Some Categorial Aspects of the Dorroh Extensions

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Abstract: Given two associative rings R and D, we say that D is a Dorroh extension of the ring R, if R is a subring of D and D = R ⊕ M for some ideal M ⊆ D. In this paper, we present some categorial aspects of the Dorroh extensions and we describe the group of units of this ring.

Keywords: bimodule; category; functor; adjoint functors; exact sequence of groups; (group) semidirect product

1 Introduction

If *R* is a commutative ring and *M* is an *R*-module then the direct sum $R \oplus M$ (with *R* and *M* regarded as abelian groups), with the product defined by $(a,x)\cdot(b,y)=(ab,bx+ay)$ is a commutative ring. This ring is called the idealization of *R* by *M* (or the trivial extension of *M*) and is denoted by $R \ltimes M$. While we do not know who first constructed an example using idealization, the idea of using idealization to extend results concerning ideals to modules is due to Nagata [12]. Nagata in the famous book, Local rings [12], presented a principle, called the principle of idealization. By this principle, modules become ideals.

We note that this ring can be introduced more generally, namely for a ring R and an (*R*, *R*) - bimodule M, considering the product $(a,x)\cdot(b,y)=(ab,xb+ay)$.

The purpose of idealization is to embed *M* into a commutative ring *A* so that the structure of *M* as *R*-module is essentially the same as an *A*-module, that is, as on ideal of *A* (called ringification). There are two main ways to do this: the idealization $R \ltimes M$ and the symmetric algebra $S_R(M)$ (see e.g. [1]). Both constructions give functors from the category of *R*-modules to the category of R-algebras.

Another construction which provides a number of interesting examples and counterexamples in algebra is the triangular ring. If R and S are two rings and M is an $(R,S)-$ bimodule, the set of (formal) matrices

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & x \\ 0 & s \end{pmatrix} : r \in R,\, s \in S,\, x \in M \right\}$$

with the component-wise addition and the (formal) matrix multiplication,

$$\begin{pmatrix} r & x \\ 0 & s \end{pmatrix} \cdot \begin{pmatrix} r' & x' \\ 0 & s' \end{pmatrix} = \begin{pmatrix} rr' & rx' + xs' \\ 0 & ss' \end{pmatrix}$$

becomes a ring, called triangular ring (see [10]). If R and S are unitary, then $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ has the unit $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If we identify R, S and M as subgroups of $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, we can regard $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ as the (abelian groups) direct sum, $R \oplus M \oplus S$. Also, R and S are left, respectively right ideals, and M, $R \oplus M$, $M \oplus S$ are two sided ideals of the ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, with $M^2 = 0$, $(R \oplus M \oplus S)/(R \oplus M) \cong S$ and $(R \oplus M \oplus S)/(M \oplus S) \cong R$. Finally, $R \oplus S$ is a subring of $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$.

If R and S are two rings and M is an (R,S)-bimodule, then M is a $(R \times S, R \times S)$-bimodule under the scalar multiplications defined by $(r,s)x = rx$ and $x(r,s) = xs$. The triangular ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is isomorphic with the trivial extension $(R \times S) \ltimes M$ and conversely, if R is a ring and M is an (R,R)-bimodule, then the trivial extension $R \ltimes M$ is isomorphic with the subring $\left\{ \begin{pmatrix} a & x \\ 0 & a \end{pmatrix} : a \in R,\, x \in M \right\}$ of the triangular ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$.

Thus, the above construction can be considered the third realization of the idealization.

The idealization construction can be generalized to what is called a semi-trivial extension. Let R be a ring and M a (R,R)-bimodule. Assume that $\varphi = [-,-] : M \otimes_A M \to R$ is an (R,R)-bilinear map such that $[x,y]z$

$= x[y,z]$ for any $x,y,z \in M$. Then we can define a multiplication on the abelian group $R \oplus M$ by $(a,x)\cdot(b,y) = (ab+[x,y], xb+ay)$ which makes $R \oplus M$ a ring called the semi-trivial extension of *R* by *M* and φ, and denoted by $R \ltimes_{\varphi} M$.

M. D'Anna and M. Fontana in [2] and [3] introduced another general construction, called the amalgamated duplication of a ring *R* along an *R*-module *M* and denoted by $R \bowtie M$. If *R* is a commutative ring with identity, $T(R)$ is the total ring of fractions and *M* an *R*-submodule of $T(R)$ such that $M \cdot M \subseteq M$, then $R \bowtie M$ is the subring $\{(a, a+x) : a \in R,\ x \in M\}$ of the ring $R \times T(R)$ (endowed with the usual componentwise operations).

More generally, given two rings *R* and *M* such that M is an (R,R)-bimodule for which the actions of *R* are compatible with the multiplication in *M*, i.e.

$$(ax)y = a(xy),\ (xy)a = x(ya),\ (xa)y = x(ay)$$

for every $a \in R$ and $x,y \in M$, we can define the multiplication

$$(a,x)\cdot(b,y) = (ab, xb+ay+xy)$$

to obtain a ring structure on the direct sum $R \oplus M$. This ring is called the Dorroh extension (it is also called an ideal extension) of *R* by *M*, and we will denote it by $R \bowtie M$. If the ring *R* has the unit 1, the ring $R \bowtie M$ has the unit $(1,0)$. Dorroh [5] first used this construction, with $R = \mathbb{Z}$, (the ring of integers), as a means of embedding a (nonunital) ring *M* without identity into a ring with identity.

In this paper, in Section 3, we give the universal property of the Dorroh-extensions that allows to construct the covariant functor $\mathbf{D} : \mathfrak{D} \to \mathfrak{Rng}$, where $\mathfrak{D}$ is the category of the Dorroh-pairs and the Dorroh-pair homomorphisms. We prove that the functor **D** has a right adjoint and this functor commute with the direct products and inverse limits. Also we establish a correspondence between the Dorroh extensions and some semigroup graded rings.

L. Salce in [13] proves that the group of units of the amalgamated duplication of the ring *R* along the *R*-module *M* is isomorphic with the direct product of the groups $\mathbf{U}(R)$ and M°. In Section 4 we prove that in the case of the Dorroh extensions, the group of units $\mathbf{U}(R \bowtie M)$ is isomorphic with the semidirect product of the groups $\mathbf{U}(R)$ and M°.

2 Some Basic Concepts

Recall that if S is semigroup, the ring R is called S-**graded** if there is a family $\{R_s : s \in S\}$ of additive subgroups of R such that $R = \bigoplus_{s\in S} R_s$ and $R_s R_t \subseteq R_{st}$ for all $s,t \in S$. For a subset $T \subseteq S$ consider $R_T = \bigoplus_{t\in T} R_t$. If T is a subsemigroup of S then R_T is a subring of R. If T is a left (right, two-sided) ideal of R then R_T is a left (right, two-sided) ideal of R.

The semidirect product of two groups is also a well-known construction in group theory.

Definition. Given the groups H and N, a group homomorphism $\varphi : H \to \operatorname{Aut} K$, if we define on the Cartesian product, the multiplication

$$(h_1,k_1)(h_2,k_2) = (h_1h_2, k_1\cdot\varphi(h_1)(k_2)) ,$$

we obtain a group, called the semidirect product of the groups H and N with respect to φ . This group is denoted by $H \times_\varphi N$.

Theorem. Let G be a group. If G contain a subgroup H and a normal subgroup N such that $H \cap K = \{1\}$ and $G = K \cdot H$, then the correspondence $(h,k) \mapsto kh$ establishes an isomorphism between the semidirect product $H \times_\varphi N$ of the groups H and N with respect to $\varphi : H \to \operatorname{Aut} K$, defined by $\varphi(h)(k) = hkh^{-1}$ and the group G.

Definition. A short exact sequence of groups is a sequence of groups and group homomorphisms

$$1 \longrightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1$$

where α is injective, β is surjective and $\operatorname{Im}\alpha = \ker\beta$. We say that the above sequence is split if there exists a group homomorphism $s : H \to G$ such that $\beta \circ s = \operatorname{id}_H$.

Theorem. Let G, H, and N be groups. Then G is isomorphic to a semidirect product of H and N if and only if there exists a split exact sequence

$$1 \longrightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1$$

3 The Dorroh Extension

To simplify the presentation, we give the following definition:

Definition 1. *A pair* (R,M) *of* $(associative)$ *rings, is called a Dorroh-pair if* M *is also an* (R,R)*-bimodule and for all* $a\in R$ *and* $x,y\in M$*, are satisfied the following compatibility conditions:*

$(ax)y=a(xy),\ (xy)a=x(ya),\ (xa)y=x(ay).$

We denote further with $\mathcal{D}$, the class of all Dorroh-pairs.

If $(R,M)\in\mathcal{D}$, on the module direct sum $R\oplus M$ we introduce the multiplication

$(a,x)\cdot(b,y)=(ab,xb+ay+xy).$

$(R\oplus M,+,\cdot)$ is a ring and it is denoted by $R\rhd\!\!\lhd M$ and it is called the Dorroh extension (or ideal extension (see [8], [11])). Moreover, $R\rhd\!\!\lhd M$ is a (R,R)-bimodule under the scalar multiplications defined by

$\alpha(a,x)=(\alpha a,\alpha x),\qquad (a,x)\alpha=(a\alpha,x\alpha)$

and $(R,R\rhd\!\!\lhd M)$ is also a Dorroh-pair.

If R has the unit 1, then $(1,0)$ is a unit of the ring $R\rhd\!\!\lhd M$. Dorroh first used this construction (see [5]), with $R=\mathbb{Z}$, as a means of embedding a ring without identity into a ring with identity.

Remark 2. *If* M *is a zero ring, the Dorroh extension* $R\rhd\!\!\lhd M$ *coincides with the trivial extension* $R\ltimes M$.

Example 3. *If* R *is a ring, then* (R,M) *is a Dorroh-pair for every ideal M of the ring R. Another example of a Dorroh-pair is* $(R,\mathcal{M}_{n\times n}(R))$.

Since the applications

$i_R: R\to R\rhd\!\!\lhd M,\quad a\mapsto(a,0)$

$i_M: M\to R\rhd\!\!\lhd M,\quad x\mapsto(0,x)$

are injective and both rings homomorphisms and (R,R) linear maps, we can identify further the element $a\in R$ with $(a,0)\in R\rhd\!\!\lhd M$ and $x\in M$ with $(0,x)\in R\rhd\!\!\lhd M$. Also, the application

$\pi_R: R\rhd\!\!\lhd M\to R,\quad (a,x)\mapsto a$

is a surjective ring homomorphism which is also (R,R) linear. Consequently, R is a subring of $R \bowtie M$, M is an ideal of the ring $R \bowtie M$, and the factor ring $(R \bowtie M)/M$ is isomorphic with R.

Remark 4. Given two associative rings R and D, we can say that D is a Dorroh extension of the ring R, if R is a subring of D and $D = R \oplus M$ for some ideal $M \subseteq D$.

If $(A,R),(A,M),(R,M) \in \mathcal{D}$, then M is an $(A \bowtie R, A \bowtie R)$-bimodule with the scalar multiplication

$$(\alpha,a)x = \alpha x + ax \quad \text{and} \quad x(\alpha,a) = x\alpha + xa,$$

respectively, $R \bowtie M$ is an (A,A)-bimodule with the scalar multiplication

$$\alpha(a,x) = (\alpha a, \alpha x) \quad \text{and} \quad (a,x)\alpha = (a\alpha, x\alpha).$$

Obviously, $(A \bowtie R, M),(A, R \bowtie M) \in \mathcal{D}$ and since,

$$((\alpha,a),x) + ((\beta,b),y) = ((\alpha+\beta, a+b), x+y),$$

$$((\alpha,a),x)\cdot((\beta,b),y) = ((\alpha\beta, \alpha b + a\beta + ab), \alpha y + ay + x\beta + xb + xy),$$

respectively,

$$(\alpha,(a,x))\cdot(\beta,(b,y)) = (\alpha+\beta,(a+b,x+y)),$$

$$(\alpha,(a,x))\cdot(\beta,(b,y)) = (\alpha\beta,(\alpha b + a\beta + ab, \alpha y + ay + x\beta + xb + xy)),$$

the rings $(A \bowtie R) \bowtie M$ and $A \bowtie (R \bowtie M)$ are isomorphic, and the isomorphism of these rings is given by the correspondence $((\alpha,a),x) \mapsto (\alpha,(a,x))$. Due to this isomorphism, further we can write simply $A \bowtie R \bowtie M$.

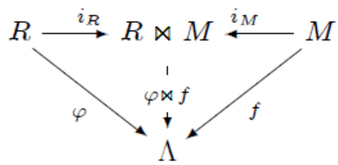
Example 5. *If $R_1,\ldots,R_n$ are rings such that (R_i, R_j) are Dorroh-pairs whenever $i \le j$, we can consider the ring $R = R_1 \bowtie R_2 \bowtie \ldots \bowtie R_n$. Since for any $i,j \in I_n$, $R_i R_j \subseteq R_{\max(i,j)}$, we can consider the ring R as a I_n-graded ring, where I_n is the monoid $\{1,\ldots,n\}$ with the operation defined by $i \vee j = \max(i,j)$. Conversely, if a ring R is I_n-graded and $R = \bigoplus_{i \in I_n} R_i$, since $R_i R_j \subseteq R_{i \vee j}$ for all $i,j \in I_n$, the subgroups $R_1,\ldots,R_n$ are subrings of R and R_j is a (R_i,R_i)-bimodule whenever $i \le j$, the rings R and $R_1 \bowtie R_2 \bowtie \ldots \bowtie R_n$ are isomorphic.*

Definition 6. *By a homomorphism between the Dorroh-pairs* (R,M) *and* $(R^{'},M^{'})$ *we mean a pair* (φ, f), *where* $\varphi : R \rightarrow R^{'}$ *and* $f : M \rightarrow M^{'}$ *are ring homomorphisms for which, for all* $\alpha \in R$ *and* $x \in M$ *we have that*

$f(\alpha \cdot x) = \varphi(\alpha) \cdot f(x)$ and $f(x \cdot \alpha) = f(x) \cdot \varphi(\alpha)$.

The Dorroh extension verifies the following universal property:

Theorem 7. *If* (R,M) *is a Dorroh-pair, then for any ring* Λ *and any Dorroh-pairs homomorphism* $(\varphi, f) : (R,M) \rightarrow (\Lambda,\Lambda)$, *there exists a unique ring homomorphism* $\varphi \bowtie f : R \bowtie M \rightarrow \Lambda$ *such that*



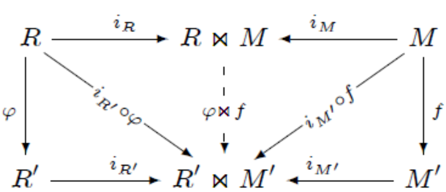
$(\varphi \bowtie f) \circ i_M = f$ and $(\varphi \bowtie f) \circ i_R = \varphi$.

Proof. It is routine to verify that the application $\varphi \bowtie f$, defined by

$(\varphi \bowtie f)(a,x) = \varphi(a) + f(x)$

is the required ring homomorphism.

Corollary 8. *If* (R,M) *and* $(R^{'},M^{'})$ *are two Dorroh-pairs, and* $(\varphi, f) : (R,M) \rightarrow (R^{'},M^{'})$ *is a Dorroh-pairs homomorphism, then there exists a unique ring homomorphism* $\varphi \bowtie f : R \bowtie M \rightarrow R' \bowtie M'$ *such that*



$(\varphi \bowtie f) \circ i_R = i_{R^{'}} \circ \varphi$ and $(\varphi \bowtie f) \circ i_M = i_{M^{'}} \circ f$.

Proof. Apply Theorem 7, considering $\Lambda = R' \bowtie M'$ and the homomorphisms pair $(i_{R'} \circ \varphi, i_{M'} \circ f)$.

Consider now the category $\mathfrak{D}$ whose objects are the class $\mathcal{D}$ of the Dorroh-pairs and the homomorphisms between two objects are the Dorroh-pairs homomorphisms and the category $\mathfrak{Rng}$ of the associative rings.

By Corollary 8, we can consider the covariant functor $\mathbf{D}:\mathfrak{D}\to\mathfrak{Rng}$, defined as follows: if (R,M) is a Dorroh-pair, then $\mathbf{D}(R,M)=R\bowtie M$, and if $(\varphi,f):(R,M)\to(R',M')$ is a Dorroh-pair homomorphism, then $\mathbf{D}(\varphi,f)$ $=\varphi\bowtie f$.

Consider also the functor $\mathbf{B}:\mathfrak{Rng}\to\mathfrak{D}$, defined as follows: if A is a ring, then $\mathbf{B}(A)=(A,A)$ and if $h:A\to B$ is a ring homomorphism, $\mathbf{B}(h)=(h,h)$.

Theorem 9. *The functor* $\mathbf{D}$ *is left adjoint of* **B.**

Proof. If $(R,M)\in Ob\mathfrak{D}$ and $\Lambda\in Ob\,\mathfrak{Rng}$, define the function

$$\phi_{(R,M),\Lambda}:Hom_{\mathfrak{Rng}}(R\bowtie M,\Lambda)\to Hom_{\mathfrak{D}}((R,M),(\Lambda,\Lambda))$$

by $\Phi\mapsto\left(\Phi|_R,\Phi|_M\right)$, which is evidently a bijection.

Since, for any Dorroh-pairs homomorphism $(\varphi,f):(R,M)\to(R',M')$ and for any ring homomorphisms $\beta:\Lambda\to\Lambda'$ and $\Psi:R'\bowtie M'\to\Lambda$ we have that

$$\begin{aligned}(\beta,\beta)\circ(\Psi|_{R'},\Psi|_{M'})\circ(\varphi,f) &= \left((\beta\circ\Psi|_{R'}\circ\varphi),(\beta\circ\Psi|_{M'}\circ f)\right)\\ &= \left((\beta\circ\Psi\circ i_{R'}\circ\varphi),(\beta\circ\Psi\circ i_{M'}\circ f)\right)\\ &= \left((\beta\circ\Psi\circ(\varphi\bowtie f)\circ i_R),(\beta\circ\Psi\circ(\varphi\bowtie f)\circ i_M)\right)\\ &=\left((\beta\circ\Psi\circ(\varphi\bowtie f))|_R,\ (\beta\circ\Psi\circ(\varphi\bowtie f))|_M\right)\end{aligned}$$

the diagram

$$\begin{array}{ccc}\mathrm{Hom}_{\mathfrak{Rng}}(R'\bowtie M',\Lambda) & \xrightarrow{\phi_{(R',M'),\Lambda}} & \mathrm{Hom}_{\mathfrak{D}}((R',M'),(\Lambda,\Lambda))\\ \Big\downarrow{\scriptstyle \mathrm{Hom}_{\mathfrak{Rng}}(\varphi\bowtie f,\beta)} & & \Big\downarrow{\scriptstyle \mathrm{Hom}_{\mathfrak{D}}((\varphi,f),(\beta,\beta))}\\ \mathrm{Hom}_{\mathfrak{Rng}}(R\bowtie M,\Lambda') & \xrightarrow{\phi_{(R,M),\Lambda'}} & \mathrm{Hom}_{\mathfrak{D}}((R,M),(\Lambda',\Lambda'))\end{array}$$

is commutative and the result follow.

Proposition 10. *Consider* $\{(R_i, M_i) : i \in I\}$ *a family of Dorroh-pairs and the ring direct products* $\prod_{i\in I} R_i$ *and* $\prod_{i\in I} M_i$ *(with the canonical projections* p_i *and* π_i*, respectively, the canonical embeddings* q_i *and* σ_i*)*

Then $\left(\prod_{i\in I} R_i, \prod_{i\in I} M_i\right)$ *is also a Dorroh-pair, for all* $i \in I$*,* (p_i, π_i) *and* (q_i, σ_i) *are Dorroh-pairs homomorphisms and*

$$\left(\prod_{i\in I} R_i\right) \bowtie \left(\prod_{i\in I} M_i\right) \cong \prod_{i\in I} (R_i \bowtie M_i).$$

Proof. Since for all $i \in I$, (R_i, M_i) are Dorroh-pairs, $\prod_{i\in I} M_i$ is a $\left(\prod_{i\in I} R_i, \prod_{i\in I} R_i\right)$-bimodule with the componentwise scalar multiplications and evidently, the compatibility conditions are satisfied. Thus $\left(\prod_{i\in I} R_i, \prod_{i\in I} M_i\right)$ is a Dorroh-pair.

If $a = (a_i)_{i\in I} \in \prod_{i\in I} R_i$ and $x = (x_i)_{i\in I} \in \prod_{i\in I} M_i$, then for all $j \in I$,

$$\pi_j(a \cdot x) = a_j \cdot x_j = p_j(a) \cdot \pi_j(a) \text{ and } \pi_j(x \cdot a) = x_j \cdot a_j = \pi_j(a) \cdot p_j(a)$$

respectively, if $i \in I$, $a_i \in R_i$ and $x_i \in M_i$, then

$$\sigma_i(a_i \cdot x_i) = q_i(a) \cdot \sigma_i(a) \text{ and } \sigma_i(x_i \cdot a_i) = \sigma_i(a) \cdot q_i(a)$$

and so (p_i, π_i) and (q_i, σ_i) are Dorroh-pairs homomorphisms.

Proposition 11. *Let I be a directed set and* $\left\{(R_i, M_i)_{i\in I}; (\varphi_{ij}, f_{ij})_{i,j\in I}\right\}$ *an inverse system of Dorroh-pairs. Then* $\left\{(R_i \bowtie M_i)_{i\in I}, (\varphi_{ij} \bowtie f_{ij})_{i,j\in I}\right\}$ *is an inverse system of rings and*

$$\lim_{\leftarrow}(R_i \bowtie M_i) \cong \left(\lim_{\leftarrow} R_i\right) \bowtie \left(\lim_{\leftarrow} M_i\right).$$

Proof. Consider the elements $i, j \in I$ such that $i \le j$. By Corolary 8, the Dorroh-pairs homomorphism $(\varphi_{ij}, f_{ij}) : (R_j, M_j) \to (R_i, M_i)$ can be extended to the ring homomorphism $\varphi_{ij} \bowtie f_{ij} : R_j \bowtie M_j \to R_i \bowtie M_i$ which is defined by

$$(\varphi_{ij} \bowtie f_{ij})(a_j, x_j) = (\varphi_{ij}(a_j), f_{ij}(x_j)), \text{ for all } (a_j, x_j) \in R_j \bowtie M_j.$$

Obviously, $\left\{\left(R_i \bowtie M_i\right)_{i\in I}, \left(\varphi_{ij} \bowtie f_{ij}\right)_{i,j\in I}\right\}$ is an inverse system of rings.

Consider now $s,t \in I$ such that $s \le t$ and $\left(a_i, x_i\right)_{i\in I} \in \varprojlim\left(R_i \bowtie M_i\right)$. Since

$$\left(a_s, x_s\right) = \left(\varphi_{st} \bowtie f_{st}\right)\left(a_t, x_t\right) = \left(\varphi_{st}\left(a_t\right), f_{st}\left(x_t\right)\right)$$

we obtain that $\left(a_i\right)_{i\in I} \in \varprojlim R_i$, $\left(x_i\right)_{i\in I} \in \varprojlim M_i$ and the correspondence

$$\left(a_i, x_i\right)_{i\in I} \mapsto \left(\left(a_i\right)_{i\in I}, \left(x_i\right)_{i\in I}\right)$$

establishes an isomorphism between $\varprojlim\left(R_i \bowtie M_i\right)$ and $\left(\varprojlim R_i\right) \bowtie \left(\varprojlim M_i\right)$.

4 The Group of Units of the Ring R⋈M

If A is a ring with identity, denote by $\mathbf{U}(A)$ the group of units of this ring.

Let (R,M) a Dorroh-pair where R is a ring with identity and consider the Dorroh extension $R \bowtie M$. In this section we will describe the group of units of the ring $R \bowtie M$. Firstly, observe that if $(a,x) \in \mathbf{U}(R \bowtie M)$, then $a \in \mathbf{U}(R)$.

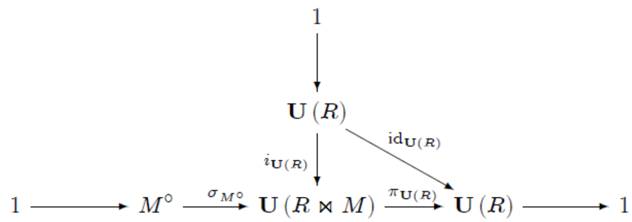
The set of all elements of M forms a monoid under the circle composition on M, $x \circ y = x + y + xy$, 0 being the neutral element. The group of units of this monoid we will denoted by M°.

Theorem 12. *The group of units* $\mathbf{U}(R \bowtie M)$ *of the Dorroh extension* $R \bowtie M$ *is isomorphic with a semidirect product of the groups* $\mathbf{U}(R)$ *and* M°.

Proof. Consider the function

$$\sigma_{M^\circ} : M^\circ \to \mathbf{U}(R \bowtie M), \quad x \mapsto (1, x),$$

which is an injective group homomorphism. Consider also the group homomorphisms $i_{\mathbf{U}(R)} : \mathbf{U}(R) \to \mathbf{U}(R \bowtie M)$ and $\pi_{\mathbf{U}(R)} : \mathbf{U}(R \bowtie M) \to \mathbf{U}(R)$ induced by the ring homomorphisms $i_R : R \to R \bowtie M$ and $\pi_R : R \bowtie M \to R$, respectively. Since the following sequences



are exacts and $\pi_{\mathbf{U}(R)} \circ i_{\mathbf{U}(R)} = id_{\mathbf{U}(R)}$, the group of units $\mathbf{U}(R \bowtie M)$ of the Dorroh extension $R \bowtie M$ is isomorphic with the semidirect product of the groups $\mathbf{U}(R)$ and M°, $\mathbf{U}(R) \times_{\delta} M^{\circ}$. The homomorphism $\delta : \mathbf{U}(R) \to \mathrm{Aut}\, M^{\circ}$, is defined by $a \mapsto \delta_a$ where $\delta_a : M^{\circ} \to M^{\circ}$, $x \mapsto axa^{-1}$ and the multiplication of the semidirect product $\mathbf{U}(R) \times_{\delta} M^{\circ}$, is defined by

$$(a,x) \cdot (b,y) = \left(ab, x \circ \left(aya^{-1}\right)\right) = \left(ab, x + aya^{-1} + xaya^{-1}\right).$$

The isomorphism between the groups $\mathbf{U}(R) \times_{\delta} M^{\circ}$ and $\mathbf{U}(R \bowtie M)$ is given by $(a,x) \mapsto (a, xa)$.

Remark 13. *If M is a ring with identity, the correspondence $x \mapsto x - 1$ establishes an isomorphism between the groups $\mathbf{U}(M)$ and M°, and therefore the group $\mathbf{U}(R \bowtie M)$ is isomorphic with a semidirect product of the groups $\mathbf{U}(R)$ and $\mathbf{U}(M)$.*

Corollary 14. *The group of units $\mathbf{U}(R \ltimes M)$ of the trivial extension $R \ltimes M$ is isomorphic with a semidirect product of the group $\mathbf{U}(R)$ with the additive group of the ring M.*

Conclusions

The Dorroh extension is a useful construction in abstract algebra being an interesting source of examples in the ring theory.

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