Left-continuous t-norms in Fuzzy Logic: an Overview*

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Abstract: In this paper we summarize some fundamental results on left-continuous t-norms. First we study the nilpotent minimum and related operations in considerable details. This is the very first example of a left-continuous but not continuous t-norm in the literature. Then we recall some recent extensions and construction methods.

Keywords: Associative operations, Triangular norm, Residual implication, Left-continuous t-norm, Nilpotent minimum.

1 Introduction

The concept of Fuzzy Logic (FL) was invented by Lotfi Zadeh [20] and presented as a way of processing data by allowing partial set membership rather than only full or non-membership. This approach to set theory was not applied to engineering problems until the 70's due to insufficient small-computer capability prior to that time.

In the context of control problems (the most successful application area of FL), fuzzy logic is a problem-solving methodology that provides a simple way to arrive at a definite conclusion based upon vague, ambiguous, imprecise, noisy, or missing input information. FL incorporates a simple, rule-based "IF X AND Y THEN Z" approach to solving a control problem rather than attempting to model a system mathematically.

When one considers fuzzy subsets of a universe, in order to generalize the Boolean set-theoretical operations like intersection, union and complement, it is quite natural to use *interpretations* of logic connectives $\wedge$, $\vee$ and $\neg$, respectively [12]. It is assumed that the conjunction $\wedge$ is interpreted by a *triangular norm* (*t-norm* for short), the disjunction $\vee$ is interpreted by a *triangular conorm* (shortly: *t-conorm*), and the negation $\neg$ by a *strong negation*.

Although engineers have learned the basics of theoretical aspects of fuzzy sets and logic, from time to time it is necessary to summarize recent developments even in such a fundamental subject. This is the main aim of the present paper.

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Therefore, we focus on recent advances on an important and rather complex subclass of t-norms: on *left-continuous t-norms*. The standard example of a left-continuous t-norm is the *nilpotent minimum* [4,15]. Starting from our more than ten years old algebraic ideas, their elegant geometric interpretations make it possible to understand more on *left-continuous t-norms with strong induced negations*, and construct a wide family of them. Studies on properties of fuzzy logics based on left-continuous t-norms, and especially on the nilpotent minimum (NM) have started only recently; see [1,14,13,18,19] along this line.

2 Preliminaries

In this section we briefly recall some definitions and results will be used later. For more details see [5,12].

A bijection φ of the unit interval onto itself preserving natural ordering is called an *automorphism* of the unit interval. It is a continuous strictly increasing function satisfying boundary conditions $\varphi(0) = 0, \varphi(1) = 1$.

A *strong negation* N is defined as a strictly decreasing, continuous function $N\colon [0,1] \to [0,1]$ with boundary conditions $N(0) = 1,\ N(1) = 0$ such that N is involutive (i.e., $N(N(x)) = x$ holds for any $x \in [0,1]$). A standard example of a strong negation is given by $N_{\mathrm{st}}(x) = 1 - x$. Any strong negation N can be represented as a φ *-transform* of the standard negation (see [17])

$$N(x) = \varphi^{-1}(1 - \varphi(x))$$

for some automorphism φ of the unit interval. In this case the strong negation is denoted by N_φ.

A *t-norm* T is defined as a symmetric, associative and nondecreasing function $T\colon [0,1]^2 \to [0,1]$ satisfying boundary condition $T(1,x) = x$ for all $x \in [0,1]$.

A *t-conorm* S is defined as a symmetric, associative and nondecreasing function $S\colon [0,1]^2 \to [0,1]$ satisfying boundary condition $S(0,x) = x$ for all $x \in [0,1]$.

For any given t-norm T and strong negation N a function S defined by $S(x,y) = N(T(N(x), N(y)))$ is a t-conorm, called the N*-dual t-conorm of* T. In this case the triplet (T, S, N) is called a *De Morgan triplet*.

Well-accepted models for conjunction (AND), disjunction (OR), negation (NOT) are given by t-norms, t-conorms, strong negations, respectively. In this paper we will focus mainly on t-norms.

The definition of t-norms does not imply any kind of continuity. Nevertheless, such a property is desirable from theoretical as well as practical points of view.

A t-norm T is *continuous* if for all convergent sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ we have

$$T\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} T(x_n, y_n).$$

The structure of continuous t-norms is well known, see [12] for more details, especially Section 3.3 on *ordinal sums*.

3 Left-continuous t-norms

In many cases, weaker forms of continuity are sufficient to consider. For t-norms, this property is *lower semicontinuity* [12, Section 1.3]. Since a t-norm T is non-decreasing and commutative, it is lower semicontinuous if and only if it is *left-continuous* in its first component. That is, if and only if for each $y \in [0,1]$ and for all non-decreasing sequences $\{x_n\}_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty} T(x_n, y) = T\left(\lim_{n\to\infty} x_n, y\right).$$

If T is a left-continuous t-norm, the operation $I_T : [0,1]^2 \to [0,1]$ defined by

$$I_T(x,y) = \sup\{t \in [0,1] \mid T(x,t) \le y\} \tag{1}$$

is called the *residual implication* (shortly: R-implication) generated by T. An equivalent formulation of left-continuity of T is given by the following property ($x, y, z \in [0,1]$):

(R) $\quad T(x,y) \le z \quad$ if and only if $\quad I_T(x,z) \ge y$.

We emphasize that the formula (1) can be computed for any t-norm T; however, the resulting operation I_T satisfies condition (R) if and only if the t-norm T is left-continuous. An interesting underlying algebraic structure of left-continuous t-norms is a commutative, residuated integral l-monoid, see [6] for more details.

4 Nilpotent Minimum and Maximum

The first known example of a left-continuous but non-continuous t-norm is the so-called *nilpotent minimum* [4] denoted as $T^{\mathbf{nM}}$ and defined by

$$T^{\mathbf{nM}}(x,y) = \begin{cases} 0 & \text{if } x+y \le 1, \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{2}$$

It can be understood as follows. We start from a t-norm (the minimum), and re-define its value below and along the diagonal $\{(x,y) \in [0,1] \mid x+y = 1\}$. So, the question is natural: if we consider any t-norm T and "annihilate" its original values below and along the mentioned diagonal, is the new operation always a t-norm? The general answer is "no" (although the contrary was "proved" in [15] where the same operation also appeared).

The definition (2) can be extended as follows. Suppose that φ is an automorphism of the unit interval. Define a binary operation on $[0,1]$ by

$$T_\varphi^{\mathbf{nM}}(x,y) = \begin{cases} 0 & \text{if } \varphi(x)+\varphi(y) \leq 1 \\ \min(x,y) & \text{if } \varphi(x)+\varphi(y) > 1 \end{cases}. \tag{3}$$

Thus defined $T_\varphi^{\mathbf{nM}}$ is a t-norm and is called the *φ-nilpotent minimum.*

Clearly, the following equivalent form of $T_\varphi^{\mathbf{nM}}$ can be obtained by using the strong negation N_φ generated by φ:

$$T_\varphi^{\mathbf{nM}}(x,y) = \begin{cases} \min(x,y) & \text{if } y > N_\varphi(x) \\ 0 & \text{otherwise} \end{cases}.$$

Extension of $T_\varphi^{\mathbf{nM}}$ for more than two arguments is easily obtained and is given by $T_\varphi^{\mathbf{nM}}(x_1,\dots,x_n) = \min_{i=1,n}\{x_i\}$ if $\min_{i\neq j}\{\varphi(x_i)+\varphi(x_j)\} > 1$, and $T_\varphi^{\mathbf{nM}}(x_1,\dots,x_n) = 0$ otherwise.

The N_φ-dual t-conorm of $T_\varphi^{\mathbf{nM}}$, called the *φ-nilpotent maximum,* is defined by

$$S_\varphi^{\mathbf{nM}}(x,y) = \begin{cases} \max(x,y) & \text{if } \varphi(x)+\varphi(y) < 1 \\ 1 & \text{otherwise} \end{cases}.$$

Clearly, $(T_\varphi^{\mathbf{nM}}, S_\varphi^{\mathbf{nM}}, N_\varphi)$ yields a De Morgan triple.

In the next theorem we list the most important properties of $T_\varphi^{\mathbf{nM}}$ and $S_\varphi^{\mathbf{nM}}$. These are easy to prove.

Theorem 1. *Suppose that φ is an automorphism of the unit interval. The t-norm $T_\varphi^{\mathbf{nM}}$ and the t-conorm $S_\varphi^{\mathbf{nM}}$ have the following properties:*
(a) The law of contradiction holds for $T_\varphi^{\mathbf{nM}}$ as follows:

$$T_\varphi^{\mathbf{nM}}(x, N_\varphi(x)) = 0 \quad \forall x \in [0,1].$$

(b) The law of excluded middle holds for $S_\varphi^{\mathbf{nM}}$:

$$S_\varphi^{\mathbf{nM}}(x, N_\varphi(x)) = 1 \quad \forall x \in [0,1].$$

(c) There exists a number α_0 depending on φ such that $0 < \alpha_0 < 1$ and $T_\varphi^{\mathbf{nM}}$ is idempotent on the interval $]\alpha_0, 1]$:

$$T_\varphi^{\mathbf{nM}}(x,x) = x \quad \forall x \in]\alpha_0, 1].$$

(d) With the previously obtained α_0, $S_\varphi^{\mathbf{nM}}$ is idempotent on the interval $[0, \alpha_0[$:

$$S_\varphi^{\mathbf{nM}}(x,x) = x \quad \forall x \in [0, \alpha_0[.$$

(e) There exists a subset X_φ of the unit square such that $(x,y) \in X_\varphi$ if and only if $(y,x) \in X_\varphi$ and the law of absorption holds on X_φ as follows:

$$S_\varphi^{\mathbf{nM}}(x, T_\varphi^{\mathbf{nM}}(x,y)) = x \quad \forall (x,y) \in X_\varphi.$$

(f) There exists a subset Y_φ of the unit square such that $(x, y) \in Y_\varphi$ if and only if $(y, x) \in Y_\varphi$ and the law of absorption holds on Y_φ as follows:

$$T_\varphi^{\mathbf{nM}}(x, S_\varphi^{\mathbf{nM}}(x, y)) = x \quad \forall (x, y) \in Y_\varphi.$$

(g) If A, B are fuzzy subsets of the universe of discourse U and the α-cuts are denoted by A_α, B_α, respectively ($\alpha \in [0, 1]$), then we have

$$A_\alpha \cap B_\alpha = [T_\varphi^{\mathbf{nM}}(A, B)]_\alpha \quad \forall \alpha \in]\alpha_0, 1]$$

and

$$A_\alpha \cup B_\alpha = [S_\varphi^{\mathbf{nM}}(A, B)]_\alpha \quad \forall \alpha \in [0, \alpha_0[,$$

where α_0 is given in (c).
(h) $T_\varphi^{\mathbf{nM}}$ is a left-continuous t-norm and $S_\varphi^{\mathbf{nM}}$ is a right-continuous t-conorm. □

4.1 Where does Nilpotent Minimum Come from?

Nilpotent minimum has been discovered not by chance. There is a study on contrapositive symmetry of fuzzy implications [4]. A particular case of those investigations yielded nilpotent minimum. Some of the related results will be cited later in the present paper.

Let T be a left-continuous t-norm and N a strong negation. Consider the residual implication I_T generated by T, defined in (1).

Contrapositive symmetry of I_T with respect to N (CPS(N) for short) is a property that can be expressed by the following equality:

$$I_T(x, y) = I_T(N(y), N(x)) \quad \forall x, y \in [0, 1]. \tag{4}$$

Unfortunately, (4) is generally not satisfied for I_T generated by a left-continuous (even continuous) t-norm T. In [4] we proved the following result.

Theorem 2 ([4]). *Suppose that T is a t-norm such that condition **(R)** is satisfied, N is a strong negation. Then the following conditions are equivalent ($x, y, z \in [0, 1]$).*

(a) I_T has contrapositive symmetry with respect to N;
(b) $I_T(x, y) = N(T(x, N(y)))$;
(c) $T(x, y) \le z$ if and only if $T(x, N(z)) \le N(y)$.

In any of these cases we have

(d) $N(x) = I_T(x, 0)$,
(e) $T(x, y) = 0$ if and only if $x \le N(y)$. □

In the case of continuous t-norms we have the following unicity result (see also [11]).

Theorem 3. *Suppose that T is a continuous t-norm. Then I_T has contrapositive symmetry with respect to a strong negation N if and only if there exists an automorphism φ of the unit interval such that*

$$T(x,y) = \varphi^{-1}(\max\{\varphi(x)+\varphi(y)-1,0\}), \tag{5}$$

$$N(x) = \varphi^{-1}(1-\varphi(x)). \tag{6}$$

In this case I_T is given by

$$I_T(x,y) = \varphi^{-1}(\min\{1-\varphi(x)+\varphi(y),1\}). \quad \square \tag{7}$$

When I_T is any R-implication and I_T does not have contrapositive symmetry then we can associate another implication with I_T. Suppose that T is a t-norm which satisfies condition **(R)**. Define a new implication associated with I_T as follows:

$$x \to_T y = \max\{I_T(x,y), I_T(N(y),N(x))\}. \tag{8}$$

If I_T has contrapositive symmetry then $x \to_T y = I_T(x,y) = I_T(N(y),N(x))$.

Define also a binary operation $*_T$ by

$$x *_T y = \min\{T(x,y), N[I_T(y,N(x))]\}. \tag{9}$$

Obviously, $*_T = T$ if (4) is satisfied by $I = I_T$. Even in the opposite case, this operation $*_T$ is a fuzzy conjunction in a broad sense and has several nice properties as we state in the next theorem.

Theorem 4. *Suppose that T is a t-norm such that* ***(R)*** *is true, N is a strong negation such that $N(x) \geq I_T(x,0)$ for all $x \in [0,1]$ and operations $\to_T$ and $*_T$ are defined by (8) and (9), respectively. Then the following conditions are satisfied:*

(a) $1 *_T y = y$;
(b) $x *_T 1 = x$;
(c) $*_T$ *is nondecreasing in both arguments;*
(d) $x \to_T y \geq z$ *if and only if* $x *_T z \leq y$. $\square$

In Table 1 we list most common t-norms and corresponding operations I_T, $*_T$, $\to_T$, with $N(x) = 1-x$.

Therefore, nilpotent minimum can be obtained as the conjunction $*_{\min}$. In general, $*_T$ is not a t-norm, not even commutative. Sufficient condition to assure that $*_T$ is a t-norm is given in the next theorem.

Theorem 5. *For a t-norm T and a strong negation N, if $y > N(x)$ implies $T(x,y) \leq N(I_T(y,N(x)))$ then $*_T$ is also a t-norm.* $\square$

T	$\min(x, y)$	$\max(x + y - 1, 0)$	xy
I_T	$1,\ x \leq y$ y otherwise	$\min(1 - x + y, 1)$	$\min\ \ 1, \dfrac{y}{x}$
$*_T$	$\min(x, y),\ x + y > 1$ $0, \qquad\quad x + y \leq 1$	$\max(x + y - 1, 0)$	$\min\ \ xy, \dfrac{x + y - 1}{y}$
$\to_T$	$1, \qquad\qquad\quad x \leq y$ $\max(1 - x, y),\ x > y$	$\min(1 - x + y, 1)$	$\max\ \ \dfrac{y}{x}, \dfrac{1 - x}{1 - y}$

Table 1. Some t-norms and associated connectives

4.2 Implications Defined by Nilpotent Minimum and Maximum

Consider the De Morgan triple $(T_\varphi^{\mathbf{nM}}, S_\varphi^{\mathbf{nM}}, N_\varphi)$ with an automorphism φ of the unit interval and define the corresponding S-implication:

$$I(x, y) = S_\varphi^{\mathbf{nM}}(N_\varphi(x), y) \tag{10}$$

$$= \begin{cases} 1, & x \leq y \\ \max(N_\varphi(x), y), & x > y \end{cases} . \tag{11}$$

One can easily prove that the R-implication defined by $T_\varphi^{\mathbf{nM}}$ coincides with the S-implication in (11).

Proposition 1. *Let φ be any automorphism of the unit interval. Then we have for all $x, y \in [0, 1]$ that*

$$I_{T_\varphi^{\mathbf{nM}}}(x, y) = S_\varphi^{\mathbf{nM}}(N_\varphi(x), y). \quad \square$$

As a trivial consequence, $I_{T_\varphi^{\mathbf{nM}}}$ always has contrapositive symmetry with respect to N_φ.

Now we list the most important and attractive properties of $I_{T_\varphi^{\mathbf{nM}}}$. Their richness is due to the fact that R- and S-implications coincide and thus advantageous features of both classes are combined.

1. $I_{T_\varphi^{\mathbf{nM}}}(x, .)$ is non-decreasing
2. $I_{T_\varphi^{\mathbf{nM}}}(., y)$ is non-increasing
3. $I_{T_\varphi^{\mathbf{nM}}}(1, y) = y$

4. $I_{T_\varphi^{\mathbf{nM}}}(0,y)=1$

5. $I_{T_\varphi^{\mathbf{nM}}}(x,1)=1$

6. $I_{T_\varphi^{\mathbf{nM}}}(x,y)=1$ if and only if $x \le y$

7. $I_{T_\varphi^{\mathbf{nM}}}(x,y)=I_{T_\varphi^{\mathbf{nM}}}(N_\varphi(y),N_\varphi(x))$

8. $I_{T_\varphi^{\mathbf{nM}}}(x,0)=N_\varphi(x)$

9. $I_{T_\varphi^{\mathbf{nM}}}(x,I_{T_\varphi^{\mathbf{nM}}}(y,x))=1$

10. $I_{T_\varphi^{\mathbf{nM}}}(x,.)$ is right-continuous

11. $I_{T_\varphi^{\mathbf{nM}}}(x,x)=1$

12. $I_{T_\varphi^{\mathbf{nM}}}(x,I_{T_\varphi^{\mathbf{nM}}}(y,z))=I_{T_\varphi^{\mathbf{nM}}}(y,I_{T_\varphi^{\mathbf{nM}}}(x,z))=I_{T_\varphi^{\mathbf{nM}}}(T_\varphi^{\mathbf{nM}}(x,y),z)$

13. $T_\varphi^{\mathbf{nM}}(x,I_{T_\varphi^{\mathbf{nM}}}(x,y))\le \min(x,y)$

14. $I_{T_\varphi^{\mathbf{nM}}}(x,y)\ge \min(x,y)$

Notice that $I_{T_\varphi^{\mathbf{nM}}}$ can also be viewed as a QL-implication defined by

$$\begin{aligned} S(x,y) &= S_\varphi^{\mathbf{nM}}(x,y),\\ N(x) &= N_\varphi(x)\\ T(x,y) &= \min(x,y) \end{aligned}$$

in (4), as one can check easily by simple calculus.

Therefore, this QL-implication (which is, in fact, an S-implication and an R-implication at the same time) also has contrapositive symmetry with respect to N_φ. Concerning this case, the following unicity result was proved in [4].

Theorem 6 ([4]). *Consider a QL-implication defined by* $\max_\varphi(N_\varphi(x),T(x,y))$, *where T is a t-norm. This implication has contrapositive symmetry with respect to N_φ if and only if $T=\min$.* □

5 Extensions and Constructions

In this section we summarize some important results on left-continuous t-norms obtained by Jenei and other researchers.

5.1 Left-continuous t-norms with Strong Induced Negations

The notions and some of the results in the above Theorem 2 were formulated in a slightly more general framework in [7]. We restrict ourselves to the case of left-continuous t-norms with strong induced negations; i.e., T is a left-continuous t-norm and the function $N_T(x) = I_T(x, 0)$ (the negation induced by T) is a strong negation.

Moreover, in a sense, a converse statement of Theorem 2 was also established in [7]: If T is a left-continuous t-norm such that $N_T(x) = I_T(x, 0)$ is a strong negation, then (a), (b) and (c) necessarily hold with $N = N_T$.

Already in [3], we studied the above algebraic property (c). Geometric interpretations of properties (b) and (c) were given in [7] under the names of *rotation invariance* and *self-quasi inverse property*, respectively. More exactly, we have the following definition.

Definition 1. Let $T : [0, 1]^2 \to [0, 1]$ be a symmetric and non-decreasing function, and let N be a strong negation. We say that T admits the *rotation invariance* property with respect to N if for all $x, y, z \in [0, 1]$ we have

$$T(x, y) \leq z \quad \text{if and only if} \quad T(y, N(z)) \leq N(x).$$

In addition, suppose T is left-continuous. We say that T admits the *self quasi-inverse* property w.r.t. N if for all $x, y, z \in [0, 1]$ we have

$$I_T(x, y) = z \quad \text{if and only if} \quad T(x, N(y)) = N(z). \quad \square$$

For left-continuous t-norms, rotation invariance is exactly property (c) in Theorem 2, while self quasi-inverse property is just a slightly reformulated version of (b) there. Nevertheless, the following geometric interpretation was given in [7]. If N is a the standard negation and we consider the transformation $\sigma : [0, 1]^3 \to [0, 1]^3$ defined by $\sigma(x, y, z) = (y, N(z), N(x))$, then it can be understood as a rotation of the unit cube with angle of $2\pi/3$ around the line connecting the points $(0, 0, 1)$ and $(1, 1, 0)$. Thus, the formula $T(x, y) \leq z \iff T(y, N(z)) \leq N(x)$ expresses that the part of the unit cube above the graph of T remains invariant under σ. This is illustrated in the first part of Figure 1.

The second part of Figure 1 is about the self quasi-inverse property which can be described as follows (for quasi-inverses of decreasing functions see [16]). For a left-continuous t-norm T, we define a function $f_x : [0, 1] \to [0, 1]$ as follows: $f_x(y) = N_T(T(x, y))$. It was proved in [7] that f_x is its own quasi-inverse if and only if T admits the self quasi-inverse property. Assume that N is the standard negation. Then the geometric interpretation of the negation is the reflection of the graph with respect to the line $y = 1/2$. Then, if it is applied to the partial mapping $T(x, \cdot)$, extend discontinuities of $T(x, \cdot)$ with vertical line segments. Then the obtained graph is invariant under the reflection with respect to the diagonal $\{(x, y) \in [0, 1] \mid x + y = 1\}$ of the unit square.

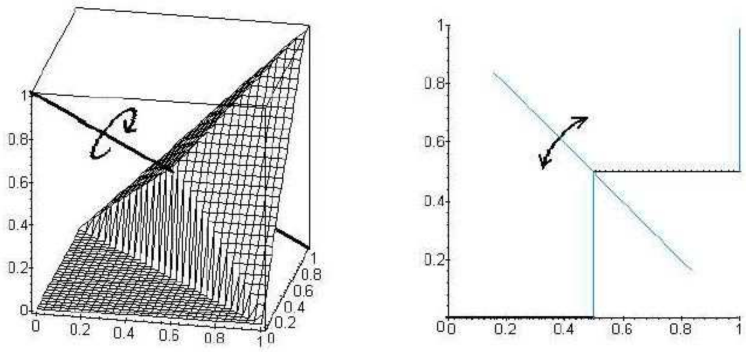

Fig. 1. Rotation invariance property (left). Self quasi-inverse property (right).

5.2 Rotation Construction

Theorem 7 ([9]). *Let N be a strong negation, t its unique fixed point and T be a left-continuous t-norm without zero divisors. Let T_1 be the linear transformation of T into $[t,1]^2$. Let $I^+ =]t,1]$, $I^- = [0,t]$, and define a function $T_{\mathbf{rot}} : [0,1]^2 \to [0,1]$ by*

$$T_{\mathbf{rot}}(x,y) = \begin{cases} T(x,y) & \text{if } x,y \in I^+, \\ N(I_{T_1}(x,N(y))) & \text{if } x \in I^+ \text{ and } y \in I^-, \\ N(I_{T_1}(y,N(x))) & \text{if } x \in I^- \text{ and } y \in I^+, \\ 0 & \text{if } x,y \in I^-. \end{cases}$$

Then $T_{\mathbf{rot}}$ is a left-continuous t-norm, and its induced negation is N.

When we start from the standard negation, the construction works as follows: take any left-continuous t-norm without zero divisors, scale it down to the square $[1/2,1]^2$, and finally rotate it with angle of $2\pi/3$ in both directions around the line connecting the points $(0,0,1)$ and $(1,1,0)$. This is illustrated in Fig. 2.

Remark that there is another recent construction method of left-continuous t-norms (called rotation-annihilation) developed in [10].

5.3 Annihilation

Let N be a strong negation (i.e., an involutive order reversing bijection of the closed unit interval). Let T be a t-norm. Define a binary operation $T_{(N)} : [0,1]^2 \to [0,1]$ as follows:

$$T_{(N)}(x,y) = \begin{cases} T(x,y) & \text{if } x > N(y) \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

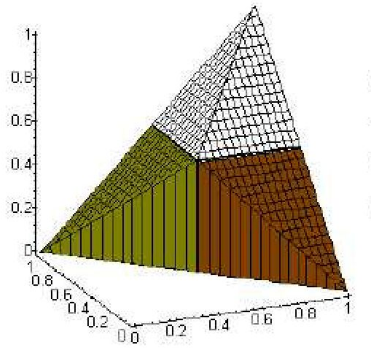

Fig. 2. $T^{\mathbf{nM}}$ as the rotation of the min, with the standard negation

We say that T *can be N-annihilated* when $T_{(N)}$ is also a t-norm. So, the question is: which t-norms can be N-annihilated? The above results show that $T = \min$ is a positive example.

A t-norm T is said to be a *trivial annihilation* (with respect to the strong negation N) if $N(x) = I_T(x, 0)$ holds for all $x \in [0, 1]$. It is easily seen that if a continuous t-norm T is a trivial annihilation then $T_{(N)} = T$.

Two t-norms T, T' are called *N-similar* if $T_{(N)} = T'_{(N)}$. Let T be a continuous non-Archimedean t-norm, and $\langle [a, b]; T_1 \rangle$ be a summand of T. We say that this summand is *in the center* (w.r.t. the strong negation N) if $a = N(b)$.

Theorem 8 ([8])**.** *(a) Let T be a continuous Archimedean t-norm. Then $T_{(N)}$ is a t-norm if and only if $T(x, N(x)) = 0$ holds for all $x \in [0, 1]$.*

(b) Let T be a continuous non-Archimedean t-norm. Then $T_{(N)}$ is a t-norm if and only if

- *either T is N-similar to the minimum,*
- *or T is N-similar to a continuous t-norm which is defined by one trivial annihilation summand in the center.* □

Interestingly enough, the nilpotent minimum can be obtained as the limit of trivially annihilated continuous Archimedean t-norms, as the following result states.

Theorem 9 ([8])**.** *There exists a sequence of continuous Archimedean t-norms T_k $(k = 1, 2, \ldots)$ such that*

$$\lim_{k\to\infty} T_k(x, y) = T^{\mathbf{nM}}(x, y) \qquad (x, y \in [0, 1]).$$

Moreover, for all k, T_k is a trivial annihilation with respect to the standard negation.

The nilpotent minimum was slightly extended in [2] by allowing a weak negation instead of a strong one in the construction. Based on this extension, monoidal t-norm based logics (MTL) were studied also in [2], together with the involutive case (IMTL). Ordinal fuzzy logic, closely related to $T^{\mathbf{nM}}$, and its application to preference modelling was considered in [1]. Properties and applications of the $T^{\mathbf{nM}}$-based implication (called R_0 implication there) were published in [14]. Linked to [2], the equivalence of IMTL logic and NM logic (i.e., nilpotent minimum based logic) was established in [13].

6 Conclusion

In this paper we have presented an overview of some fundamental results on left-continuous t-norms. The origin and basic properties of the very first left-continuous (and not continuous) t-norm called *nilpotent minimum* was recalled in some details. Extensions and general construction methods for left-continuous t-norms were also reviewed from the literature.

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References

1. B. De Baets, F. Esteva, J. Fodor and Ll. Godo, Systems ofordinal fuzzy logic with application to preference modelling, *Fuzzy Sets and Systems* **124** (2001) 353-359.
2. F. Esteva and L. Godo, Monoidal t-norm based logic: towards a logic for left-continuous t-norms, *Fuzzy Sets and Systems* **124** (2001) 271–288.
3. J.C. Fodor, A new look at fuzzy connectives, *Fuzzy Sets and Systems* **57** (1993) 141–148.
4. J.C. Fodor, Contrapositive symmetry of fuzzy implications, *Fuzzy Sets and Systems* **69** (1995) 141–156.
5. J.C. Fodor and M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support*, (Kluwer Academic Publishers, Dordrecht, 1994).
6. U. Höhle, Commutative, residuated l-monoids, in: U. Höhle and E.P. Klement, Eds., *Non-Classical Logics and their Applications to Fuzzy Subsets. A Handbook of the Mathematical Foundations of Fuzzy Set Theory.* (Kluwer Academic Publishers, Boston, 1995).
7. S. Jenei, Geometry of left-continuous t-norms with strong induced negations, *Belg. J. Oper. Res. Statist. Comput. Sci.* **38** (1998) 5–16.
8. S. Jenei, New family of triangular norms via contrapositive symmetrization of residuated implications, *Fuzzy Sets and Systems* **110** (2000) 157–174.
9. S. Jenei, Structure of left-continuous t-norms with strong induced negations. (I) Rotation construction, *J. Appl. Non-Classical Logics* **10** (2000) 83–92.

10. S. Jenei, Structure of left-continuous triangular norms with strong induced negations. (II) Rotation-annihilation construction, *J. Appl. Non-Classical Logics*, **11** (2001) 351–366.
11. P. Smets and P. Magrez, Implications in fuzzy logic, *Int. J. Approximate Reasoning* **1** (1987) 327–347.
12. E.P. Klement, R. Mesiar, and E. Pap, *Triangular Norms*, (Kluwer Academic Publishers, Dordrecht, 2000).
13. D. Pei, On equivalent forms of fuzzy logic systems NM and IMTL, *Fuzzy Sets and Systems* **138** (2003) 187-195.
14. D. Pei. R_0 implication: characteristics and applications, *Fuzzy Sets and Systems* **131** (2002) 297 - 302.
15. P. Perny, *Modélisation, agrégation et exploitation des préférences floues dans une problématique de rangement*, (PhD thesis, Université Paris-Dauphine, Paris, 1992).
16. B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, (North-Holland, New York, 1983).
17. E. Trillas, Sobre funciones de negación en la teoría de conjuntos difusos, *Stochastica* **III** (1979) 47–60.
18. S.-M. Wang, B.-S. Wang and F. Ren, NMŁ, a schematic extension of F.Esteva and L.Godos logic MTL, *Fuzzy Sets and Systems* (to appear)
19. S.-M. Wang, B.-S. Wang and X.-Y. Wang, A characterization of truth-functions in the nilpotent minimum logic, *Fuzzy Sets and Systems* **145** (2004) 253-266.
20. L.A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965) 338–353.