The Analysis of the Principal Eigenvector of Pairwise Comparison Matrices

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Abstract. This paper develops the spectral properties of pairwise comparison matrices (PCM) used in the multicriteria decision making method called analytic hierarchy process (AHP). Perturbed PCMs are introduced which may result in a reversal of the rank order of the decision alternatives. The analysis utilizes matrix theory to derive the principal eigenvector components of perturbed PCMs in explicit form. Proofs are presented for the existence of rank reversals. Intervals over which such rank reversals occur are also established as function of a continuous perturbation parameter. It is proven that this phenomenon is inherent in AHP even in the case of the slightest departure from consistency. The results are demonstrated through a sample illustration.

Keywords: multiple criteria decision making, algebraic eigenvalue-eigenvector problem, rank reversal issue

1 Introduction

The analytic hierarchy process (AHP) is a multicriteria decision making method that employs a procedure of multiple comparisons to rank order alternative solutions to a multiobjective decision problem. Ever since the development of the AHP in the late 1970's by Saaty [14], a great number of criticisms of this approach have appeared in the literature. One of its most controversial aspects is the phenomenon of rank reversal of the decision alternatives. Both proponents and opponents of the AHP agree that rank reversal may occur, but disagree on its legitimacy. The problem has been considered by many authors and a persistent debate has followed; see Watson and Freeling [22], Saaty and Vargas [18], Belton and Gear [3], Vargas [21], Harker and Vargas [10], Dyer [5], Saaty [17], Harker and Vargas [11], Salo and Hämäläinen [19] and Pérez [13].

Despite the amount of work done on the subject, there are virtually no papers presenting a formal study of the algebraic eigenvalue-eigenvector problem of AHP's pairwise comparison matrix (PCM). This paper provides a rigorous mathematical presentation of this problem and gives proofs for the existence of rank reversal for a certain case. The foregoing research has been shown that a rank reversal may occur in AHP, (*i*) by introducing continuous perturbation(s) at one or more pairs of elements of a consistent PCM (see e.g., Watson and Freeling [22], Dyer and Wendell

[6]), or, (*ii*) by adding a new alternative to a perturbed PCM that is a replica (copy) of any of the old alternatives (see e.g., Belton and Gear [2], Dyer and Wendell [6]) and (*iii*) due to the normalization when aggregating the weights of the alternatives from the data even if the PCMs are each consistent to determine the overall priorities of the alternatives (see e.g., Barzilai and Golany [1]). In this paper, intervals are also established for the case (*i*) over which such rank reversals occur for situations when a PCM departs from perfect consistency even in only an arbitrarily small degree. The paper considers PCM's with a single criterion only.

Definition 1 A square matrix **A** of order n is called a *symmetrically reciprocal* (SR) matrix if its elements a_{ij} are nonzero complex numbers and

$$a_{ij}a_{ji}=1, \quad \text{for } i{\neq}j; \quad i,j{=}1,2,...,n,$$
$$a_{ii}{=}1, \quad \text{for } i{=}1,2,...,n. \tag{1}$$

Definition 2 A square matrix **A** of order n is called a *transitive* matrix if its elements a_{ij} are nonzero complex numbers and

$$a_{ij}a_{jk}=a_{ik}, \quad \text{for all } i,j,k. \tag{2}$$

Definition 3 A square matrix **A** of order n is a *one-rank* matrix if its elements a_{ij} can be expressed as

$$a_{ij}=p_iq_j, \quad \text{for all } i,j. \tag{3}$$

Theorem 1 *Let* $\mathbf{A}{=}[a_{ij}]$ *be a square matrix of order n, $n{\geq}3$. (i) If* **A** *is transitive, then* **A** *is a one-rank* SR*, as well. (ii) If* **A** *is a* SR *matrix, then* **A** *is transitive if and only if it is a one-rank matrix.*
(The proof of Theorem 1 is given in Farkas, Rózsa and Stubnya [8].)

The concept of a SR matrix defined by relation (1) was introduced by Saaty [14], who used the term reciprocal matrix. We prefer to designate this property according to Definition 1 since reciprocal matrices are the equivalent terms for the inverse matrices. In the framework of AHP, Saaty [14] developed such a SR matrix, $\mathbf{A}{=}[a_{ij}]$, called a *pairwise (paired) comparison* matrix, entries of which represent the relative importance ratios of the alternative A_i over the alternative A_j, $i,j{=}1,2,...,n$, with respect to a common criterion. Elements of **A** are *positive, real* numbers. Saaty [14] called **A** a *consistent* matrix if the transitivity property (2) holds for **A** as well (cardinal consistency). In the AHP, every decision maker should provide ratio estimates for each possible pair of the alternatives [$n(n{-}1)/2$].

Using an eigenvalue-eigenvector approach, for a finite set of alternatives the AHP develops weights (and thus the priority ranking) of the alternatives on a *ratio scale*. Due to the properties of most of the decision problems occurring in practice the *rank order* of the alternatives, however, is usually generated on an *ordinal scale*. As it is well known, an ordinal ranking is said to be complete (it contains no ties) if the ordinal transitivity condition (ordinal consistency) holds, i.e.,

$$A_i \rightarrow A_j \text{ and } A_j \rightarrow A_k \text{ imply } A_i \rightarrow A_k \quad \text{for all } i,j,k, \tag{4}$$

where, the symbol $A_i \rightarrow A_j$ is interpreted as A_i is preferred to A_j.

Saaty [15, p.848] proved that the weight (priority score) of an alternative, what he called the *relative dominance* of the ith alternative A_i, is the ith component of the principal right eigenvector of $\mathbf{A}$, u_i, provided that $\mathbf{A}$ is consistent, i.e., $\mathbf{A}$ is a transitive PCM. The *principal* right eigenvector belongs to the eigenvalue of largest modulus. The eigenvalue of largest modulus will be called *maximal* eigenvalue. By Perron's theorem, for matrices with positive elements, the maximal eigenvalue is always positive, simple and the components of its associated eigenvector are positive (see e.g., in Horn and Johnson [12]). Saaty [15, p.853] claimed to prove that this result also holds for a SR matrix that is *not* necessarily consistent, i.e., if it is *not* transitive. At this point the question can be raised, whether or not the components of the principal right eigenvector produce the true relative dominance of the alternatives, if the PCM is perturbed. Therefore, in this paper we shall study the behavior of the components of the principal eigenvectors of perturbed PCMs.

In Section 2, PCMs of specific form are defined and their spectral properties are described. In Section 3, PCMs of general form are introduced and their spectral properties are developed. In Section 4, the issue of rank reversal is examined for the specific versus the simple perturbed case. Proofs are given for the intervals of rank reversals. A sample illustration is provided in Section 5. The characteristic polynomials of PCMs of general form and the development of their principal eigenvector components are presented in Appendices A and B.

2 Pairwise Comparison Matrices of Specific Form

Definition 4 A square matrix with positive entries is called a *specific* PCM denoted by $\mathbf{A}$, if it is transitive.

According to Theorem 1, any transitive matrix can be expressed as the product of a column vector $\mathbf{u}$ and a row vector $\mathbf{v}^{\mathrm{T}}$ as :

$$\mathbf{A} = \mathbf{u}\mathbf{v}^{\mathrm{T}}, \tag{5}$$

where

$$\mathbf{v}^{\mathrm{T}} = \left[1, x_1, x_2, ..., x_{n-1}\right], \text{ and } \mathbf{u}^{\mathrm{T}} = \left[1, \frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_{n-1}}\right].$$

Introducing the diagonal matrix $\mathbf{D} = \mathrm{diag}\langle 1, 1/x_1, 1/x_2, ..., 1/x_{n-1}\rangle$ and the vector $\mathbf{e}^{\mathrm{T}} = [1, 1, ..., 1]$, obviously $\mathbf{D}^{-1}\mathbf{A}\mathbf{D} = \mathbf{e}\mathbf{e}^{\mathrm{T}}$. It is easy to see that the characteristic polynomial of $\mathbf{A}$, $p_n(\lambda)$, can be obtained in the following from

$$p_n(\lambda) \equiv \det[\lambda\mathbf{I}_n - \mathbf{A}] = \det[\lambda\mathbf{I}_n - \mathbf{e}\mathbf{e}^{\mathrm{T}}] = \lambda^{n-1}(\lambda - n), \tag{6}$$

where $\mathbf{I}_n$ is the identity matrix of order n. From (6), it is apparent that $\mathbf{A}$ has a zero eigenvalue with multiplicity $n-1$ and one simple positive eigenvalue, $\lambda=n$, with the corresponding right and left eigenvectors, $\mathbf{u}$ and $\mathbf{v}^{\mathrm{T}}$, respectively. The relative dominance of the alternatives are given by the components of $\mathbf{u}$. Conventionally, this solution is normalized so that its components sum to unity.

3 Pairwise Comparison Matrices of General Form

In applied problems, decision makers give subjective judgements on the relative importance ratios. As a common consequence, usually, a failure of the relation (2) to hold is manifested in their PCMs. Hence, it seems to be apparent to explore how the maximal eigenvalue and its associated eigenvector vary when matrix $\mathbf{A}$ is perturbed such that it remains in SR, however, its transitivity is lost.

Definition 5 A square matrix with positive entries is called a *perturbed* PCM and denoted by $\mathbf{A}_{\mathrm{p}}$, if the matrix is symmetrically reciprocal and not transitive.

Consider now the transitive matrix $\mathbf{A}=\mathbf{D}\mathbf{e}\mathbf{e}^{\mathrm{T}}\,\mathbf{D}^{-1}$ with the elements $a_{ij}=1/a_{ji}=x_j/x_i$, $i,j=0,1,2,...,n-1$. Let the elements of matrix $\mathbf{A}$ be perturbed in its first row and in its first column in a multiplicative way. This perturbed SR matrix $\mathbf{A}_{\mathrm{P}}$ can be written as

$$\mathbf{A}_{\mathrm{P}}=\begin{bmatrix} 1 & x_1\delta_1 & x_2\delta_2 & . & . & . & x_{n-1}\delta_{n-1} \\[2mm] \dfrac{1}{x_1\delta_1} & 1 & \dfrac{x_2}{x_1} & . & . & . & \dfrac{x_{n-1}}{x_1} \\[2mm] \dfrac{1}{x_2\delta_2} & \dfrac{x_1}{x_2} & 1 & . & . & . & \dfrac{x_{n-1}}{x_2} \\[1mm] . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\[1mm] \dfrac{1}{x_{n-1}\delta_{n-1}} & \dfrac{x_1}{x_{n-1}} & \dfrac{x_2}{x_{n-1}} & . & . & . & 1 \end{bmatrix}, \qquad (7)$$

where $\delta_1,\delta_2,...,\delta_{n-1}$ are arbitrary positive numbers with $\delta_i\neq1$, $i=1,2,...,n-1$. Performing a similarity transformation, the characteristic polynomial of $\mathbf{A}_{\mathrm{P}}$, $p_n^{\mathrm{P}}(\lambda)$, is obtained as

$$p_n^{\mathrm{P}}(\lambda)\equiv\det[\lambda\mathbf{I}_n-\mathbf{A}_{\mathrm{P}}]=\det[\lambda\mathbf{I}_n-\mathbf{D}^{-1}\mathbf{A}_{\mathrm{P}}\mathbf{D}]=\det\mathbf{K}_{\mathrm{P}}(\lambda), \qquad (8)$$

where

$$\det \mathbf{K}_{\mathrm{P}}(\lambda) = \begin{vmatrix} \lambda-1 & -\delta_1 & -\delta_2 & . & . & . & -\delta_{n-1} \\ -\dfrac{1}{\delta_1} & \lambda-1 & -1 & . & . & . & -1 \\ -\dfrac{1}{\delta_2} & -1 & \lambda-1 & . & . & . & -1 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ -\dfrac{1}{\delta_{n-1}} & -1 & -1 & . & . & . & \lambda-1 \end{vmatrix}.$$

The matrix $\mathbf{K}_{\mathrm{P}}(\lambda) = \lambda \mathbf{I}_n - \mathbf{D}^{-1}\mathbf{A}_{\mathrm{P}}\mathbf{D}$ may be interpreted in the form of a modified matrix: with the notation

$$\mathbf{U}_{\mathrm{P}} = \begin{bmatrix} 0 & 1 \\ 1-\dfrac{1}{\delta_1} & 0 \\ . & . \\ . & . \\ . & . \\ 1-\dfrac{1}{\delta_{n-1}} & 0 \end{bmatrix} \text{ and } \mathbf{V}_{\mathrm{P}}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1-\delta_1 & . & . & . & 1-\delta_{n-1} \end{bmatrix}.$$

To find the inverse of a matrix that is modified by a one-rank matrix [see the determinant (A1) in Appendix A] through applying the Sherman-Morrison formula [20, p.126] let us introduce the matrix $\mathbf{T}_{\mathrm{P}}(\lambda)$ as

$$\mathbf{T}_{\mathrm{P}}(\lambda) = \lambda \mathbf{I}_n + \mathbf{U}_{\mathrm{P}}\mathbf{V}_{\mathrm{P}}^{\mathrm{T}}. \tag{10}$$

Thus, the modified matrix $\mathbf{K}_{\mathrm{P}}(\lambda)$ can now be described as

$$\mathbf{K}_{\mathrm{P}}(\lambda) = \mathbf{T}_{\mathrm{P}}(\lambda) - \mathbf{e}\mathbf{e}^{\mathrm{T}}. \tag{11}$$

It is shown in Appendix A that the determinant of matrix $\mathbf{K}_{\mathrm{P}}(\lambda)$, i.e. the characteristic polynomial $p_{\mathrm{n}}^{\mathrm{P}}(\lambda)$, yields

$$p_n^{\mathrm{P}}(\lambda) \equiv \det \mathbf{K}_{\mathrm{P}}(\lambda) = \lambda^{n-3}\left\{ \lambda^3 - n\lambda^2 + (n-1)\sum_{i=1}^{n-1}(1-\delta_i)(1-\frac{1}{\delta_i}) - \sum_{i=1}^{n-1}(1-\frac{1}{\delta_i})\sum_{i=1}^{n-1}(1-\delta_i) \right\}. \quad (12)$$

Expression (12) shows clearly, that even if all elements are perturbed in one (arbitrary) row and in its corresponding column of matrix $\mathbf{A}$, then, $\mathbf{A}_{\mathrm{P}}$ has a zero eigenvalue with multiplicity $\geq n-3$, if $n>2$ and a trinomial equation is obtained for the nonzero eigenvalues. From (12), the characteristic polynomial, $p_{\mathrm{n}}^{\mathrm{P}}(\lambda)$, can be rewritten in the simplified form

$$p_n^{\mathrm{P}}(\lambda) \equiv \lambda^{n-3}\left\{ \lambda^3 - n\lambda^2 - C \right\}, \quad (13)$$

where the constant term C contains the perturbation factors $\delta_i \neq 1$, $i=1,2,...,n-1$, as

$$C = -(n-1)\sum_{i=1}^{n-1}(1-\delta_i)(1-\frac{1}{\delta_i}) + \sum_{i=1}^{n-1}(1-\frac{1}{\delta_i})\sum_{i=1}^{n-1}(1-\delta_i).$$

From now on, we restrict our investigations to PCMs with *one* SR perturbed pair of elements only, say $\delta_1 = \delta \neq 1$, while $\delta_i = 1$ for $i \neq 1$.

Definition 6 If one pair of elements, a_{12} and a_{21} of a specific PCM has the form $a_{12} = x_1\delta$, $a_{21} = 1/x_1\delta$, and $\delta > 0$, then it is called a *simple perturbed* PCM, denoted by $\mathbf{A}_{\mathrm{S}}$.

In this special case, we have the simple perturbed matrix $\mathbf{A}_{\mathrm{S}}$ as

$$\mathbf{A}_{\mathrm{S}} = \begin{bmatrix} 1 & x_1\delta & x_2 & . & . & . & x_{n-1} \\ \dfrac{1}{x_1\delta} & 1 & \dfrac{x_2}{x_1} & . & . & . & \dfrac{x_{n-1}}{x_1} \\ \dfrac{1}{x_2} & \dfrac{x_1}{x_2} & 1 & . & . & . & \dfrac{x_{n-1}}{x_2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \dfrac{1}{x_{n-1}} & \dfrac{x_1}{x_{n-1}} & \dfrac{x_2}{x_{n-1}} & . & . & . & 1 \end{bmatrix}. \quad (14)$$

Performing a similarity transformation [see (6) and (8)], the characteristic polynomial of $\mathbf{A}_S, p_n^P(\lambda)$, can be written as

$$p_n^S(\lambda) \equiv \det[\lambda \mathbf{I}_n - \mathbf{A}_S] = \det[\lambda \mathbf{I}_n - \mathbf{D}^{-1} \mathbf{A}_S \mathbf{D}] = \det \mathbf{K}_S(\lambda), \tag{15}$$

where

$$\det \mathbf{K}_S(\lambda) = \begin{vmatrix} \lambda-1 & 1-\delta & -1 & . & . & . & -1 \\ 1-\dfrac{1}{\delta} & \lambda-1 & -1 & . & . & . & -1 \\ -1 & -1 & \lambda-1 & . & . & . & -1 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ -1 & -1 & -1 & . & . & . & \lambda-1 \end{vmatrix}.$$

Similarly to (9), the matrix $\mathbf{K}_S(\lambda)$ in (15)

$$\mathbf{K}_S(\lambda) = \lambda \mathbf{I}_n - \mathbf{D}^{-1} \mathbf{A}_S \mathbf{D}, \tag{16}$$

may be interpreted as a modified matrix

$$\mathbf{K}_S(\lambda) = \lambda \mathbf{I}_n + \mathbf{U}_S \mathbf{V}_S^T - \mathbf{e} \mathbf{e}^T, \tag{17}$$

with the notation

$$\mathbf{U}_S = \begin{bmatrix} 0 & 1 \\ 1-\dfrac{1}{\delta} & 0 \\ . & . \\ . & . \\ . & . \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{V}_S^T = \begin{bmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1-\delta & . & . & . & 0 \end{bmatrix}.$$

Introducing

$$\mathbf{T}_S(\lambda) = \lambda \mathbf{I}_n + \mathbf{U}_S \mathbf{V}_S^T = \begin{bmatrix} \lambda & 1-\delta & 0 & . & 0 \\ 1-\dfrac{1}{\delta} & \lambda & 0 & . & 0 \\ 0 & 0 & \lambda & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & \lambda \end{bmatrix}, \tag{18}$$

the modified matrix $\mathbf{K}_S(\lambda)$ can be written as

$$\mathbf{K}_S(\lambda) = \mathbf{T}_S(\lambda) - \mathbf{e}\mathbf{e}^T. \tag{19}$$

In this special case, the characteristic polynomial (12) has the form

$$p_n^S(\lambda) \equiv \lambda^{n-3}[\lambda^3 - n\lambda^2 - C_S], \tag{20}$$

where, the constant term, C_S, now becomes

$$C_S = -(n-2)(1-\delta)(1-\frac{1}{\delta}) = (n-2)Q,$$

and Q is expressed as a function of the perturbation factor δ as

$$Q = \delta + \frac{1}{\delta} - 2, \quad \delta > 0 \; (\delta \neq 1). \tag{21}$$

Let r denote the maximal eigenvalue of a simple perturbed PCM, $\mathbf{A}_S$. Then, r can be obtained from the equation [cf.(20)]:

$$r^3 - nr^2 - (n-2)Q = 0, \tag{22}$$

where Q is given by (21). Since $Q > 0$, from (22) it is easy to see that $r > n$. The proof can be found in Farkas, Rózsa and Stubnya [8]. The components of the principal eigenvector can be obtained from the one-rank matrix

$$\mathrm{adj}(r\mathbf{I}_n - \mathbf{A}_S) = [u_{ij}^S(r)], \tag{23}$$

since any column of the adjoint gives the elements of the principal eigenvector. In Appendix B, we show that the elements, $u_{ij}^S(r)$, of the principal eigenvector for the simple perturbed case are:

$$
\begin{bmatrix} u_{11}^S(r) \\ u_{21}^S(r) \\ \dots \\ u_{i1}^S(r) \\ \dots \end{bmatrix} = \begin{bmatrix} r^{n-2}[r-(n-1)] \\ \dfrac{1}{x_1}r^{n-3}\left\{r-\left(1-\dfrac{1}{\delta}\right)[r-(n-2)]\right\} \\ \dots \\ \dfrac{1}{x_{i-1}}r^{n-3}\left\{r-\left(1-\dfrac{1}{\delta}\right)\right\} \\ \dots \end{bmatrix} ; \quad i=3,4,\dots,n, \tag{24}
$$

$$
\begin{bmatrix} u_{12}^S(r) \\ u_{22}^S(r) \\ \dots \\ u_{i2}^S(r) \\ \dots \end{bmatrix} = \begin{bmatrix} r^{n-3}\{r+(\delta-1)[r-(n-2)]\} \\ \dfrac{1}{x_1}r^{n-2}[r-(n-1)] \\ \dots \\ \dfrac{1}{x_{i-1}}r^{n-3}\{r+(\delta-1)\} \\ \dots \end{bmatrix} x_1; \quad i=3,4,\dots,n, \tag{25}
$$

and

$$
\begin{bmatrix} u_{1j}^S(r) \\ u_{2j}^S(r) \\ \dots \\ u_{ij}^S(r) \\ \dots \end{bmatrix} = \begin{bmatrix} r^{n-3}[r+(\delta-1)] \\ \dfrac{1}{x_1}r^{n-3}\left\{r-\left(1-\dfrac{1}{\delta}\right)\right\} \\ \dots \\ \dfrac{1}{x_{i-1}}r^{n-2}\left\{\dfrac{r-2}{n-2}\right\} \\ \dots \end{bmatrix} x_{j-1}; \quad i,j=3,4,\dots,n. \tag{26}
$$

4 The Issue of Rank Reversal

The concept of rank reversal is now introduced. Consider the simple perturbed matrix $\mathbf{A}_S$ defined by (14). In the specific versus the simple perturbed case, the maximal eigenvalue r of matrix $\mathbf{A}_S$ can be determined from (22), where $r>n$ ($n \geq 3$) always holds [8]. The components of the principal eigenvector can be obtained from the one-rank matrix (23). Since any column of this matrix gives the elements of the

principal eigenvector, let us choose the nth column, hence, let $j=n$. Suppose that for two consecutive elements, u_i and u_{i+1} of the principal eigenvector of a *specific* PCM

$$u_i < u_{i+1} \tag{27}$$

holds. Furthermore, suppose that for the corresponding two elements, $u_{in}^{S}(r)$ and $u_{i+1,n}^{S}(r)$, of the adjoint matrix (23), i.e., for those of the principal eigenvector of a *simple perturbed* PCM

$$u_{in}^{S}(r) > u_{i+1,n}^{S}(r) \tag{28}$$

holds. If this case occurs, then, the rank order of the alternatives A_i and A_{i+1} has been reversed. This phenomenon is called the *rank reversal* of the alternatives which are in question.

It is well known in the cardinal theory of decision making that an opposite order of the corresponding components of the principal eigenvector *cannot* be yielded. In contrary to this, in the sequel, we give proofs for the occurrence of such rank reversals in the AHP between the alternatives A_1 and A_2. For this purpose, it will be sufficient to compare the order of the first two components of the principal eigenvectors.

For the specific case, the maximal eigenvalue of $\mathbf{A}$ equals n. The first two components of the principal eigenvector of $\mathbf{A}$ are as follows [cf. (5)]

$$1 \quad ; \quad \frac{1}{x_1} \tag{29}$$

i.e., the components of the principal eigenvector are monotonously increasing for $x_1 < 1$, whereas they are monotonously decreasing for $x_1 > 1$. In Theorem 2, necessary and sufficient condition is given for the occurrence of a rank reversal in the specific versus the simple perturbed case.

Theorem 2 *Let $\mathbf{A}=[a_{ij}]$ be a transitive (consistent) pairwise comparison matrix of order n, n $\geq$3. Between the alternatives A_1 and A_2 when the elements a_{12} and a_{21} of are perturbed, a rank reversal occurs if and only if*

$$1 > x_1 > \frac{r-1+\dfrac{1}{\delta}}{r-1+\delta} = 1 - \frac{\delta - \dfrac{1}{\delta}}{\delta + (r-1)}, \quad for \ \delta > 1, \tag{30}$$

or

$$1 < x_1 < \frac{r-1+\dfrac{1}{\delta}}{r-1+\delta} = 1 + \frac{\dfrac{1}{\delta} - \delta}{\delta + (r-1)}, \quad for \ 0 < \delta < 1, \tag{31}$$

Proof. Using (B5) given in Appendix B, after performing the necessary algebraic manipulations the first two elements of the *n*th column of $\text{adj}(r\mathbf{I}_n - \mathbf{D}^{-1}\mathbf{A}_S\mathbf{D})$, i.e., the cofactors corresponding to the first two elements of the *n*th row $(r\mathbf{I}_n - \mathbf{D}^{-1}\mathbf{A}_S\mathbf{D})$ are obtained as [cf. (26)]

$$\left\{ \text{adj}(r\mathbf{I}_n - \mathbf{D}^{-1}\mathbf{A}_S\mathbf{D}) \right\}_{1n} = r^{n-3}[r - (1 - \delta)] \;\; ; \;\; \left\{ \text{adj}(r\mathbf{I}_n - \mathbf{D}^{-1}\mathbf{A}_S\mathbf{D}) \right\}_{2n} = r^{n-3}[r - (1 - \frac{1}{\delta})]. \quad (32)$$

Taking into account (B6) given in Appendix B, the first two components of the principal right eigenvector of the simple perturbed PCM, $\mathbf{A}_S$, are proportional to A rank reversal occurs if the elements in (33) are monotonously decreasing for A_i and $A_{i+1} < 1$, or they are monotonously increasing for $x_1 > 1$ [cf. (29)].

$$r - 1 + \delta \;\; ; \;\; \frac{1}{x_1}(r - 1 + \frac{1}{\delta}). \quad (33)$$

Depending on than whether δ is greater unity, or δ is less than unity, two cases are distinguished:

(*i*) if $\delta > 1$ and $x_1 < 1$, then the elements in (33) are monotonously *decreasing* if x_1 resides in the interval given by (30), and

(*ii*) if $0 < \delta < 1$ and $x_1 > 1$, then the elements in (33) are monotonously *increasing* if x_1 resides in the interval given by (31).

This means that the condition is *necessary*. Furthermore, since all operations in the proof can be performed in the opposite direction, the condition is *sufficient* as well.

We note that according to (21) and (22), r is dependent on the value of δ. This fact, however, has no impact on the *existence* of the intervals (30) and (31), over which a rank reversal occurs. □

Concerning the other elements of the principal eigenvector, they can be obtained by making similar considerations. As a result, for these elements we have

$$u_{in}^P = \frac{1}{x_{i-1}} r \frac{r-2}{n-2}, \quad i = 3,4,...,n. \quad (34)$$

From (34), it is obvious that rank reversal cannot occur between any pair of the alternatives $A_3, A_4,...,A_n$. The occurrence of a rank reversal between alternatives A_1 and A_i, $i = 3,4,...,n$, or between A_2 and A_i, $i = 3,4,...,n$, could be analyzed in a similar way as was shown above. This investigation, however, is left to the reader.

5 A Sample Illustration

A widely used concept of measuring the degree of consistency of a perturbed PCM in the AHP framework is to calculate the *consistency index*, *CI*. Saaty (1977) introduced the following formula

$$CI = \frac{r-n}{n-1}. \qquad (35)$$

The *consistency ratio*, *CR*, can be computed by comparing the *CI* with the corresponding random consistency index, *RI*, derived from a sample of 500, of randomly generated PCMs using the scale of [1/9,1/8,...,1,...,8,9] (see in Saaty [16]). He proposed that if this consistency ratio $CR=CI/RI$ is less than or equal to 0.10, then the results be accepted. Otherwise, the problem should be studied again and its corresponding PCM revised. He also stated that such a small error does not affect the order of magnitude of the alternatives and hence, the relative dominance would be about the same.

Given *n*, and specifying a value for *CI*, from (35), the maximal eigenvalue *r*, of a simple perturbed PCM, $\mathbf{A_S}$, given by (14) can be obtained as

$$r = n + CI(n-1), \qquad (36)$$

then, from (22), for the term *Q* we have

$$Q = \frac{n-1}{n-2} r^2 CI. \qquad (37)$$

Next, using (21), the roots of the following equation can be calculated from

$$\delta^2 - (2+Q)\delta + 1 = 0. \qquad (38)$$

Finally, using (30) and (31), the intervals for the values of x_1 over which a rank reversal occurs are

$$1 > x_1 > \frac{(n-1)(1+CI)+\frac{1}{\delta}}{(n-1)(1+CI)+\delta} = \frac{r-1+\frac{1}{\delta}}{r-1+\delta}, \quad for \ \delta > 1, \qquad (39)$$

and

$$1 < x_1 < \frac{(n-1)(1+CI)+\frac{1}{\delta}}{(n-1)(1+CI)+\delta} = \frac{r-1+\frac{1}{\delta}}{r-1+\delta}, \quad for \ 0 < \delta < 1. \qquad (40)$$

Consider a simple perturbed PCM of order *n*=3, that departs from consistency arbitrarily small. Let *CI*=0.01. Using the appropriate table in Saaty [16], the corresponding *RI*=0.58. Thus, *CR*=0.017. From (36), (37), and (38), the computed parameters are, *r*=3.02, *Q*=0.1824, δ=1.5279, 1/δ=0.6545, respectively. Using (39) and (40), the values of x_1, with any of which a rank reversal occurs lie in the interval 0.7538 to 1.3266. This result demonstrates that due to the fact that $\mathbf{A_S}$ is an inconsistent PCM even in the slightest degree, yet there exists a relatively large

interval, over which rank reversal occurs between alternatives A_1 and A_2. That is, the fundamental ordinal transitivity relation given by (4) is being violated by this phenomenon. The occurrence of such a rank reversal might be serious in practice when an undesired alternative is chosen by the decision maker as the best.

At this point the question might be raised as to whether it would be meaningful to revise a given perturbed PCM and then, to make attempts to reduce its *CR* measure. It is remarkable, that meanwhile in the literature, several other more promising ways have been proposed for improving the measure of inconsistency of a general PCM (see Salo and Hämäläinen [19], Genest and Zhang [9] and Bozóki and Rapcsák [4]). The study of these approaches is, however, left to the reader.

Conclusions

This paper presented a matrix theory based analysis for the eigenvalue-eigenvector approach of the AHP. It was shown that this approach produces a perfect solution to the decision making problem if the PCM is consistent. However, the method cannot give the true ranking of the alternatives if the PCM is inconsistent, i.e., if it is *not* transitive. Therefore, if a PCM is inconsistent, even in the slightest degree, then the principal eigenvector components do not give the true relative dominance of the alternatives. Obviously, this result can be extended to PCMs with arbitrary number of perturbed pairs of elements, since, in the practical applications of the AHP, neither the cardinal consistency, nor the ordinal consistency of the expert's judgements can be ensured a'priori.

Appendix A

In order to obtain the characteristic polynomial, $p_n^P(\lambda)$, of the perturbed PCM, $\mathbf{A}_p$, [see (8)], let us write the determinant of the modified matrix $\mathbf{K}_p(\lambda)$ given by (9), in the following form

$$\det\mathbf{K}_P(\lambda)=\det\left[(\lambda\mathbf{I}_n+\mathbf{U}_P\mathbf{V}_P^T)-\mathbf{e}\mathbf{e}^T\right]=\det(\lambda\mathbf{I}_n+\mathbf{U}_P\mathbf{V}_P^T)\det\left[\mathbf{I}_n-(\lambda\mathbf{I}_n+\mathbf{U}_P\mathbf{V}_P^T)^{-1}\mathbf{e}\mathbf{e}^T\right]. \quad (A1)$$

It is easy to show that

$$\det[\mathbf{I}_n+\mathbf{W}\mathbf{Z}^T]=\det[\mathbf{I}_m+\mathbf{Z}^T\mathbf{W}],$$

where $\mathbf{W}$ is an $(n\times m)$ matrix and $\mathbf{Z}^T$ is an $(m\times n)$ matrix. Rewriting (A1), then using (10) and (11) we get

$$\det\mathbf{K}_P(\lambda)=\det(\lambda\mathbf{I}_n+\mathbf{U}_P\mathbf{V}_P^T)\left[1-\mathbf{e}^T(\lambda\mathbf{I}_n+\mathbf{U}_P\mathbf{V}_P^T)^{-1}\mathbf{e}\right]=\det\mathbf{T}_P(\lambda)\left[1-\mathbf{e}^T\mathbf{T}_P^{-1}(\lambda)\mathbf{e}\right]. \quad (A2)$$

The inverse of a matrix modified by a low-rank matrix may be written in the following form (see in Woodbury, [23])

$$T_P^{-1}(\lambda)=(\lambda I_n+U_P V_P^T)^{-1}=\frac{1}{\lambda}I_n-\frac{1}{\lambda}U_P(\lambda I_2+V_P^T U_P)^{-1}V_P^T. \tag{A3}$$

Using (A2) and by performing the necessary operations in (A3) (see in [8, p.426]), the characteristic polynomial of the perturbed PCM is obtained in the form

$$p_n^P(\lambda)\equiv\lambda^{n-3}\left\{\lambda^3-n\lambda^2+(n-1)\sum_{i=1}^{n-1}(1-\delta_i)(1-\frac{1}{\delta_i})-\sum_{i=1}^{n-1}(1-\frac{1}{\delta_i})\sum_{i=1}^{n-1}(1-\delta_i)\right\}. \tag{A4}$$

Remark. It is easy to show that, if the number of the rows (and their corresponding columns) which contain at least one perturbed pair of elements in the specific PCM, **A** [see matrix (7)], is $m\le(n-1)/2$, then, the rank of matrix **A** increases by $2m$, i.e., the multiplicity of the zero eigenvalues becomes $n-2m-1$, and we obtain an equation of degree $2m+1$ for the nonzero eigenvalues.

Appendix B

To develop the principal eigenvector of the simple perturbed PCM, A_S, let us calculate the one-rank matrix

$$\text{adj}(rI_n-A_S)=ab^T, \tag{B1}$$

any column of which produces the principal (right) eigenvector. First, the proof of the following lemma will be given that refers to the calculation of the adjoint of a modified matrix.

Lemma *If T_P is a nonsingular matrix of order n, furthermore, **a** and **b** are column vectors of order n, then the adjoint of the modified matrix T_P-ab^T can be obtained in the form* (see in Elsner and Rózsa [7]):

$$\text{adj}[T_P-ab^T]=\text{adj}T_P\left\{(1-b^T T_P^{-1}a)I_n+ab^T T_P^{-1}\right\}. \tag{B2}$$

Proof. By the Sherman-Morrison formula [20, p.126)], the inverse of the modified nonsingular matrix T_P-ab^T exists if

$$1-b^T T_P^{-1}a\neq 0,$$

and it can be written as

$$(T_P-ab^T)^{-1}=T_P^{-1}+\frac{T_P^{-1}ab^T T_P^{-1}}{1-b^T T_P^{-1}a}. \tag{B3}$$

By (A2), the determinant of a nonsingular matrix $\mathbf{T}_P$ modified by a one-rank matrix $\mathbf{ab}^T$ is given as

$$\det[\mathbf{T}_P - \mathbf{ab}^T] = (1 - \mathbf{b}^T \mathbf{T}_P^{-1} \mathbf{a}) \det \mathbf{T}_P. \tag{B4}$$

Multiplying (B4) by the inverse (B3), the formula (B2) for the adjoint follows. □

Corollary *Since the determinant is a continuous function of its elements,* (B2) *is valid also in the case if* $1 - \mathbf{b}^T \mathbf{T}_P^{-1} \mathbf{a} = 0$, i.e.,

$$\mathrm{adj}[\mathbf{T}_P - \mathbf{ab}^T] = (\mathrm{adj}\,\mathbf{T}_P) \mathbf{ab}^T \mathbf{T}_P^{-1}, \quad \text{if} \quad 1 - \mathbf{b}^T \mathbf{T}_P^{-1} \mathbf{a} = 0. \tag{B5}$$

Let us now apply these results for the *simple perturbed* PCM, $\mathbf{A}_S$. Making use of (16), it is easy to show that

$$\mathrm{adj}[\lambda \mathbf{I}_n - \mathbf{A}_S] = \mathbf{D}\left\{\mathrm{adj}[\lambda \mathbf{I}_n - \mathbf{D}^{-1} \mathbf{A}_S \mathbf{D}]\right\} \mathbf{D}^{-1} = \mathbf{D}\left\{\mathrm{adj}[\mathbf{K}_S(\lambda)]\right\} \mathbf{D}^{-1}. \tag{B6}$$

Let us introduce the notation $\mathbf{P}_S(\lambda) = \mathrm{adj}[\mathbf{K}_S(\lambda)]$. According to (B2), by letting $\mathbf{a} = \mathbf{b} = \mathbf{e}$, and using (19) we can write that

$$\mathbf{P}_S(\lambda) \equiv \mathrm{adj}[\mathbf{K}_S(\lambda)] = \mathrm{adj}[\mathbf{T}_S(\lambda) - \mathbf{ee}^T]. \tag{B7}$$

Substituting r for λ, by (22) and (21) it is obvious that $1 - \mathbf{e}^T \mathbf{T}_S^{-1}(r) \mathbf{e} = 0$. Thus, (B5) can be applied, and for the adjoint $\mathbf{P}_S(r)$ we have

$$\mathbf{P}_S(r) = [p_{ij}^S(r)] = \mathrm{adj}[\mathbf{K}_S(r)] = \left\{\mathrm{adj}\,\mathbf{T}_S(r)\right\} \mathbf{ee}^T \mathbf{T}_S^{-1}(r). \tag{B8}$$

Consequently, $\mathbf{P}_S(r)$ is a rank-one matrix, and therefore, any (column) vector of $\mathrm{adj}[\mathbf{T}_S(r)]$ is the principal eigenvector corresponding to the maximal eigenvalue r of the simple perturbed PCM $\mathbf{A}_S$. Hence, making use of (18), (B6), and (B8) the eigenvectors $u_{ij}^S(r)$, given by formulas (24), (25) and (26), can be obtained from

$$\mathrm{adj}(r\mathbf{I}_n - \mathbf{A}_S) = \mathbf{D}\left\{ r^{n-3} \begin{bmatrix} r^2 + (\delta - 1)r \\ r^2 - \left(1 - \dfrac{1}{\delta}\right)r \\ r^2 + Q \\ . \\ . \\ . \\ r^2 + Q \end{bmatrix} \begin{bmatrix} \dfrac{r - \left(1 - \dfrac{1}{\delta}\right)}{r^2 + Q}, & \dfrac{r + \delta - 1}{r^2 + Q}, & \dfrac{1}{r}, & . & . & ., & \dfrac{1}{r} \end{bmatrix} \right\} \mathbf{D}^{-1}, \tag{B9}$$

as the kth column of $\mathbf{P}_S(r)$ is premultiplied by $\mathbf{D}$ and is multiplied by x_{k-1}, $k = 1, 2, ..., n$. In (B9), Q is given by (21).

References

[1]   Barzilai,J and B.Golany, "AHP Rank Reversal, Normalization and Aggregation Rules", *INFOR*, **32**, (1994), 57-64.

[2]   Belton,V. and T.Gear, "On a Short-coming on Saaty's Method of Analytic Hierarchies", *Omega*, **11**, (1983), 228-230.

[3]   Belton,V. and T.Gear, "The Legitimacy of Rank Reversal - A Comment", *Omega*, **13**, (1985), 143-144.

[4]   Bozóki,S. and T.Rapcsák, "On Saaty's and Koczkodaj's inconsistencies of pairwise comparison matrices," *Journal of Global Optimization*, (2007), [in press].

[5]   Dyer,J.S. "Remark on the Analytic Hierarchy Process", *Management Sci.*, **36**, (1990), 249-258.

[6]   Dyer,J.S., and R.E. Wendell, "A Critique of the Analytic Hierarchy Process", Working Paper. 84/85-424. Department of Management, The University of Texas at Austin, 1985.

[7]   Elsner,L. and P.Rózsa, "On Eigenvectors and Adjoints of Modified Matrices," *Linear and Multilinear Algebra*, **10**, (1981), 235-247.

[8]   Farkas,A., Rózsa,P. and E.Stubnya, "Transitive Matrices and Their Applications", *Linear Algebra and its Applications*. **302-303**, (1999), 423-433.

[9]   Genest,C. and S.S.Zhang, "A Graphical Analysis of Ratio-scaled Pairwise Comparison Data", *Management Sci.*, **42**, (1996), 335-349.

[10] Harker,P.T. and L.G.Vargas, "The Theory of Ratio Scale Estimation: Saaty's Analytic Hierarchy Process", *Management Sci.*, **33**, (1987), 1383-1403.

[11] Harker,P.T. and L.G.Vargas, "Reply to 'Remarks on the Analytic Hierarchy Process'", *Management Sci.*, **36**, (1990), 269-273.

[12] Horn,R.A. and C.R.Johnson, Matrix Analysis. Cambridge University Press, Cambridge, 1985.

[13] Pérez,J., "Some Comments on Saaty's AHP", *Management Sci.*, **41**, (1995), 1091-1095.

[14] Saaty,T.L., "A Scaling Method for Priorities in Hierarchical Structures", *Journal of Math. Psychology*, **15**, (1977), 234-281.

[15] Saaty,T.L., "Axiomatic Foundation of the Analytic Hierarchy Process", *Management Sci.*, **32**, (1986), 841-855.

[16] Saaty,T.L.,"The Analytic Hierarchy Process-What It Is and How It Is Used.", *Math. Modelling*, **9**, (1987), 161-176.

[17] Saaty,T.L., "An Exposition of the AHP in Reply to the Paper 'Remarks on the Analytic Hierarchy Process'", *Management Sci.*, **36**, (1990), 259-268.

[18] Saaty,T.L. and L.G.Vargas, "The Legitimacy of Rank Reversal", *Omega*, **12**, (1984), 514-516.

[19] Salo,A.A. and R.P.Hämäläinen, "On the Measurement of Preferences in the Analytic Hierarchy Process", Research Report. A47, Helsinki University of Technology, Systems Analysis Laboratory, Espoo, Finland, 1992.

[20] Sherman,J. and W.J.Morrison,"Adjustment of an Inverse Matrix Corresponding to Changes in a given Column or a given Row of the Original Matrix", *Ann. Math. Stats.*, **21**, (1949), 124-127.

[21] Vargas,L.G., "A Rejoinder", *Omega*, **13**, (1985), p.249.

[22] Watson,S.R. and A.N.S.Freeling, "Comment on: Assessing Attribute Weights by Ratios", *Omega*, **11**, (1983), p.13.

[23] Woodbury,M.A.,"Inverting Modified Matrices", Memorandum Report. 42, Statistical Research Group. Institute for Advanced Study, Princeton, New Jersey, June 14, 1950 .