Some Categorial Aspects of the Dorroh Extensions

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Abstract: Given two associative rings $R$ and $D$, we say that $D$ is a Dorroh extension of the ring $R$, if $R$ is a subring of $D$ and $D = R \oplus M$ for some ideal $M \subseteq D$. In this paper, we present some categorial aspects of the Dorroh extensions and we describe the group of units of this ring.

Keywords: bimodule; category; functor; adjoint functors; exact sequence of groups; (group) semidirect product

1 Introduction

If $R$ is a commutative ring and $M$ is an $R$-module then the direct sum $R \oplus M$ (with $R$ and $M$ regarded as abelian groups), with the product defined by $(a, x) \cdot (b, y) = (ab, bx + ay)$ is a commutative ring. This ring is called the idealization of $R$ by $M$ (or the trivial extension of $M$) and is denoted by $R \times M$.

While we do not know who first constructed an example using idealization, the idea of using idealization to extend results concerning ideals to modules is due to Nagata [12]. Nagata in the famous book, Local rings [12], presented a principle, called the principle of idealization. By this principle, modules become ideals.

We note that this ring can be introduced more generally, namely for a ring $R$ and an $(R, R)$-bimodule $M$, considering the product $(a, x) \cdot (b, y) = (ab, xb + ay)$.

The purpose of idealization is to embed $M$ into a commutative ring $A$ so that the structure of $M$ as $R$-module is essentially the same as an $A$-module, that is, as on ideal of $A$ (called ringification). There are two main ways to do this: the idealization $R \times M$ and the symmetric algebra $S_R(M)$ (see e.g. [1]). Both constructions give functors from the category of $R$-modules to the category of $R$-algebras.
Another construction which provides a number of interesting examples and counterexamples in algebra is the triangular ring. If \( R \) and \( S \) are two rings and \( M \) is an \((R,S)\)-bimodule, the set of (formal) matrices

\[
\begin{pmatrix}
R & M \\
0 & S
\end{pmatrix} = \left\{ \begin{pmatrix} r & x \\ 0 & s \end{pmatrix} : r \in R, s \in S, x \in M \right\}
\]

with the component-wise addition and the (formal) matrix multiplication,

\[
\begin{pmatrix} r & x \\ 0 & s \end{pmatrix} \cdot \begin{pmatrix} r' & x' \\ 0 & s' \end{pmatrix} = \begin{pmatrix} rr' & rx' + xs' \\ 0 & ss' \end{pmatrix}
\]

becomes a ring, called triangular ring (see [10]). If \( R \) and \( S \) are unitary, then \( \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \) has the unit \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). If we identify \( R, S \) and \( M \) as subgroups of \( \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \), we can regard \( \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \) as the (abelian groups) direct sum, \( R \oplus M \oplus S \). Also, \( R \) and \( S \) are left, respectively right ideals, and \( M, \ R \oplus M, \ M \oplus S \) are two sided ideals of the ring \( \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \), with \( M^2 = 0 \),

\[
(R \oplus M \oplus S)/(R \oplus M) \cong S \text{ and } (R \oplus M \oplus S)/(M \oplus S) \cong R.
\]

Finally, \( R \oplus S \) is a subring of \( \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \).

If \( R \) and \( S \) are two rings and \( M \) is an \((R,S)\)-bimodule, then \( M \) is a \((R \times S, R \times S)\)-bimodule under the scalar multiplications defined by \( (r,s)x = rx \) and \( x(r,s) = xs \). The triangular ring \( \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \) is isomorphic with the trivial extension \((R \times S) \times M\) and conversely, if \( R \) is a ring and \( M \) is an \((R,R)\)-bimodule, then the trivial extension \( R \times M \) is isomorphic with the subring \( \left\{ \begin{pmatrix} a & x \\ 0 & a \end{pmatrix} : a \in R, x \in M \right\} \) of the triangular ring \( \begin{pmatrix} R & M \\ 0 & R \end{pmatrix} \).

Thus, the above construction can be considered the third realization of the idealization.

The idealization construction can be generalized to what is called a semi-trivial extension. Let \( R \) be a ring and \( M \) a \((R,R)\)-bimodule. Assume that \( \varphi = [\cdot, -] : M \otimes_R M \rightarrow R \) is an \((R,R)\)-bilinear map such that \([x,y]z\)
= x[y, z] for any \( x, y, z \in M \). Then we can define a multiplication on the abelian group \( R \oplus M \) by \((a, x) \cdot (b, y) = (ab + [x, y], xb + ay)\) which makes \( R \oplus M \) a ring called the semi-trivial extension of \( R \) by \( M \) and \( \varphi \), and denoted by \( R \ltimes_{\varphi} M \).

M. D’Anna and M. Fontana in [2] and [3] introduced another general construction, called the amalgamated duplication of a ring \( R \) along an \( R \)-module \( M \) and denoted by \( R \triangleright:\triangleleft M \). If \( R \) is a commutative ring with identity, \( T(R) \) is the total ring of fractions and \( M \) an \( R \)-submodule of \( T(R) \) such that \( M \cdot M \subseteq M \), then \( R \triangleright:\triangleleft M \) is the subring \( \{(a, a + x) : a \in R, x \in M\} \) of the ring \( R \times T(R) \) (endowed with the usual componentwise operations).

More generally, given two rings \( R \) and \( M \) such that \( M \) is an \((R, R)\)-bimodule for which the actions of \( R \) are compatible with the multiplication in \( M \), i.e.

\[(ax)y = a(xy), \quad (xy)a = x(ya), \quad (xa)y = x(ay)\]

for every \( a \in R \) and \( x, y \in M \), we can define the multiplication

\[(a, x) \cdot (b, y) = (ab, xb + ay + xy)\]

to obtain a ring structure on the direct sum \( R \oplus M \). This ring is called the Dorroh extension (it is also called an ideal extension) of \( R \) by \( M \), and we will denote it by \( R \triangleright:\triangleleft M \). If the ring \( R \) has the unit 1, the ring \( R \triangleright:\triangleleft M \) has the unit \((1, 0)\).

Dorroh [5] first used this construction, with \( R = \mathbb{Z} \), (the ring of integers), as a means of embedding a (nonunital) ring \( M \) without identity into a ring with identity.

In this paper, in Section 3, we give the universal property of the Dorroh-extensions that allows to construct the covariant functor \( D : \mathcal{D} \rightarrow \text{Ring} \), where \( \mathcal{D} \) is the category of the Dorroh-pairs and the Dorroh-pair homomorphisms. We prove that the functor \( D \) has a right adjoint and this functor commute with the direct products and inverse limits. Also we establish a correspondence between the Dorroh extensions and some semigroup graded rings.

L. Salce in [13] proves that the group of units of the amalgamated duplication of the ring \( R \) along the \( R \)-module \( M \) is isomorphic with the direct product of the groups \( U(R) \) and \( M^* \). In Section 4 we prove that in the case of the Dorroh extensions, the group of units \( U(R \triangleright:\triangleleft M) \) is isomorphic with the semidirect product of the groups \( U(R) \) and \( M^* \).
2 Some Basic Concepts

Recall that if $S$ is semigroup, the ring $R$ is called \textbf{$S$-graded} if there is a family $\{R_s : s \in S\}$ of additive subgroups of $R$ such that $R = \bigoplus_{s \in S} R_s$ and $R_s R_t \subseteq R_{st}$ for all $s, t \in S$. For a subset $T \subseteq S$ consider $R_T = \bigoplus_{t \in T} R_t$. If $T$ is a subsemigroup of $S$ then $R_T$ is a subring of $R$. If $T$ is a left (right, two-sided) ideal of $R$ then $R_T$ is a left (right, two-sided) ideal of $R$.

The semidirect product of two groups is also a well-known construction in group theory.

\textbf{Definition.} Given the groups $H$ and $N$, a group homomorphism $\varphi : H \to \text{Aut} \, K$, if we define on the Cartesian product, the multiplication

$$(h_1,k_1)(h_2,k_2) = (h_1 h_2, k_1 \varphi(h_1)(k_2)),$$

we obtain a group, called the semidirect product of the groups $H$ and $N$ with respect to $\varphi$. This group is denoted by $H \times_\varphi N$.

\textbf{Theorem.} Let $G$ be a group. If $G$ contain a subgroup $H$ and a normal subgroup $N$ such that $H \cap K = \{1\}$ and $G = K \cdot H$, then the correspondence $(h,k) \mapsto kh$ establishes an isomorphism between the semidirect product $H \times_\varphi N$ of the groups $H$ and $N$ with respect to $\varphi : H \to \text{Aut} \, K$, defined by $\varphi(h)(k) = hkh^{-1}$ and the group $G$.

\textbf{Definition.} A short exact sequence of groups is a sequence of groups and group homomorphisms

$$1 \longrightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1$$

where $\alpha$ is injective, $\beta$ is surjective and $\text{Im} \alpha = \ker \beta$. We say that the above sequence is split if there exists a group homomorphism $s : H \to G$ such that $\beta \circ s = \text{id}_H$.

\textbf{Theorem.} Let $G$, $H$, and $N$ be groups. Then $G$ is isomorphic to a semidirect product of $H$ and $N$ if and only if there exists a split exact sequence

$$1 \longrightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1$$

3 The Dorroh Extension

To simplify the presentation, we give the following definition:
Definition 1. A pair \((R, M)\) of (associative) rings, is called a Dorroh-pair if \(M\) is also an \((R, R)\)-bimodule and for all \(a \in R\) and \(x, y \in M\), are satisfied the following compatibility conditions:

\[(ax)y = a(xy),\quad (xy)a = x(ya),\quad (xa)y = x(ay).\]

We denote further with \(\mathcal{D}\), the class of all Dorroh-pairs.

If \((R, M) \in \mathcal{D}\), on the module direct sum \(R \oplus M\) we introduce the multiplication

\[(a, x) \cdot (b, y) = (ab, xb + ay + xy).\]

\((R \oplus M, +, \cdot)\) is a ring and it is denoted by \(R \triangleright\triangleleft M\) and it is called the Dorroh extension (or ideal extension (see [8], [11])). Moreover, \(R \triangleright\triangleleft M\) is a \((R, R)\)-bimodule under the scalar multiplications defined by

\[\alpha(a, x) = (\alpha a, \alpha x),\quad (a, x)\alpha = (a\alpha, x\alpha)\]

and \((R, R \triangleright\triangleleft M)\) is also a Dorroh-pair.

If \(R\) has the unit 1, then \((1, 0)\) is a unit of the ring \(R \triangleright\triangleleft M\). Dorroh first used this construction (see [5]), with \(R = \mathbb{Z}\), as a means of embedding a ring without identity into a ring with identity.

Remark 2. If \(M\) is a zero ring, the Dorroh extension \(R \triangleright\triangleleft M\) coincides with the trivial extension \(R \times M\).

Example 3. If \(R\) is a ring, then \((R, M)\) is a Dorroh-pair for every ideal \(M\) of the ring \(R\). Another example of a Dorroh-pair is \((R, M_{w,n}(R))\).

Since the applications

\[i_R : R \rightarrow R \triangleright\triangleleft M,\quad a \mapsto (a, 0)\]

\[i_M : M \rightarrow R \triangleright\triangleleft M,\quad x \mapsto (0, x)\]

are injective and both rings homomorphisms and \((R, R)\) linear maps, we can identify further the element \(a \in R\) with \((a, 0) \in R \triangleright\triangleleft M\) and \(x \in M\) with \((0, x) \in R \triangleright\triangleleft M\). Also, the application

\[\pi_R : R \triangleright\triangleleft M \rightarrow R,\quad (a, x) \mapsto a\]
is a surjective ring homomorphism which is also \((R,R)\) linear. Consequently, \(R\) is a subring of \(R \triangleright \triangleleft M\), \(M\) is an ideal of the ring \(R \triangleright \triangleleft M\), and the factor ring \((R \triangleright \triangleleft M)/M\) is isomorphic with \(R\).

**Remark 4.** Given two associative rings \(R\) and \(D\), we can say that \(D\) is a Dorroh extension of the ring \(R\), if \(R\) is a subring of \(D\) and \(D = R \oplus M\) for some ideal \(M \subseteq D\).

If \((A,R),(A,M),(R,M) \in D\), then \(M\) is an \((A,A)\)-bimodule with the scalar multiplication
\[
(\alpha,a)x = \alpha x + ax \quad \text{and} \quad x(\alpha,a) = x\alpha + xa,
\]
respectively, \(R \triangleright \triangleleft M\) is an \((A,A)\)-bimodule with the scalar multiplication
\[
\alpha(a,x) = (\alpha a, \alpha x) \quad \text{and} \quad (a,x)\alpha = (a\alpha, x\alpha).
\]
Obviously, \((A \triangleleft \triangleright R, M), (A,R \triangleright \triangleleft M) \in D\) and since,
\[
((\alpha,a),x) + ((\beta,b),y) = ((\alpha + \beta, a + b), x + y),
\]
\[
((\alpha,a),x) \cdot ((\beta,b),y) = ((\alpha \beta, ab + a\beta + ab), \alpha y + ay + x\beta + xb + xy),
\]
respectively,
\[
(\alpha,(a,x)) \cdot (\beta,(b,y)) = (\alpha + \beta, (a + b, x + y)),
\]
\[
(\alpha,(a,x)) \cdot (\beta,(b,y)) = (\alpha \beta, (ab + a\beta + ab, \alpha y + ay + x\beta + xb + xy)),
\]
the rings \((A \triangleright \triangleright R) \triangleright \triangleleft M\) and \(A \triangleright \triangleright (R \triangleright \triangleleft M)\) are isomorphic, and the isomorphism of these rings is given by the correspondence \((\alpha,(a,x)) \mapsto (\alpha,(a,x))\). Due to this isomorphism, further we can write simply \(A \triangleright \triangleright R \triangleright \triangleleft M\).

**Example 5.** If \(R_1,\ldots,R_n\) are rings such that \((R_i,R_j)\) are Dorroh-pairs whenever \(i \leq j\), we can consider the ring \(R = R_1 \triangleright \triangleright R_2 \triangleright \triangleright \ldots \triangleright \triangleright R_n\). Since for any \(i, j \in I_n\), \(R_i R_j \subseteq R_{\max(i,j)}\), we can consider the ring \(R\) as a \(I_n\)-graded ring, where \(I_n\) is the monoid \(\{1,\ldots,n\}\) with the operation defined by \(i \vee j = \max(i, j)\). Conversely, if a ring \(R\) is \(I_n\)-graded and \(R = \bigoplus_{i \in I_n} R_i\), since \(R_i R_j \subseteq R_{i \vee j}\) for all \(i, j \in I_n\), the subgroups \(R_1,\ldots,R_n\) are subrings of \(R\) and \(R_j\) is a \((R_i,R_i)\)-bimodule whenever \(i \leq j\), the rings \(R\) and \(R_i \triangleright \triangleright R_2 \triangleright \triangleright \ldots \triangleright \triangleright R_n\) are isomorphic.
Definition 6. By a homomorphism between the Dorroh-pairs \((R, M)\) and \((R', M')\) we mean a pair \((\varphi, f)\), where \(\varphi : R \to R'\) and \(f : M \to M'\) are ring homomorphisms for which, for all \(\alpha \in R\) and \(x \in M\) we have that
\[
f(\alpha \cdot x) = \varphi(\alpha) \cdot f(x) \quad \text{and} \quad f(x \cdot \alpha) = f(x) \cdot \varphi(\alpha).
\]

The Dorroh extension verifies the following universal property:

**Theorem 7.** If \((R, M)\) is a Dorroh-pair, then for any ring \(\Lambda\) and any Dorroh-pairs homomorphism \((\varphi, f) : (R, M) \to (\Lambda, \Lambda)\), there exists a unique ring homomorphism \(\varphi \bowtie f : R \bowtie M \to \Lambda\) such that
\[
\begin{align*}
R & \xrightarrow{i_R} R \times M \xleftarrow{i_M} M \\
\varphi \times f & \downarrow \quad \quad f \uparrow \\
\Lambda & \xrightarrow{f} \Lambda
\end{align*}
\]

\((\varphi \bowtie f) \circ i_M = f\) and \((\varphi \bowtie f) \circ i_R = \varphi\).

**Proof.** It is routine to verify that the application \(\varphi \bowtie f\), defined by
\[
(\varphi \bowtie f)(a, x) = \varphi(a) + f(x)
\]
is the required ring homomorphism.

**Corollary 8.** If \((R, M)\) and \((R', M')\) are two Dorroh-pairs, and \((\varphi, f) : (R, M) \to (R', M')\) is a Dorroh-pairs homomorphism, then there exists a unique ring homomorphism \(\varphi \bowtie f : R \bowtie M \to R' \bowtie M'\) such that
\[
\begin{align*}
R & \xrightarrow{i_R} R \times M \xleftarrow{i_M} M \\
\varphi \times f & \downarrow \quad \quad f \uparrow \\
R' & \xleftarrow{i_{R'}} R' \times M' \xrightarrow{i_{M'}} M'
\end{align*}
\]

\((\varphi \bowtie f) \circ i_{M'} = i_{M'} \circ f\) and \((\varphi \bowtie f) \circ i_R = i_R \circ \varphi\).

**Proof.** Apply Theorem 7, considering \(\Lambda = R' \bowtie M'\) and the homomorphisms pair \((i_R \circ \varphi, i_{M'} \circ f)\).
Consider now the category \( \mathcal{D} \) whose objects are the class \( \mathcal{D} \) of the Dorroh-pairs and the homomorphisms between two objects are the Dorroh-pairs homomorphisms and the category \( \mathcal{Rng} \) of the associative rings.

By Corollary 8, we can consider the covariant functor \( \mathcal{D} : \mathcal{D} \to \mathcal{Rng} \), defined as follows: if \( (R,M) \) is a Dorroh-pair, then \( \mathcal{D}(R,M) = R \downarrow M \), and if \( (\varphi,f):(R,M) \to (R',M') \) is a Dorroh-pair homomorphism, then \( \mathcal{D}(\varphi,f) = \varphi \downarrow f \).

Consider also the functor \( \mathcal{B} : \mathcal{Rng} \to \mathcal{D} \), defined as follows: if \( A \) is a ring, then \( \mathcal{B}(A) = (A,A) \) and if \( h : A \to B \) is a ring homomorphism, \( \mathcal{B}(h) = (h,h) \).

**Theorem 9.** The functor \( \mathcal{D} \) is left adjoint of \( \mathcal{B} \).

**Proof.** If \( (R,M) \in \text{Ob}\mathcal{D} \) and \( \Lambda \in \text{Ob}\mathcal{Rng} \), define the function

\[
\phi_{(R,M),\Lambda} : \text{Hom}_{\mathcal{Rng}}(R \downarrow M, \Lambda) \to \text{Hom}_\mathcal{D}((R,M), (\Lambda, \Lambda))
\]

by \( \Phi \mapsto (\Phi|_R, \Phi|_M) \), which is evidently a bijection.

Since, for any Dorroh-pairs homomorphism \( (\varphi,f):(R,M) \to (R',M') \) and for any ring homomorphisms \( \beta : \Lambda \to \Lambda' \) and \( \Psi : R' \downarrow M' \to \Lambda \) we have that

\[
(\beta,\beta) \circ (\Psi|_{R'}, \Psi|_{M'}) \circ (\varphi,f) = ((\beta \circ \Psi|_{R'} \circ \varphi), (\beta \circ \Psi|_{M'} \circ f))
\]

\[
= ((\beta \circ \Psi \circ i_{R'}, \varphi), (\beta \circ \Psi \circ i_{M'} \circ f))
\]

\[
= ((\beta \circ \Psi \circ (\varphi \downarrow f) \circ i_{R'}, (\beta \circ \Psi \circ (\varphi \downarrow f) \circ i_{M'}))
\]

\[
= ((\beta \circ \Psi \circ (\varphi \downarrow f)|_R, (\beta \circ \Psi \circ (\varphi \downarrow f)|_M)
\]

the diagram

\[
\text{Hom}_{\mathcal{Rng}}(R' \ltimes M', \Lambda) \xrightarrow{\phi_{(R',M'),\Lambda}} \text{Hom}_\mathcal{D}((R',M'), (\Lambda, \Lambda))
\]

\[
\text{Hom}_{\mathcal{Rng}}((\varphi \ltimes f), \beta) \xrightarrow{\phi_{(R,M),\Lambda'}} \text{Hom}_\mathcal{D}((R,M), (\Lambda', \Lambda'))
\]

is commutative and the result follow.
Proposition 10. Consider \( \{(R_i, M_i) : i \in I\} \) a family of Dorroh-pairs and the ring direct products \( \prod_{i \in I} R_i \) and \( \prod_{i \in I} M_i \) (with the canonical projections \( p_i \) and \( \pi_i \), respectively, the canonical embeddings \( q_i \) and \( \sigma_i \)).

Then \( \left( \prod_{i \in I} R_i, \prod_{i \in I} M_i \right) \) is also a Dorroh-pair, for all \( i \in I \), \( \left( p_i, \pi_i \right) \) and \( \left( q_i, \sigma_i \right) \) are Dorroh-pairs homomorphisms and

\[
\left( \prod_{i \in I} R_i \right) \triangleleft \left( \prod_{i \in I} M_i \right) \cong \prod_{i \in I} \left( R_i \triangleleft M_i \right).
\]

Proof. Since for all \( i \in I \), \( (R_i, M_i) \) are Dorroh-pairs, \( \prod_{i \in I} M_i \) is a \( \left( \prod_{i \in I} R_i, \prod_{i \in I} R_i \right) \)-bimodule with the componentwise scalar multiplications and evidently, the compatibility conditions are satisfied. Thus \( \left( \prod_{i \in I} R_i, \prod_{i \in I} M_i \right) \) is a Dorroh-pair.

If \( a = (a_i)_{i \in I} \in \prod_{i \in I} R_i \) and \( x = (x_i)_{i \in I} \in \prod_{i \in I} M_i \), then for all \( j \in I \),

\[
\pi_j (a \cdot x) = a_j \cdot x_j = p_j (a) \cdot \pi_j (a) \quad \text{and} \quad \pi_j (x \cdot a) = x_j \cdot a_j = \pi_j (a) \cdot p_j (a)
\]

respectively, if \( i \in I \), \( a_i \in R_i \) and \( x_i \in M_i \), then

\[
\sigma_i (a_i \cdot x_i) = q_i (a_i) \cdot \sigma_i (a_i) \quad \text{and} \quad \sigma_i (x_i \cdot a_i) = \sigma_i (a_i) \cdot q_i (a_i)
\]

and so \( (p_i, \pi_i) \) and \( (q_i, \sigma_i) \) are Dorroh-pairs homomorphisms.

Proposition 11. Let \( I \) be a directed set and \( \left\{ (R_i, M_i)_{i \in I} : (\varphi_{ij}, f_{ij})_{i,j \in I} \right\} \) an inverse system of Dorroh-pairs. Then \( \left\{ (R_i \triangleleft M_i)_{i \in I} : (\varphi_{ij} \triangleleft f_{ij})_{i,j \in I} \right\} \) is an inverse system of rings and

\[
\lim_{\leftarrow} (R_i \triangleleft M_i) \cong \left( \lim_{\leftarrow} R_i \right) \triangleleft \left( \lim_{\leftarrow} M_i \right).
\]

Proof. Consider the elements \( i, j \in I \) such that \( i \leq j \). By Corollary 8, the Dorroh-pairs homomorphism \( (\varphi_{ij}, f_{ij}) : (R_j, M_j) \rightarrow (R_i, M_i) \) can be extended to the ring homomorphism \( \varphi_j \triangleleft f_{ij} : R_j \triangleleft M_j \rightarrow R_i \triangleleft M_i \) which is defined by

\[
(\varphi_j \triangleleft f_{ij})(a_j, x_j) = \left( \varphi_j (a_j), f_{ij} (x_j) \right), \text{ for all } (a_j, x_j) \in R_j \triangleleft M_j.
\]
Obviously, \(\left\{ (R_i \triangleright \triangleleft M_i)_{i \in I}, (\varphi_{ij} \triangleright \triangleleft f_{ij})_{i,j \in I}\right\}\) is an inverse system of rings.

Consider now \(s,t \in I\) such that \(s \leq t\) and \((a_i,x_i)_{i \in I} \in \varprojlim R_i \triangleright \triangleleft M_i\). Since
\[
(a_s,a_t) = (\varphi_{st} \triangleright \triangleleft f_{st})(a_i,x_i) = (\varphi_{st}(a_i), f_{st}(x_i))
\]
we obtain that \((a_i)_{i \in I} \in \varprojlim R_i\), \((x_i)_{i \in I} \in \varprojlim M_i\) and the correspondence
\[
(a_i,x_i)_{i \in I} \mapsto ((a_i)_{i \in I}, (x_i)_{i \in I})
\]
establishes an isomorphism between \(\varprojlim (R_i \triangleright \triangleleft M_i)\) and \((\varprojlim R_i) \triangleright \triangleleft (\varprojlim M_i)\).

### 4 The Group of Units of the Ring \(R \triangleright \triangleleft M\)

If \(A\) is a ring with identity, denote by \(U(A)\) the group of units of this ring.

Let \((R,M)\) a Dorroh-pair where \(R\) is a ring with identity and consider the Dorroh extension \(R \triangleright \triangleleft M\). In this section we will describe the group of units of the ring \(R \triangleright \triangleleft M\). Firstly, observe that if \((a,x) \in U(R \triangleright \triangleleft M)\), then \(a \in U(R)\).

The set of all elements of \(M\) forms a monoid under the circle composition on \(M\), \(x \circ y = x + y + xy\), \(0\) being the neutral element. The group of units of this monoid we will denote by \(M^*\).

**Theorem 12.** The group of units \(U(R \triangleright \triangleleft M)\) of the Dorroh extension \(R \triangleright \triangleleft M\) is isomorphic with a semidirect product of the groups \(U(R)\) and \(M^*\).

**Proof.** Consider the function
\[
\sigma_{M^*} : M^* \to U(R \triangleright \triangleleft M), \quad x \mapsto (1,x),
\]
which is an injective group homomorphism. Consider also the group homomorphisms \(i_{U(R)} : U(R) \to U(R \triangleright \triangleleft M)\) and \(\pi_{U(R)} : U(R \triangleright \triangleleft M) \to U(R)\) induced by the ring homomorphisms \(i_R : R \to R \triangleright \triangleleft M\) and \(\pi_R : R \triangleright \triangleleft M \to R\), respectively. Since the following sequences
are exacts and \( \pi_{U(R)} \circ i_{U(R)} = id_{U(R)} \), the group of units \( U(R \triangleright \triangleleft M) \) of the Dorroh extension \( R \triangleright \triangleleft M \) is isomorphic with the semidirect product of the groups \( U(R) \) and \( M^* \), \( U(R) \times_{\delta} M^* \). The homomorphism \( \delta : U(R) \to \text{Aut} M^* \), is defined by \( a \mapsto \delta_a \) where \( \delta_a : M^* \to M^* \), \( x \mapsto axa^{-1} \) and the multiplication of the semidirect product \( U(R) \times_{\delta} M^* \), is defined by

\[
(a,x) \cdot (b,y) = \left( ab, x \circ (aya^{-1}) \right) = (ab, x + aya^{-1} + xaya^{-1}).
\]

The isomorphism between the groups \( U(R) \times_{\delta} M^* \) and \( U(R \triangleright \triangleleft M) \) is given by \( (a,x) \mapsto (a,xa) \).

**Remark 13.** If \( M \) is a ring with identity, the correspondence \( x \mapsto x^{-1} \) establishes an isomorphism between the groups \( U(M) \) and \( M^* \), and therefore the group \( U(R \triangleright \triangleleft M) \) is isomorphic with a semidirect product of the groups \( U(R) \) and \( U(M) \).

**Corollary 14.** The group of units \( U(R \times M) \) of the trivial extension \( R \times M \) is isomorphic with a semidirect product of the group \( U(R) \) with the additive group of the ring \( M \).

**Conclusions**

The Dorroh extension is a useful construction in abstract algebra being an interesting source of examples in the ring theory.

**References**


