\in T}$ is *contracting* or *nonexpansive semigroup with respect to the distance d generated by the total variation norm in the class* $\widetilde{\mathscr{M}} \subset \mathscr{M}_{sig}$ if the following condition holds

$$d(P^t\mu_1, P^t\mu_2) \le d(\mu_1,\mu_2) \qquad \mu_1,\mu_2 \in \widetilde{\mathscr{M}};\ t \in T. \tag{16}$$

A contracting semigroup $(P^t)_{t\in T}$ will be called *strongly contracting with respect to the distance d generated by the total variation norm in the class* $\widetilde{\mathscr{M}} \subset \mathscr{M}_{sig}$ if and only if for every $\mu_1,\mu_2 \in \widetilde{\mathscr{M}}$, $\mu_1 \neq \mu_2$ there is a number $t_0 \in T$ such that

$$d(P^{t_0}\mu_1, P^{t_0}\mu_2) < d(\mu_1,\mu_2).$$

Let $(P^t)_{t\in T}$ be a semidynamical system which has at least one sequentially compact trajectory and $\mathscr{Z}$ – the set of all $\mu \in \mathscr{M}_{sig}$ such that the trajectory $(P^t\mu)$ is sequentially compact. $\mathscr{Z}$ is a nonempty set, so

$$\Omega = \bigcup_{\mu\in\mathscr{Z}} \omega(\mu) \neq = \emptyset.$$

In the proof of the main result of this paper – Theorem 3.2 – we will use the following criterion for the asymptotic stability of trajectories

Theorem 2.2. *Let* $x_* \in \Omega$ *be fixed. Assume that for every* $x \in \Omega$, $x \neq x_*$ *there is* $t(x) \in T$ *such that*

$$d(S^{t(x)}x, S^{t(x)}x_*) < d(x, x_*). \tag{17}$$

Further assume that the semidynamical system $(S^t)_{t\in T}$ *is nonexpansive with respect to distance d, i.e.,*

$$d(S^t x, S^t y) \leq d(x,y) \qquad \text{for} \qquad x,y \in \mathscr{M}_{sig} \qquad \text{and} \qquad t \in T. \tag{18}$$

Then x_* *is a stationary point of* $(S^t)_{t\in T}$ *and*

$$\lim_{t\to\infty} d(S^t z, x_*) = 0 \qquad \text{for} \qquad z \in Z. \tag{19}$$

where Z is the set of all $z \in \mathscr{M}_{sig}$ *such that the trajectory* $(S^t z)$ *is compact.*

This criterion is a special case, adapted to the distance generated by the total variation norm, of the more general result (for any distance), which may be found in [4], p. 28–30.

3 Main result - asymptotic stability

In this section we show that the equation (7) may by considered in a convex closed subset of a vector space of signed measures. This approach seems to be quite natural and it is related to the classical results concerning the semigroups and differential equations on convex subsets of Banach spaces (see [3], [11]).

Let $(E, \|\cdot\|)$ be a Banach space and let $\tilde{D}$ be a closed, convex, nonempty subset of E. In the space E we consider *an evolutionary differential equation*

$$\frac{du}{dt} = -u + \tilde{P}u \qquad \text{for} \qquad t \in \mathbb{R}_+ \tag{20}$$

with the initial condition

$$u(0) = u_0, \qquad u_0 \in \tilde{D}, \tag{21}$$

where $\tilde{P} : \tilde{D} \to \tilde{D}$ is a given operator.

A function $u : \mathbb{R}_+ \to E$ is called a solution of problem (20), (21) if it is strongly differentiable on $\mathbb{R}_+$, $u(t) \in \tilde{D}$ for all $t \in \mathbb{R}_+$ and u satisfies relations (20), (21).

We start from the following theorem which is usually stated in the case $E = \tilde{D}$.

Theorem 3.1. *Assume that the operator* $\tilde{P} : \tilde{D} \to \tilde{D}$ *satisfies the Lipschitz condition*

$$\|\tilde{P}v - \tilde{P}w\| \leq l\,\|v- = w\| \qquad \text{for} \qquad u,w \in \tilde{D}, \tag{22}$$

where l *is a nonnegative constant. Then for every* $u_0 \in \tilde{D}$ *there exists a unique solution* u *of problem (20), (21).*

The standard proof of the Theorem 3.1 is based on the fact, that a function $u : \mathbb{R}_+ \to \tilde{D}$ is the solution of (20), (21) if and only if it is continuous and satisfies the integral equation

$$u(t) = e^{-t} u_0 + \int_0^t e^{-(t-s)} \tilde{P} u(s)\, ds \qquad \text{for} \qquad t \in \mathbb{R}_+. \tag{23}$$

Due to completeness of $\tilde{D}$ the integral on the right hand side is well defined and equation (23) may be solved by the method of successive approximations.

Observe that, thanks to the properties of $\tilde{D}$, for every $u_0 \in \tilde{D}$ and every continuous function $u : \mathbb{R}_+ \to \tilde{D}$ the right hand side of (23) is also a function with values in $\tilde{D}$.

The solutions of (23) generate a semigroup of operators $(\tilde{P}^{\,t})_{t\geq 0}$ on $\tilde{D}$ given by the formula

$$\tilde{P}^{\,t} u_0 = u(t) \qquad \text{for} \qquad t \in \mathbb{R}_+, \qquad u_0 \in \tilde{D}. \tag{24}$$

We can now come to the main result of the paper – a sufficient condition for the asymptotic stability of solutions of the equation (7) with respect to the total variation metric.

At the beginning we return to equation (7) and give the precise definition of P.

We start from recalling that the *convolution of measures* $\mu, \nu \in \mathscr{M}_{sig}$ is a unique measure $\mu * \nu$ satisfying

$$(\mu * \nu)(A) := \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} 1_A(x+y) \mu(dx) \nu(dy) \qquad \text{for} \qquad A \in \mathscr{B}_X. \tag{25}$$

(see [9]).

A linear operator $P_{*2} : \mathscr{M}_{sig} \mapsto \mathscr{M}_{sig}$ is defined by

$$P_{*2}\mu := \mu * \mu \qquad\qquad \text{for } \mu \in \mathscr{M}_{sig}. \tag{26}$$

It is easy to verify that $P_{*2}(\mathscr{M}_1) \subset \mathscr{M}_1$. Moreover the maps $P_{*2}\big|_{\mathscr{M}_1}$ have a simple probabilistic interpretation. Namely, if ξ_1, ξ_2 are independent random variables with the same distribution μ, then $P_{*2}\mu$ is the distribution of the sum $\xi_1 + \xi_2$.

The second class of operators we are going to study is related to the multiplication of random variables (see [9]). The formal definition is as follows. Given two measures $\mu, \nu \in \mathscr{M}_{sig}$, we define the *elementary product* $\mu \circ \nu$ by the formula

$$(\mu \circ \nu)(A) : \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} 1_A(xy) \mu(dx) \nu(dy) \qquad \text{for} \qquad A \in \mathscr{B}_{\mathbb{R}_+}. \tag{27}$$

For fixed $\varphi \in \mathscr{M}_1$ we define the linear operator $P_\varphi : \mathscr{M}_{sig} \to \mathscr{M}_{sig}$ by the formula

$$P_\varphi \mu := \varphi \circ \mu \qquad \text{for} \qquad \mu \in \mathscr{M}_{sig}. \tag{28}$$

Again, as in the case of convolution, from this definition it follows that $P_\varphi(\mathscr{M}_1) \subset \mathscr{M}_1$. For $\mu \in \mathscr{M}_1$ the measure $P_\varphi\,\mu$ has an immediate probabilistic interpretation. If φ and μ are the distributions of random variables ξ and η respectively, then $P_\varphi\,\mu$ is the distribution of the product $\xi\,\eta$.

Now we may return to the equation (7) and give the precise definition of P. Namely we define

$$P := P_\varphi\, P_{*2}, \tag{29}$$

where $\varphi \in \mathscr{M}_1$ and $m_1(\varphi) = \frac{1}{2}$. From equality (29) it follows that $P(\mathscr{M}_1) \subset \mathscr{M}_1$. Further using (26) and (28) it is easy to verify that for $\mu \in D$

$$m_1(P_{*2}\mu) = 2 \qquad \text{and} \qquad m_1(P_\varphi\mu) = \frac{1}{2}, \tag{30}$$

where D is defined by the formula (9).

From the definition of the set D and operator P, we obtain the following properties:

1. The set D is a convex subset of $\mathscr{M}_{\mathrm{sig},1}$.
2. The set D with distance d is a complete metric space.
3. If $\varphi \in \mathscr{M}_1$ and $m_1(\varphi) = 1/2$, $m_1(\nu_0) = 1$, then $P(D) \subset D$.

Equation (7) together with the initial condition (8) may be considered in a convex subset D of the vector space of signed measures. From the properties (1), (2), (3) and the results of [3] it follows immediately that for every $\psi_0 \in D$ the initial value problem (7), (8) has exactly one solution ψ satisfying the integral equation

$$\psi(t) = e^{-t}\,\psi_0 + \int_0^t e^{-(t-s)}\,P\,\psi(s)\,ds \qquad \text{for } t \in \mathbb{R}_+. \tag{31}$$

Corollary 3.1. *If $\varphi \in \mathscr{M}_1$ and $m_1(\varphi) = 1/2$ then for every $\psi_0 \in D$ there exists an unique solution ψ of problem* (7), (8).

The solutions of (31) generate a semigroup of Markov operators $(P^{\,t})_{t\ge 0}$ on D given by

$$\psi(t) = P^{\,t}\,\psi_0 \qquad \text{for } t \in \mathbb{R}_+,\ \psi_0 \in D. \tag{32}$$

Now using criterion for the asymptotic stability of trajectories Theorem 2.2 jointly with the maximum principle for total variation metric from the Theorem 2.1, we can easily derive the following main result of this paper:

Theorem 3.2. *Let P be the operator given by (29). Moreover, let φ be a probability measure with $m_1(\varphi) = 1/2$ and let 0 be accumulation point of* $\operatorname{supp}\varphi$*. If P has a fixed point $\psi_* \in D$ such that*

$$\operatorname{supp}\psi_* = \mathbb{R}_+, \tag{33}$$

then

$$\lim_{t\to\infty}\|\psi(t)-\psi_*\|_T = 0 \tag{34}$$

for every compact solution ψ *of* (7), (8).

Proof. It is sufficient to verify condition (17) of Theorem 2.2.

From (31) it follows immediately that

$$\begin{aligned}\|P^t\psi_0-\psi_*\|_T &\le e^{-t}\|\psi_0-\psi_*\|_T\\ &+\int_0^t e^{-(t-s)}\|P^s\psi_0-\psi_*\|_T\,ds \qquad \text{for } \psi_0\in D \text{ and } t>0.\end{aligned}$$

This may be rewritten in the form

$$\begin{aligned}\|P^t\psi_0-\psi_*\|_T &\le e^{-t}\|\psi_0-\psi_*\|_T+(1-e^{-t})\|\psi_0-\psi_*\|_T\\ &= \|\psi_0-\psi_*\|_T \qquad \text{for } \psi_0\in D \text{ and } t>0.\end{aligned} \tag{35}$$

Condition (33) is equivalent to the fact that the measures $P^t\psi_0, \psi_*\in D$ overlap supports for $t>0$ and $\psi_0\in D$. Applying Theorem 2.1 for P^t, we will get that Markov operator P^t is strongly contracting. Consequently, in (35) we have a strict inequality, because $P^t(\psi_*)=\psi_*$. An application of Theorem 2.2 completes the proof.

□

Remark 1. We showed that if equation (7) has a stationary solution μ_* such that $\operatorname{supp}\mu_*=\mathbb{R}_+$, then this measure is asymptotically stable. The positivity of u_* plays an important role in the proof of the stability. Namely, it allows us to apply the maximum principle in order to show that the total variation distance between u_* and an arbitrary solution u is decreasing in time.

Moreover, in [4] p. 34. the following result was shown:

Let φ *be a probability measure and let* $m_1(\varphi)=\frac{1}{2}$. *Assume that:*

(i) There is $\sigma_0>0$ *such that*

$$(0,\sigma_0)\subset\operatorname{supp}\varphi. \tag{36}$$

(ii) The operator P *has a fixed point* $v\in\mathscr{M}$ *such that* $v\neq\delta_0$.
Then

$$\operatorname{supp} v=\mathbb{R}_+. \tag{37}$$

From above it follows that the assumption (33) can be replaced by the more effective condition (36).

Observe that in the case of the classical linear Tjon–Wu type equation (1) the measure φ is absolutely continuous with density $\mathbf{1}_{[0,1]}$. Moreover, $u_*(t,x) := \exp(-x)$ is the density function of the stationary solution of (1). This is a simple illustration of the situation described by Theorem 3.2.

Moreover, the condition (33) is not particularly restrictive because in Lasota's and Traple's paper (see [12]) it has been proved that the stationary solution ϕ_* has the following property: Either ψ_* is supported at one point or $supp\,\psi_* = \mathbb{R}_+$. The first case holds if and only if $\varphi = \delta_{\frac{1}{2}}$. But this case is forgettable as a physical model of particle collisions because it is more restrictive than the model described by the classical Tjon-Wu equation.

Remark 2. It is worth noting that:

1. Every solution of the equation $P\mu = \mu$ is a stationary solution of equation (7).
2. We have many possibilities to apply the criterion written in Theorem 3.2. For example, if we consider the equation (7) with the following assumption:

$$2m_r(\varphi) < 1, \text{ where } r > 1,$$

then for every $\psi_0 \in D$ the solution of (7) and (8) is compact (see [4]).

Summary

The Boltzmann equation in the general form gives us information about time, position and velocity of particles in the dilute gas. This equation is a base for many mathematical models of colliding particles.

In particular, in [2] authors described the homogeneous model where a small number of particles is introduced into a gas which contains many more particles, at equilibrium. The solution of the considered in [2] equation in the time t informs us about an energy state of the introduced particles in t.

In present paper authors consider the homogeneous model in the dilute gas with a possibility of collisions of two particles. The solution of the equation describing this model, (7), in time t, gives an information about an energy change between colliding particles in t.

In the future, it is planned to describe the mathematical model of colliding particles with a possibility of collisions of arbitrary many particles. Moreover, the external forces may exist.

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