

On the Non-Uniqueness of the Solution to the Least-Squares Optimization of Pairwise Comparison Matrices

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Abstract: This paper develops the “best” rank one approximation matrix to a general pairwise comparison matrix (PCM) in a least-squares sense. Such quadratic matrices occur in the multi-criteria decision making method called the analytic hierarchy process (AHP). These matrices may have positive entries only. The lack of uniqueness of the stationary values of the associated nonlinear least-squares optimization problem is investigated. Sufficient conditions for the non-uniqueness of the solution to the derived system of nonlinear equations are given. Results are illustrated through a great number of numerical examples.

Keywords: nonlinear optimization, least-squares method, multiple-criteria decision making

1 Introduction

Let $\mathbf{A}=[a_{ij}]$ denote an $n \times n$ matrix with all entries positive numbers. \mathbf{A} is called a *symmetrically reciprocal (SR)* matrix if the entries satisfy $a_{ij}a_{ji}=1$ for $i \neq j, i, j=1, 2, \dots, n$, and $a_{ii}=1, i=1, 2, \dots, n$. These matrices were introduced by Saaty [1] in his multi-criteria decision making method called the analytic hierarchy process (AHP). Here an entry a_{ij} represents a *ratio*, i.e. the element a_{ij} indicates the strength with which decision alternative A_i dominates decision alternative A_j with respect to a given criterion. Such a *pairwise comparison matrix (PCM)* is usually constructed by eliciting experts' judgements. Then the fundamental objective is to derive implicit *weights*, w_1, w_2, \dots, w_n , for the given set of decision alternatives according to relative importance (priorities) measured on a ratio-scale.

Let $\mathbf{B}=[b_{ij}]$ denote an $n \times n$ matrix with all entries positive numbers. \mathbf{B} is called a *transitive* matrix if $b_{ij}b_{jk}=b_{ik}$, for $i,j,k=1,2, \dots,n$. In [2] it is proven that any transitive matrix is necessarily SR and has rank one. Two strongly related notations will be used for the weights: $\mathbf{W}=\text{diag}[w_i]$, $i=1,2, \dots,n$, is the diagonal matrix with the diagonal entries w_1, w_2, \dots, w_n , and the (column) vector from \mathbb{R}^n with elements w_1, w_2, \dots, w_n is denoted by \mathbf{w} . Thus, \mathbf{W} is a positive definite diagonal matrix if and only if \mathbf{w} is an elementwise positive column vector. With these notations, and defining the n vector $\mathbf{e}^T=[1,1, \dots,1]$ to be the row vector of \mathbb{R}^n , any transitive matrix \mathbf{B} can now be written in the form

$$\mathbf{B} = \mathbf{W}^{-1} \mathbf{e} \mathbf{e}^T \mathbf{W} = \begin{bmatrix} w_j \\ w_i \end{bmatrix}, \quad i, j = 1, 2, \dots, n. \quad (1)$$

Using (1) it is easy to show that

$$\mathbf{B} \mathbf{W}^{-1} \mathbf{e} = n \mathbf{W}^{-1} \mathbf{e}. \quad (2)$$

From (2) it is seen that the only nonzero (dominant) eigenvalue of \mathbf{B} is n and its associated Perron-eigenvector is $\mathbf{W}^{-1} \mathbf{e}$, i.e. a vector whose elements are the reciprocals of the weights.

In decision theory, a transitive matrix \mathbf{B} is termed *consistent* matrix. Otherwise a PCM is termed *inconsistent*. Saaty [1] showed that the weights for a consistent PCM are determined by the elements, u_i , $i=1,2, \dots,n$, of the principal right eigenvector \mathbf{u} of matrix \mathbf{B} , if \mathbf{B} is a *consistent* matrix, i.e. if it is transitive. This solution for the weights is *unique* up to a multiplicative constant. Hence, this Perron-eigenvector becomes $\mathbf{u}=\mathbf{W}^{-1} \mathbf{e}$. (In the applications of the AHP these components are usually normalized so that their sum is unity.)

During the last decades several authors have advocated particular best ways for approximating a *general* (not transitive) SR matrix \mathbf{A} . There are various possible ways to generate approximations for \mathbf{A} in some sense. Saaty [3] proposed the eigenvector approach for finding the weights, even if \mathbf{A} is an inconsistent PCM. Extremal methods have also been considered, like the direct least-squares method [4], the weighted least-squares method [5],[6], the logarithmic least-squares method [7],[8], furthermore, the logarithmic least-absolute-values method [9]. A graphical technique that is based on the construction of the Gower-plots was also proposed which produces the “best” rank two matrix approximation to \mathbf{A} [10]. The most comprehensive comparative study that has appeared thus far both in terms of the number of these scaling methods and the number of the evaluation criteria used was presented by Golany and Kress [11]. They concluded that these methods have different weaknesses and advantages, hence, none of them is dominated by the other.

2 The Least-Squares Optimization Method

The authors have developed a method that generates a “best” transitive (rank one) matrix \mathbf{B} to approximate a general SR matrix \mathbf{A} , where the “best” is assessed in a least-squares (LS) sense [12]. There it is shown that a common procedure to find a positive vector of the weights can be done by minimizing the expression

$$S^2(\mathbf{w}) := \|\mathbf{A} - \mathbf{B}\|_F^2 = \sum_{i=1}^n \sum_{j=1}^n \left(a_{ij} - \frac{w_j}{w_i} \right)^2. \quad (3)$$

(Here, the subscript F denotes the Frobenius norm; the square root of the sum of squares of the elements, i.e. the error.)

Given the subjective estimates a_{ij} for a particular PCM, it is always desired that $a_{ij} \approx w_j/w_i$. In other words, the *weights* w_i , and thus the *consistency adjustments*, $a_{ij} - b_{ij}$, $i, j=1, 2, \dots, n$, should be determined such that the sum of the *consistency adjustment error*, S , is minimized. In [12], with appropriately chosen initial values, the Newton-Kantorovich (NK) method was applied for this optimization procedure due to its computational advantages. There the authors asserted that a stationary value \mathbf{w} , of the error functional $S^2(\mathbf{w})$, called a *stationary vector*, satisfies the following *homogeneous* nonlinear equation

$$\mathbf{R}(\mathbf{w})\mathbf{w} = \mathbf{0}, \quad (4)$$

where

$$\mathbf{R}(\mathbf{w}) = \mathbf{W}^{-2}(\mathbf{A} - \mathbf{W}^{-1}\mathbf{e}\mathbf{e}^T\mathbf{W}) - (\mathbf{A} - \mathbf{W}^{-1}\mathbf{e}\mathbf{e}^T\mathbf{W})^T\mathbf{W}^{-2} \quad (5)$$

is a variable dependent, *skew-symmetric* matrix. In [13] it is shown that expression (5) can be more generally used in the approximation of merely *positive* matrices (which are not necessarily SR).

In the present paper, with regard to the basic properties of a PCM, investigations are made for general *symmetrically reciprocal* (SR) matrices \mathbf{A} with positive entries. If matrix \mathbf{A} is in SR, the homogeneous system of n nonlinear equations (4) can be written in the form

$$f = (w_1, w_2, \dots, w_n) = \sum_{k=1}^n \left[\frac{a_{ik}}{w_i^2} + \frac{1}{a_{ik} w_k^2} - 2 \left(\frac{w_i}{w_k^3} + \frac{w_k}{w_i^3} \right) \right] w_k = 0, \quad i = 1, 2, \dots, n. \quad (6)$$

Note that each of the n equations in (6) represents a type of homogeneous function in the variables as

$$f_i(cw_1, cw_2, \dots, cw_n) = c^v f_i(w_1, w_2, \dots, w_n), \quad i = 1, 2, \dots, n, \quad (7)$$

where c is an arbitrary constant and $v=-1$ is the degree of the homogeneous function in (6). Substituting (7) for the set of equations (6) we get

$$\frac{1}{c} \sum_{k=1}^n \left[\frac{a_{ik}}{w_i^2} + \frac{1}{a_{ik} w_k^2} - 2 \left(\frac{w_i}{w_k^3} + \frac{w_k}{w_i^3} \right) \right] w_k = 0, \quad i = 1, 2, \dots, n. \quad (8)$$

It is apparent from (8) that any constant multiple of the solution to the homogeneous nonlinear system (4) would produce another solution. To circumvent this difficulty, equation (4) can be reformulated, as any one of its n scalar equations can be dropped without affecting the solution set [12]. Denoting the j th row of any matrix \mathbf{M} by \mathbf{M}_{j*} and introducing the nonzero vector $\mathbf{c} \in \mathbb{R}^n$, let (4) have a positive solution \mathbf{w} normalized so that $\mathbf{c}^T \mathbf{w} = 1$. Then, for any j , $1 \leq j \leq n$, apparently, the stationary vector \mathbf{w} is a solution to the following *inhomogeneous* system of n equations

$$\mathbf{c}^T \mathbf{w} = 1, \quad \mathbf{R}_{k*}(\mathbf{w}) \mathbf{w} = \mathbf{0}, \quad k \neq j, \quad 1 \leq k \leq n. \quad (9)$$

Here it is convenient to use $j=1$. Thus $\mathbf{c}^T = [1, 0, \dots, 0]$, i.e. the normalization condition in (9) is then $w_1 = 1$.

Although a great number of numerical experiments showed that authors strategy always determined a convergent process for the NK iteration, however, a possible *non-uniqueness* of this solution (local minima) have also experienced. The occurrence of such alternate stationary vectors for a PCM was first reported by Jensen [4, p.328]. He argued that it is possible to specify PCMs “that have certain symmetries and high levels of response inconsistency that result in multiple solutions yielding minimum least-squares error.” Obviously, an eventual occurrence of a possible multiple solution may not seem surprising since it is well-known in the theory of nonlinear optimization that $S^2(\mathbf{w})$ does not necessarily have a unique minimum. In what follows now, some respective results of the authors is discussed for this problem.

3 On the Non-Uniqueness of the Solution to the Least-Squares Optimization Problem

In this Section *sufficient* conditions for a *multiple* solution to the inhomogeneous system of n equations (9) are given. The following matrices will play an important role in this subject matter:

Definition 1 An $n \times n$ matrix $\mathbf{Z}=[z_{ij}]$ is said to be *persymmetric* if its entries satisfy

$$z_{ij} = z_{n+1-j, n+1-i}, \quad i, j = 1, 2, \dots, n, \quad (10)$$

i.e., if its elements are symmetric about the counterdiagonal (secondary diagonal).

Definition 2 An $n \times n$ matrix \mathbf{P}_n is called a *permutation* matrix and is described by $\mathbf{P}_n = [\mathbf{e}_{j_1} \ \mathbf{e}_{j_2} \ \dots \ \mathbf{e}_{j_n}]$, where the n numbers in the indices, $p=(j_1 \ j_2 \ \dots \ j_n)$, indicate a particular permutation from the standard order of the numbers $1, 2, \dots, n$.

It is easy to see that any permutation matrix \mathbf{P}_n is an orthogonal matrix, since

$$\mathbf{P}_n^T \mathbf{P}_n = \mathbf{I}_n, \quad (11)$$

where \mathbf{I}_n denotes the $n \times n$ identity matrix.

Definition 3 An $n \times n$ matrix $\mathbf{M}=[m_{ij}]$, $i, j=1, 2, \dots, n$, is called a *symmetric permutation invariant* (SPI) matrix if there exists an $n \times n$ permutation matrix \mathbf{P}_n such that

$$\mathbf{P}_n^T \mathbf{M} \mathbf{P}_n = \mathbf{M}. \quad (12)$$

is satisfied [14].

Definition 4 By a *circulant matrix*, or *circulant* for short, is meant an $n \times n$ matrix $\mathbf{C}=[c_{jk}]$, $j, k=1, 2, \dots, n$, where

$$c_{jk} = \begin{cases} c_{1, k+1-j}, & \text{if } j \leq k, \\ c_{1, n+k+1-j}, & \text{if } j > k. \end{cases} \quad (13)$$

The elements of each row of \mathbf{C} in (13) are identical to those of the previous row, but are moved one position to the right and wrapped around. Thus, the whole circulant is evidently determined by the first row as

$$\mathbf{C} = \text{circ}[c_{11}, c_{12}, \dots, c_{1n}]. \quad (14)$$

It is meaningful to use a different notation for a special class of the permutation matrices. Among the permutation matrices the following matrix plays a fundamental role in the theory of circulants. This refers to the forward shift permutation, that is to the cycle $p=(1, 2, \dots, n)$ generating the cyclic group of order n , since its factorization consists of one cycle of full length n (see in [15]).

Definition 5 The special $n \times n$ permutation matrix $\mathbf{\Omega}_1$ of the form

$$\mathbf{\Omega}_1 = [\mathbf{e}_n \quad \mathbf{e}_1 \quad \mathbf{e}_2 \dots \mathbf{e}_{n-1}], \quad (15)$$

is said to be the *elementary (primitive) circulant* matrix, i.e. $\mathbf{\Omega}_1 = \text{circ}[0, 1, 0, \dots, 0]$. The other $n \times n$ *circulant* permutation matrices $\mathbf{\Omega}_k$ of the form

$$\mathbf{\Omega}_k = [\mathbf{e}_{n-k+1} \quad \mathbf{e}_{n-k+2} \dots \mathbf{e}_n \quad \mathbf{e}_1 \dots \mathbf{e}_{n-k}], \quad k = 1, 2, \dots, n, \quad (16)$$

are the powers of matrix $\mathbf{\Omega}_1$ defined by (15).

Notice in (16) that the relation $\mathbf{\Omega}_k = \mathbf{\Omega}_1^k$, holds for all $k=1, \dots, n-1$, and, obviously, $\mathbf{\Omega}_1^n = \mathbf{I}_n$. It follows from (14) that a circulant \mathbf{C} is invariant to a cyclic (simultaneous) permutation of the rows and the columns, hence

$$\mathbf{\Omega}_k^T \mathbf{C} \mathbf{\Omega}_k = \mathbf{C}, \quad k = 1, 2, \dots, n-1, \quad (17)$$

where $\mathbf{\Omega}_k$ is a particular circulant permutation matrix. Thus, by Definition 3, any circulant matrix is an SPI matrix. Also, it can be readily shown that a circulant \mathbf{C} may be expressed as a polynomial of the elementary circulant matrix in the form of

$$\mathbf{C} = \text{circ}[c_{11}, c_{12}, \dots, c_{1n}] = c_{11} \mathbf{I}_n + c_{12} \mathbf{\Omega}_1 + c_{13} \mathbf{\Omega}_1^2 + \dots + c_{1n} \mathbf{\Omega}_1^{n-1}. \quad (18)$$

Definition 6 The special $n \times n$ permutation matrix \mathbf{K}_n , which has 1's on the main counterdiagonal and 0's elsewhere, i.e.,

$$\mathbf{K}_n = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix}, \quad (19)$$

is called a *counteridentity* matrix.

Using (19), it may be easily shown that the following expression,

$$\mathbf{K}_n \mathbf{A} \mathbf{K}_n = \mathbf{A}^T, \quad (20)$$

holds for a persymmetric SR matrix \mathbf{A} .

Remark 1 The special $n \times n$ permutation matrices, $\mathbf{\Omega}_k$ and \mathbf{K}_n , defined by (16) and (19), respectively, are persymmetric matrices.

In the sequel we will provide *sufficient* conditions for the occurrence of multiple solutions to the inhomogeneous system of n equations (9).

Proposition 1 Let $\mathbf{A}=[a_{ij}]$ be an $n \times n$ SR matrix with positive entries. Let a (positive) stationary vector of the error functional (3) be derived and be denoted by \mathbf{w}^* . If \mathbf{A} is a symmetric permutation invariant (SPI) matrix to a certain permutation matrix \mathbf{P}_n , then $\mathbf{P}_n^T \mathbf{w}^*$ produces an alternate stationary vector, provided that $\mathbf{P}_n^T \mathbf{w}^*$ and \mathbf{w}^* are linearly independent. If this permutation is consecutively repeated (not more than n times over) then the vectors, $\mathbf{P}_n^T \mathbf{w}^*, \mathbf{P}_n^{T^2} \mathbf{w}^*, \mathbf{P}_n^{T^3} \mathbf{w}^*, \dots$ represent alternate stationary vectors, provided that they are linearly independent.

Proof. Write the Frobenius norm of the nonlinear LS optimization problem (3) in the form

$$S^2(\mathbf{w}) = \left\| \mathbf{A} - \mathbf{W}^{-1} \mathbf{e} \mathbf{e}^T \mathbf{W} \right\|_F^2. \quad (21)$$

Let \mathbf{P}_n be an arbitrary $n \times n$ permutation matrix. Considering the fact that the sum of squares of the elements of a matrix is not affected by any permutation of the rows and the columns of this matrix the Frobenius norm does not vary by postmultiplying the matrix $(\mathbf{A} - \mathbf{W}^{-1} \mathbf{e} \mathbf{e}^T \mathbf{W})$ by an arbitrarily chosen permutation matrix \mathbf{P}_n , and then by premultiplying it by its transpose \mathbf{P}_n^T . Therefore,

$$S^2(\mathbf{w}) = \|\mathbf{P}_n^T (\mathbf{A} - \mathbf{W}^{-1} \mathbf{e} \mathbf{e}^T \mathbf{W}) \mathbf{P}_n\|_F^2 = \|\mathbf{P}_n^T \mathbf{A} \mathbf{P}_n - \mathbf{P}_n^T \mathbf{W}^{-1} \mathbf{P}_n \mathbf{P}_n^T \mathbf{e} \mathbf{e}^T \mathbf{P}_n \mathbf{P}_n^T \mathbf{W} \mathbf{P}_n\|_F^2. \quad (22)$$

Observe that in (22) $\mathbf{P}_n^T \mathbf{e} = \mathbf{e}$ and $\mathbf{e}^T \mathbf{P}_n = \mathbf{e}^T$. For an SPI matrix \mathbf{A} , by (12), $\mathbf{P}_n^T \mathbf{A} \mathbf{P}_n = \mathbf{A}$ holds. Thus,

$$S^2(\mathbf{w}) = \|\mathbf{A} - \mathbf{P}_n^T \mathbf{W}^{-1} \mathbf{P}_n \mathbf{e} \mathbf{e}^T \mathbf{P}_n^T \mathbf{W} \mathbf{P}_n\|_F^2. \quad (23)$$

In (23), the terms $\mathbf{P}_n^T \mathbf{W} \mathbf{P}_n$ and $\mathbf{P}_n^T \mathbf{W}^{-1} \mathbf{P}_n$ represent the permutations of the elements of \mathbf{W} and \mathbf{W}^{-1} , respectively. After they have been permuted by the permutation matrix $\mathbf{P}_n = [\mathbf{e}_{j_1} \ \mathbf{e}_{j_2} \ \dots \ \mathbf{e}_{j_n}]$, the elements of $\mathbf{P}_n^T \mathbf{W} \mathbf{P}_n$ (and the elements of $\mathbf{P}_n^T \mathbf{W}^{-1} \mathbf{P}_n$) are: $w_{j_1}, w_{j_2}, \dots, w_{j_n}$, (and their inverses). If the derived stationary vector, \mathbf{w}^* is linearly independent of the vector $\mathbf{P}_n^T \mathbf{w}^*$, i.e., if $\mathbf{P}_n^T \mathbf{w}^* \neq c \mathbf{w}^*$, where c is an arbitrary constant, then

$$\mathbf{P}_n^T \mathbf{W} \mathbf{P}_n \mathbf{e} = \mathbf{P}_n^T \mathbf{W} \mathbf{e}$$

becomes an alternate stationary vector. By repeating this procedure we may get

$$\mathbf{P}_n^{T^2} \mathbf{W} \mathbf{e},$$

which constitutes an other stationary vector, provided that this solution is linearly independent of both of the previous solutions. This way, the process can be continued as long as new linearly independent solutions are obtained. This completes the proof. \diamond

Corollary 1 If an $n \times n$ SR matrix \mathbf{A} is a circulant matrix then its factorization consists of one cycle of full length by the circulant permutations, $\mathbf{Q}_k \mathbf{w}^*$, $k=1, 2, \dots, n$, (i.e. if \mathbf{P}_n^T is an elementary circulant matrix) and the total number of alternate stationary vectors of the error functional (9) is n .

It is well-known that any permutation, $p=(j_1 \ j_2 \ \dots \ j_n)$, may be expressed as the product of the circulant permutations, $p=(p_1)(p_2)(p_3) \ \dots \ (p_r)$, where p_i is a circulant permutation of s_i elements called a *cyclic group*, where $\sum_{i=1}^r s_i = n$. Thus, after an

appropriate rearrangement of the rows and the columns, any permutation matrix \mathbf{P}_n may be written in the form of a *block diagonal* matrix with the circulants $\mathbf{\Omega}^{(s_i)}$ of order s_i , $i=1,2, \dots, r$, being placed on its main diagonal as follows

$$\mathbf{P}_n = \begin{bmatrix} \mathbf{\Omega}^{(s_1)} & & & & \\ & \mathbf{\Omega}^{(s_2)} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \mathbf{\Omega}^{(s_r)} \end{bmatrix}. \quad (24)$$

Using this notation, after performing an appropriate rearrangement of the rows and the columns, it implies that any SPI matrix \mathbf{M} , defined by (12), can be partitioned in the form

$$\mathbf{M} = [\mathbf{M}_{ij}],$$

where every $s_i \times s_j$ block, \mathbf{M}_{ij} , $i, j=1,2, \dots, r$, satisfies the relation

$$\mathbf{\Omega}^{(s_i)\top} \mathbf{M}_{ij} \mathbf{\Omega}^{(s_j)} = \mathbf{M}_{ij}, \quad i, j = 1, 2, \dots, r. \quad (25)$$

Apply now the above considerations to an SPI matrix \mathbf{A} , which is in SR. Perform a (simultaneous) rearrangement of the rows and the columns of \mathbf{A} . Let the resulting matrix be denoted by $\tilde{\mathbf{A}}$. Then, obviously, for $\tilde{\mathbf{A}}$ the following relation holds

$$\mathbf{\Omega}^{(s_i)\top} \tilde{\mathbf{A}}_{ij} \mathbf{\Omega}^{(s_j)} = \tilde{\mathbf{A}}_{ij}, \quad i, j = 1, 2, \dots, r. \quad (26)$$

Hence, in case of $s_i=s_j$, the matrix $\tilde{\mathbf{A}}$ has circulant SR matrices of order s_i for the blocks, $\tilde{\mathbf{A}}_{ii}$, on the main diagonal, where the order s_i is odd (otherwise an SR matrix cannot be a circulant), or all elements of $\tilde{\mathbf{A}}_{ii}$ are equal to 1, if the order s_i is even. It follows from the definition of an SR matrix that any other block, $\tilde{\mathbf{A}}_{ij}$, ($i \neq j$, $s_i=s_j$), might be a circulant of order s_i satisfying

$$\tilde{\mathbf{A}}_{ji} = \tilde{\mathbf{A}}_{ij}^{-\top}, \quad (i \neq j), \quad (27)$$

where $\sum_{i=1}^r s_i = n$ and $\tilde{\mathbf{A}}_{ij}^{-T}$ denotes the transpose of the block containing the reciprocals of the elements of $\tilde{\mathbf{A}}_{ij}$. If for any pair (i, j) , $s_i \neq s_j$, $i, j = 1, 2, \dots, r$, then the off-diagonal block, $\tilde{\mathbf{A}}_{ij}$, is an $s_i \times s_j$ rectangular block. Since for $s_i \neq s_j$,

$$\mathbf{\Omega}^{(s_i)T} \tilde{\mathbf{A}}_{ij} \mathbf{\Omega}^{(s_j)} = \tilde{\mathbf{A}}_{ij} \quad (28)$$

holds, a sufficient condition for $\tilde{\mathbf{A}}_{ij}$ is that its elements are equal.

Corollary 2 Let $\tilde{\mathbf{A}}$ be an $n \times n$ positive SR matrix whose rows and columns have been appropriately rearranged to be an SPI matrix. Let a (positive) stationary vector of the error functional (3) be determined. Let this solution be denoted by \mathbf{w}^* . Then the permutations $\mathbf{P}_n^T \mathbf{w}^*$, $\mathbf{P}_n^{T^2} \mathbf{w}^*$, $\mathbf{P}_n^{T^3} \mathbf{w}^*$, ... are also solutions, where \mathbf{P}_n^T is defined by (24). The total number of the alternate stationary vectors as solutions to equation (9) cannot exceed the least common multiple of s_1, s_2, \dots, s_r . (see the proof in [13].)

Proposition 2 Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ SR matrix with positive entries. Let a (positive) stationary vector of the error functional (3) be determined and let this solution of eq. (9) be denoted by $\mathbf{w}^{*T(1)} = [1, w_2^{*(1)}, \dots, w_n^{*(1)}]$. If \mathbf{A} is a persymmetric matrix, then

$$\mathbf{w}^{*T(2)} = \left[\frac{1}{w_{n-1}^{*(1)}}, \frac{1}{w_{n-2}^{*(1)}}, \dots, 1 \right], \quad (29)$$

is an alternate stationary vector as an other solution of equation (9), provided that the latter solution $\mathbf{w}^{*(2)}$ is linearly independent of $\mathbf{w}^{*(1)}$, i.e., if

$$w_n^{*(1)} \neq (w_i^{*(1)})(w_{n+1-i}^{*(1)}), \quad i = 1, 2, \dots, n. \quad (30)$$

Proof. Write the Frobenius norm of the nonlinear LS optimization problem (3) in the form

$$S^2(\mathbf{w}) = \left\| \mathbf{A} - \mathbf{W}^{-1} \mathbf{e} \mathbf{e}^T \mathbf{W} \right\|_F^2. \quad (31)$$

Consider the $n \times n$ counteridentity (permutation) matrix, \mathbf{K}_n defined by (19). Since \mathbf{K}_n

is an involutory matrix, therefore, $\mathbf{K}_n^2 = \mathbf{I}_n$. Let \mathbf{P}_n be an arbitrary $n \times n$ permutation matrix. Recognize that $\mathbf{I}_n = \mathbf{P}_n \mathbf{P}_n^T = \mathbf{P}_n \mathbf{K}_n \mathbf{K}_n \mathbf{P}_n^T$. Now apply the same technique that was used for the proof of Proposition 1. Thus, one may write that

$$S^2(\mathbf{w}) = \|\mathbf{K}_n \mathbf{A} \mathbf{K}_n - \mathbf{K}_n \mathbf{W}^{-1} \mathbf{K}_n \mathbf{e} \mathbf{e}^T \mathbf{K}_n \mathbf{W} \mathbf{K}_n\|_F^2 = \|\mathbf{A}^T - \mathbf{K}_n \mathbf{W}^{-1} \mathbf{K}_n \mathbf{e} \mathbf{e}^T \mathbf{K}_n \mathbf{W} \mathbf{K}_n\|_F^2. \quad (32)$$

Making use of (20), the transpose of the matrix in the right hand side of (32) is, apparently,

$$S^2(\mathbf{w}) = \|\mathbf{A} - \mathbf{K}_n \mathbf{W} \mathbf{K}_n \mathbf{e} \mathbf{e}^T \mathbf{K}_n \mathbf{W}^{-1} \mathbf{K}_n\|_F^2. \quad (33)$$

It is obvious from (33) that the elements of the matrix $\mathbf{K}_n \mathbf{W}^{-1} \mathbf{K}_n$ are composed of the elements of a vector $\mathbf{w}^{*(2)}$, which also constitutes a stationary vector. If this solution is linearly independent of $\mathbf{w}^{*(1)}$, then it must represent an alternate stationary vector as the entries of $\mathbf{K}_n \mathbf{W}^{-1} \mathbf{K}_n$ are: $\frac{1}{w_{n-1}^{*(1)}}, \frac{1}{w_{n-2}^{*(1)}}, \dots, 1$. If (30) is satisfied, then they are linearly independent. This completes the proof. \diamond

Corollary 3 Suppose that for the stationary vector $\mathbf{w}^{*(1)}$ the equality

$$\left[1, \frac{w_n^{*(1)}}{w_{n-1}^{*(1)}}, \frac{w_n^{*(1)}}{w_{n-2}^{*(1)}}, \dots, w_n^{*(1)} \right] = [1, w_2^{*(1)}, w_3^{*(1)}, \dots, w_n^{*(1)}] \quad (34)$$

is satisfied, i.e., the relation

$$w_n^{*(1)} = (w_i^{*(1)})(w_{n+1-i}^{*(1)}), \quad i = 1, 2, \dots, n, \quad (35)$$

holds. Then, (34) provides *one* solution to the nonlinear optimization problem (3). In this case no trivial alternate stationary vector can be found. It should be noted, however, that one might not call this solution a *unique* solution until the *necessary* conditions for the non-uniqueness problem of the solution of equation (9) have not been found, because, at this point, the existence of an other stationary value cannot be excluded.

To summarize the results of the developments made in this Section the following theorem gives *sufficient* conditions for the occurrence of a *non-unique* stationary vector of the error functional (3) as a solution to equation (9).

Theorem 1 Let \mathbf{A} be a general $n \times n$ SR matrix with positive entries.

- (i) If \mathbf{A} is a *circulant* matrix, or
- (ii) if $\mathbf{A}=[\mathbf{A}_{ij}]$ is a block SR matrix with $s_i \times s_j$ blocks, where \mathbf{A}_{ii} are circulant SR matrices, \mathbf{A}_{ij} are circulant matrices for $i \neq j$, $s_i = s_j$ and \mathbf{A}_{ij} has equal entries for $i \neq j$, $s_i \neq s_j$ and all blocks satisfy (27), or
- (iii) if \mathbf{A} is a *persymmetric* matrix, and for a given solution the relation

$$w_i^* \neq \frac{w_n^*}{w_{n+1-i}^*}, \quad i = 1, 2, \dots, n, \quad (36)$$

is satisfied, and

- (iv) if a (positive) solution to equation (9), under the condition (i), or (ii), or (iii), represents a *stationary vector* $\mathbf{w}^* = [1, w_2^*, \dots, w_n^*]$ (a local minimum),

then, this solution, \mathbf{w}^* , of the nonlinear least-squares optimization problem (3) is a *non-unique* stationary point.

4 Numerical Illustrations

The illustrations presented in this Section were selected to demonstrate the results of our paper. The numerical computations are made by “Mathematica”. Seven examples for given positive SR matrices, \mathbf{A} , (PCMs) are discussed in some detail below. For these examples Saaty’s reciprocal nine-point scale: $[1/9, \dots, 1/2, 1, 2, \dots, 9]$ is used for the numerical values of the entries of \mathbf{A} . The numerical experiments reported below include computation of the Hessian matrices. In every case they were found to be positive definite, thus ensuring that each stationary value computed was a local minimum.

EXAMPLE 1. The first example concerns data of a 5×5 PCM and demonstrates the occurrence of alternate optima, if \mathbf{A} is a circulant matrix. Since matrix \mathbf{A} is in SR by definition, for sake of simplicity the entries below the main diagonal are not depicted. This response matrix \mathbf{A}_1 is specified as

$$\mathbf{A}_1 = \begin{bmatrix} 1 & a & 1/a & a & 1/a \\ & 1 & a & 1/a & a \\ & & 1 & a & 1/a \\ & & & 1 & a \\ & & & & 1 \end{bmatrix}.$$

Let $a=9$. Applying the (elementary) circulant permutation matrix, $\mathbf{P}_n = \mathbf{\Omega}_k$, $k=1,2, \dots,5$, the following linearly independent solutions (five alternate minima) are obtained (see Corollary 1):

$$\mathbf{w}^{*T(1)} = [1, 0.3354, 1.9998, 0.5000, 2.9814],$$

$$\mathbf{w}^{*T(2)} = [1, 5.9623, 1.4909, 8.8890, 2.9814],$$

$$\mathbf{w}^{*T(3)} = [1, 5.9623, 1.9998, 0.6708, 3.9993],$$

$$\mathbf{w}^{*T(4)} = [1, 0.3354, 0.1125, 0.6708, 0.1677],$$

$$\mathbf{w}^{*T(5)} = [1, 0.2500, 1.4909, 0.5000, 0.1677],$$

with the same error, $S(\mathbf{w}^{*(i)}) = 23.8951$, $i = 1,2, \dots,5$.

Since each of the above stationary vectors, $\mathbf{w}^{*(i)}$, $i=1,2, \dots,5$, directly gives the first rows of their corresponding “best” approximating (rank one) transitive matrices, thus these approximation matrices, $\mathbf{B}_1^{(i)}$, $i=1,2, \dots,5$, to \mathbf{A}_1 could now be easily constructed.

EXAMPLE 2. The second example refers to a 6×6 PCM whose rows and columns have been rearranged appropriately. This response matrix $\tilde{\mathbf{A}}_2$ is specified as

$$\tilde{\mathbf{A}}_2 = \begin{bmatrix} 1 & 9 & 1/9 & \vdots & 5 & 4 & 1/3 \\ 1/9 & 1 & 9 & \vdots & 1/3 & 5 & 4 \\ 9 & 1/9 & 1 & \vdots & 4 & 1/3 & 5 \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots \\ 1/5 & 3 & 1/4 & \vdots & 1 & 7 & 1/7 \\ 1/4 & 1/5 & 3 & \vdots & 1/7 & 1 & 7 \\ 3 & 1/4 & 1/5 & \vdots & 7 & 1/7 & 1 \end{bmatrix}.$$

Observe that $\tilde{\mathbf{A}}_2$ contains two 3×3 circulant SR block matrices along its main diagonal. Note that $\tilde{\mathbf{A}}_2$ consists of a circulant off-diagonal block of size 3×3 as well. Using an appropriate permutation matrix, \mathbf{P}_n , consisting of two circulant permutation matrices along its main diagonal the following linearly independent solutions (three alternate minima) are obtained:

$$\mathbf{w}^{*T(1)} = [1, 5.5572, 1.9022, 5.8397, 3.9612, 2.5166],$$

$$\mathbf{w}^{*T(2)} = [1, 0.5257, 2.9215, 1.3230, 3.0700, 2.0825],$$

$$\mathbf{w}^{*T(3)} = [1, 0.3423, 0.1799, 0.7128, 0.4529, 1.0508],$$

with the same error, $S(\mathbf{w}^*)^{(i)} = 18.7968, \quad i = 1,2,3.$

As for EXAMPLE 1, the “best” approximating transitive matrices, $\mathbf{B}_2^{(i)}, \quad i=1,2,3,$ to $\tilde{\mathbf{A}}_2$ could readily be constructed. EXAMPLE 2 demonstrates the occurrence of a multiple solution for an SR matrix $\tilde{\mathbf{A}}_2$ which is neither circulant nor persymmetric. The reason that even in such a case alternate stationary vectors may occur is attributed to the SPI property (see Definition 3) of the SR matrix $\tilde{\mathbf{A}}_2$, which could be permuted by elementary circulants since it consists of cyclic groups.

EXAMPLE 3. The third example contains data of a 7×7 PCM whose rows and columns have been rearranged appropriately. This response matrix $\tilde{\mathbf{A}}_3$ is specified as

$$\tilde{\mathbf{A}}_3 = \begin{bmatrix} 1 & 9 & 1/9 & \vdots & 9 & \vdots & 1 & 9 & 1/9 \\ 1/9 & 1 & 9 & \vdots & 9 & \vdots & 1/9 & 1 & 9 \\ 9 & 1/9 & 1 & \vdots & 9 & \vdots & 9 & 1/9 & 1 \\ \dots & \dots & \dots & \vdots & \dots & \vdots & \dots & \dots & \dots \\ 1/9 & 1/9 & 1/9 & \vdots & 1 & \vdots & 9 & 9 & 9 \\ \dots & \dots & \dots & \vdots & \dots & \vdots & \dots & \dots & \dots \\ 1 & 9 & 1/9 & \vdots & 1/9 & \vdots & 1 & 9 & 1/9 \\ 1/9 & 1 & 9 & \vdots & 1/9 & \vdots & 1/9 & 1 & 9 \\ 9 & 1/9 & 1 & \vdots & 1/9 & \vdots & 9 & 1/9 & 1 \end{bmatrix}.$$

Observe that $\tilde{\mathbf{A}}_3$ consists of two 3×3 circulant block matrices along its main diagonal and a single element on its midpoint. Note that $\tilde{\mathbf{A}}_3$ has a 3×3 circulant off-diagonal block with the same entries as those of the block matrices along the main diagonal. Using an appropriate permutation matrix, \mathbf{P}_n , consisting of two circulant permutation matrices on its main diagonal and the unity at the midpoint (in other words, there are three cyclic groups here: two of size three and one fix-point), the following linearly independent solutions (six alternate minima) are obtained:

$$\mathbf{w}^{*T(1)} = [1, 2.6729, 2.1324, 0.8259, 1.7821, 7.6524, 4.9210],$$

$$\mathbf{w}^{*T(2)} = [1, 0.4689, 1.2535, 0.3873, 2.3077, 0.8357, 3.5886],$$

$$\mathbf{w}^{*T(3)} = [1, 0.7978, 0.3741, 0.3090, 2.8629, 1.8410, 0.6667],$$

$$\mathbf{w}^{*T(4)} = [1, 0.6431, 2.7613, 5.9584, 2.3077, 1.8410, 4.9210],$$

$$\mathbf{w}^{*T(5)} = [1, 4.2940, 1.5551, 9.2657, 2.8629, 7.6524, 3.5886],$$

$$\mathbf{w}^{*T(6)} = [1, 0.3621, 0.2329, 2.1578, 1.7821, 0.8357, 0.6667],$$

with the same error, $S(\mathbf{w}^{*})^{(i)} = 32.0030$, $i = 1, \dots, 6$.

Similarly to the previous examples the “best” approximating transitive matrices, $\mathbf{B}_3^{(i)}$, $i=1, \dots, 6$, to $\tilde{\mathbf{A}}_3$ may be constructed easily.

EXAMPLE 4. The fourth example shows data of a 8×8 PCM whose rows and columns have been rearranged appropriately. This response matrix $\tilde{\mathbf{A}}_4$ is specified as

$$\tilde{\mathbf{A}}_4 = \begin{bmatrix} 1 & 9 & 1/9 & \vdots & 9 & 9 & 9 & 9 & 9 \\ 1/9 & 1 & 9 & \vdots & 9 & 9 & 9 & 9 & 9 \\ 9 & 1/9 & 1 & \vdots & 9 & 9 & 9 & 9 & 9 \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots & \dots & \dots \\ 1/9 & 1/9 & 1/9 & \vdots & 1 & 9 & 1/9 & 9 & 1/9 \\ 1/9 & 1/9 & 1/9 & \vdots & 1/9 & 1 & 9 & 1/9 & 9 \\ 1/9 & 1/9 & 1/9 & \vdots & 9 & 1/9 & 1 & 9 & 1/9 \\ 1/9 & 1/9 & 1/9 & \vdots & 1/9 & 9 & 1/9 & 1 & 9 \\ 1/9 & 1/9 & 1/9 & \vdots & 9 & 1/9 & 9 & 1/9 & 1 \end{bmatrix}.$$

Observe here that $\tilde{\mathbf{A}}_4$ contains two circulant block matrices along its main diagonal of size 3×3 and 5×5 , respectively. Note that $\tilde{\mathbf{A}}_4$ consists of a rectangular off-diagonal block of size 3×5 with identical entries. It represents a “trivial” circulant matrix. By applying an appropriate permutation matrix, \mathbf{P}_n , which consists of two circulant

permutation matrices along its main diagonal, then the following linearly independent solutions (fifteen alternate minima) are obtained:

$$\mathbf{w}^{*T(1)} = [1, 1.1943, 1.0028, 3.3427, 2.0231, 5.9230, 5.3371, 5.9300],$$

$$\mathbf{w}^{*T(2)} = [1, 0.9972, 1.1910, 2.0175, 5.9066, 5.3224, 5.9136, 3.3334],$$

$$\mathbf{w}^{*T(3)} = [1, 0.8396, 0.8373, 4.9593, 4.4687, 4.9651, 2.7988, 1.6939],$$

⋮

$$\mathbf{w}^{*T(15)} = [1, 0.8396, 0.8373, 4.9651, 2.7988, 1.6939, 4.9593, 4.4687],$$

with the same error, $S(\mathbf{w}^{*(i)}) = 29.3584$, $i = 1, \dots, 15$.

Now, the “best” approximating transitive matrices, $\mathbf{B}_4^{(i)}$, $i=1, \dots, 15$, to $\tilde{\mathbf{A}}_4$ may be constructed easily.

EXAMPLE 5. The fifth example exhibits data of a 5×5 PCM. Since \mathbf{A} is in SR by definition, for sake of simplicity the entries below the main diagonal are not depicted. This response matrix \mathbf{A}_5 is specified as

$$\mathbf{A}_5 = \begin{bmatrix} 1 & 9 & 8 & 5 & 1/9 \\ & 1 & 3 & 2 & 5 \\ & & 1 & 3 & 8 \\ & & & 1 & 9 \\ & & & & 1 \end{bmatrix}.$$

Observe that \mathbf{A}_5 is a persymmetric SR matrix. Applying Proposition 2, the following linearly independent solutions (two alternate minima) are obtained:

$$\mathbf{w}^{*T(1)} = [1, 0.9988, 0.6409, 0.5834, 4.4176],$$

$$\mathbf{w}^{*T(2)} = [1, 7.5719, 6.8926, 4.4229, 4.4176],$$

with the same error, $S(\mathbf{w}^{*(i)}) = 16.0449$, $i = 1, 2$.

The “best” approximating transitive matrices, $\mathbf{B}_5^{(i)}$, $i=1,2$, to \mathbf{A}_5 are:

$$\mathbf{B}_5^{(1)} = \begin{bmatrix} 1 & 0.9988 & 0.6409 & 0.5834 & 4.4176 \\ & 1 & 0.6417 & 0.5841 & 4.4229 \\ & & 1 & 0.9103 & 6.8926 \\ & & & 1 & 7.5719 \\ & & & & 1 \end{bmatrix},$$

and

$$\mathbf{B}_5^{(2)} = \begin{bmatrix} 1 & 7.5719 & 6.8926 & 4.4229 & 4.4176 \\ & 1 & 0.9103 & 0.5841 & 0.5834 \\ & & 1 & 0.6417 & 0.6409 \\ & & & 1 & 0.9988 \\ & & & & 1 \end{bmatrix}.$$

Observe here that neither $\mathbf{B}_5^{(1)}$ nor $\mathbf{B}_5^{(2)}$ is a persymmetric matrix. The two independent solutions, $\mathbf{w}^{*(1)}$ and $\mathbf{w}^{*(2)}$, are in the first rows and they also appear in the last columns of $\mathbf{B}_5^{(1)}$ and $\mathbf{B}_5^{(2)}$ in opposite order.

EXAMPLE 6. The sixth example shows data of a 6×6 PCM. Since \mathbf{A} is in SR by definition, for sake of simplicity the entries below the main diagonal are not depicted. This response matrix \mathbf{A}_6 is specified as

$$\mathbf{A}_6 = \begin{bmatrix} 1 & 9 & 8 & 7 & 6 & 1/9 \\ & 1 & 1 & 3 & 9 & 6 \\ & & 1 & 1 & 3 & 7 \\ & & & 1 & 1 & 8 \\ & & & & 1 & 9 \\ & & & & & 1 \end{bmatrix}.$$

Observe that \mathbf{A}_6 is a persymmetric SR matrix. By applying Corollary 3, the following solution (a local minimum) is obtained:

$$\mathbf{w}^{*T} = [1, 0.8753, 1.7818, 3.0458, 6.2006, 5.4271],$$

with the error, $S(\mathbf{w}^*) = 18.8874$.

The “best” approximating transitive matrix, \mathbf{B}_6 , to \mathbf{A}_6 is:

$$\mathbf{B}_6 = \begin{bmatrix} 1 & 0.8753 & 1.7818 & 3.0458 & 6.2006 & 5.4271 \\ & 1 & 2.0358 & 3.4799 & 7.0843 & 6.2006 \\ & & 1 & 1.7094 & 3.4799 & 3.0458 \\ & & & 1 & 2.0358 & 1.7818 \\ & & & & 1 & 0.8753 \\ & & & & & 1 \end{bmatrix}.$$

Observe here that now \mathbf{B}_6 is a persymmetric matrix, i.e. its entries are symmetric about its counterdiagonal (secondary diagonal). Therefore, in such a case, no trivial alternate stationary vector can be found. It is easy to check that each of the conditions (35) for the elements of the stationary vector (which are in the first row *and* in opposite order in the last column of matrix \mathbf{B}_6) holds.

EXAMPLE 7. The authors carried out a comprehensive analysis for a large set of different 3×3 SR matrices \mathbf{A} . Although in the applications of the AHP these matrices represent the simplest cases only, yet they seem to be adequate to show us a certain tendency of the occurrences of non-unique solutions to the nonlinear LS optimization problem (3). For this purpose we utilize some of our results presented in Section 3. Let these response matrices \mathbf{A}_7 be given in the form

$$\mathbf{A}_7 = \begin{bmatrix} 1 & a & b \\ 1/a & 1 & a \\ 1/b & 1/a & 1 \end{bmatrix}.$$

Observe that \mathbf{A}_7 is persymmetric. Here the entries a correspond to $a_{12}=a_{23}$ and entry b corresponds to a_{13} . Using Saaty’s nine-point scale for a great number of appropriately chosen 3×3 PCM’s the multiple stationary vectors (global minima) have been determined and are displayed in Figure 1 over the entire range of the possible values of these entries (recall that the respective entry in \mathbf{A} expresses the relative strength with which decision alternative A_i dominates alternative A_j). Here a particular solution represents, in fact, a *global* minimum. Namely, by the NK method, applying a heuristics approach *all* solutions were generated and examined in a numerical way for each interval by using “Mathematica”. Consider now Figure 1. Here the selected scale-values of the entries, $a_{12}=[1, 2, \dots, 9]$ (or for $a_{12}=[1/9, 1/8, \dots, 1]$) are plotted as function of the NE corner entry, a_{13} . The locations of the black dots indicate the numerical values with which a 3×3 PCM, \mathbf{A}_7 becomes a circulant matrix (here at these

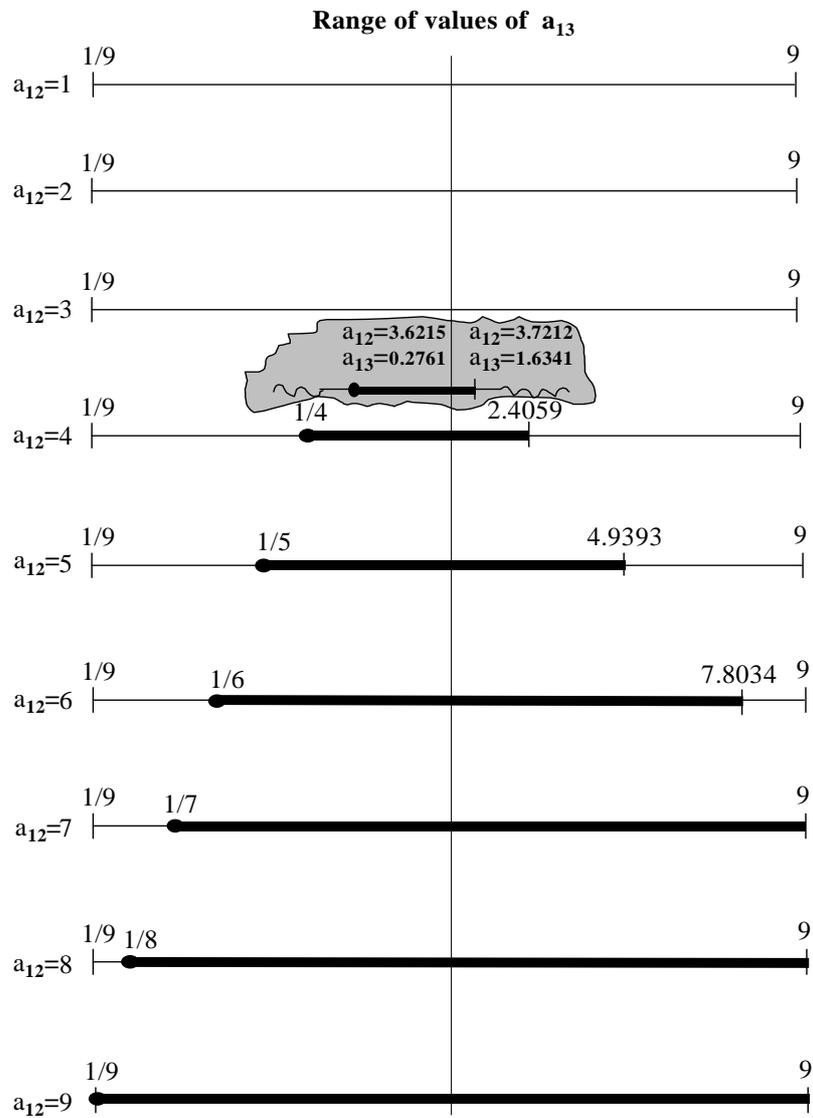


Figure 1. Domains of multiple optima for 3x3 SR matrices A (PCMs)

points there are three alternate optima). The intervals drawn by heavy lines indicate the regions over which linearly independent solutions occur (here, there are two alternate optima). The regions drawn by solid lines indicate the intervals over which there is one solution (a global minimum). Bozóki [16] used the resultant method for analyzing the non-uniqueness problem of 3×3 SR matrices. It is interesting to note that on applying our method to EXAMPLE 7 exactly the same results are obtained; giving confidence in the appropriateness of both approaches.

Figure 1 exhibits a remarkable tendency concerning the likelihood of a multiple solution. Note that with a growing level of inconsistency (as the entries $a_{12}=a_{23}$ are increased relative to the entry a_{13}) the range of values over which a multiple solution occurs will be greater and greater. Note that for $a_{12}=9$, only multiple solution occurs within the whole possible range of the values of entry a_{13} . One may recognize from this chart that initially (i.e. at low levels of inconsistency of the matrix \mathbf{A}_7) the optimal solution (global minimum) is unique. It is interesting to note that up to a turning point the solution yields $\mathbf{w}^{*T}=[1,1,1]$ evaluated at the entries of a and $b=1/a$. If, however, the entries of \mathbf{A}_7 are increased to: $a_{12}=a_{23}=3.6215$ and $a_{13}=1/a_{12}=1/a_{23}=0.2761$, then three other independent stationary vectors achieve the minimum in (3) also (thus at this intersection point there are four alternate optima). For \mathbf{A}_7 , the numerical values of the entries $a_{12}=a_{23}$ and a_{13} when a unique solution switches to a multiple solution, or reversed, when it switches back to a unique one, can be determined explicitly (see in the Appendix the formulation of the system of nonlinear equations which considers a more general case than is discussed by EXAMPLE 7).

Conclusions

A system of nonlinear equations has been used to determine the entries of a transitive matrix which is the best approximation to a general pairwise comparison matrix in a least-squares sense. The nonlinear minimization problem as the solution of a set of inhomogeneous equations has been examined for its uniqueness properties. Sufficient conditions for possible non-uniqueness of the solution to this optimization problem have been developed and the related proofs have also been presented. For a great number of different sized positive SR matrices having certain properties, the results have been demonstrated by the numerical experiments. Further research will include the investigation for finding the necessary conditions for the non-uniqueness of the solution to the nonlinear optimization problem.

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Appendix

Suppose that the positive $n \times n$ SR matrix \mathbf{A} is specified as

$$\mathbf{A} = \begin{bmatrix} 1 & a & a & a & \cdot & \cdot & \cdot & b \\ & 1 & 1 & 1 & \cdot & \cdot & \cdot & a \\ & & 1 & 1 & \cdot & \cdot & \cdot & a \\ & & & 1 & \cdot & \cdot & \cdot & a \\ & & & & \cdot & & & \cdot \\ & & & & & \cdot & & \cdot \\ & & & & & & \cdot & \cdot \\ & & & & & & & 1 \end{bmatrix}.$$

Let the entries of \mathbf{A} be denoted by $a = a_{ij} = a_{jn}$, $j=2, \dots, n-1$, and $b = a_{1n}$, the linearly independent solutions to equation (9) given in Proposition 2, by $\mathbf{w}^{*(1)}$ and $\mathbf{w}^{*(2)}$, respectively, and the solution (34) by \mathbf{v}^* . At a stationary point of $S^2(\mathbf{w})$ where a multiple optima occurs the elements of the unknown vectors $\mathbf{w}^{*(1)}$, $\mathbf{w}^{*(2)}$, \mathbf{v}^* and the entries of \mathbf{A} can be determined by solving the following constrained nonlinear optimization problem [for a particular problem, the size of matrix \mathbf{A} has to be properly adjusted, the weights v_i and v_k should be inserted in the equations according to relation (35) and recall that $\mathbf{w}^{*(2)}$ can be obtained from (29)]:

$$\sum_{i=1}^n \sum_{k=1}^n \left(a_{ik} - \frac{w_k}{w_i} \right)^2 - \sum_{i=1}^n \sum_{k=1}^n \left(a_{ik} - \frac{v_k}{v_i} \right)^2 = 0,$$

$$\sum_{k=1}^n \left[\frac{a_{ik}}{w_i^2} + \frac{1}{a_{ik} w_k^2} - 2 \left(\frac{w_i}{w_k^3} + \frac{w_k}{w_i^3} \right) \right] w_k = 0, \quad i=2, \dots, n,$$

$$\sum_{k=1}^n \left[\frac{a_{ik}}{v_i^2} + \frac{1}{a_{ik} v_k^2} - 2 \left(\frac{v_i}{v_k^3} + \frac{v_k}{v_i^3} \right) \right] v_k = 0, \quad i=2, \dots, n,$$

$$\mathbf{c}^T \mathbf{w} = 1, \quad \text{and} \quad \mathbf{c}^T \mathbf{v} = 1,$$

where $\mathbf{w}^T := \{w_1, w_2, \dots, w_n\}$, $\mathbf{v}^T := \{v_1, v_2, \dots, v_n\}$, and $\mathbf{c}^T := \{1, 0, 0, \dots, 0\}$.

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