Adaptive Fuzzy Control Design

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Abstract: An application of fuzzy systems to nonlinear system adaptive control design is proposed in this paper. The fuzzy system is constructed to approximate the nonlinear system dynamics. Based on this fuzzy approximation suitable adaptive control laws and appropriate parameter update algorithms for nonlinear uncertain (or unknown) systems are developed to achieve H_∞ *tracking performance. It is shown that the effects of approximation errors and external disturbance can be attenuated to a specific attenuation level using the proposed adaptive fuzzy control scheme. The nonlinear gradient law guarantees the convergence of the training algorithm.*

Keywords: adaptive fuzzy control, Riccati equation, uncertain system, nonlinear systems

1 Introduction

Fuzzy logic controllers are in general considered being applicable to plants that are mathematically poorly understood and where the experienced human operators are available [1]. In indirect adaptive fuzzy control, the fuzzy logic systems are used to model the plant. Then a controller is constructed assuming that the fuzzy logic system approximately represents the true plant.

Feedback linearization techniques for nonlinear control system design have been developed in the last two decades [2], [3]. However, these techniques can only be applied to nonlinear systems whose parameters are known exactly. If the nonlinear system contains unknown or uncertain parameters then the feedback linearization is no longer utilizable. In this situation, the adaptive strategies are used to simplify the problem and to allow a suitable solution. At present, a number of adaptive control design techniques for nonlinear systems based on the feedback linearization can be found in literature [4], [5]. These approaches simplify the nonlinear systems by assuming either linearly or nonlinearly parametrized structures. However, these assumptions are not sufficient for many practical applications. Recently, the fuzzy systems have been employed successfully in the adaptive control design problems of nonlinear systems. According to the universal approximation theorem [6], [7], many important adaptive fuzzy-based control schemes have been developed to incorporate the expert information directly and systematically and various stable performance criteria are guaranteed by theoretical analysis [6], [8]-[12].

In this paper we combine the characteristics of fuzzy systems, the technique of feedback linearization, the adaptive control scheme and the H_∞ optimal control theory with aim to solve the tracking control design problem for nonlinear systems with bounded unknown or uncertain parameters and external disturbances. H_∞ optimal control theory is well known as an efficient tool for robust stabilization and disturbance rejection problems [13], [14].

More specifically, we propose the fuzzy adaptive algorithm equipped with a gradient projection law. The resulting controller performances can be improved by incorporating some linguistic rules describing the plant dynamic behavior.

The paper is organized as follows. First, the problem formulation is presented in Section 2. In Section 3, the adaptive fuzzy control is proposed. Simulation results for the proposed control concept are shown in Section 4. Finally, the paper is concluded in Section 5.

2 Problem Statement

We consider the n-th order nonlinear dynamic single input single output (SISO) system with $n \geq 2$ of the following form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_n &= f(\underline{x}) + g(\underline{x})u + d \\ y &= x_1 \end{aligned} \tag{1}$$

or equivalently

$$\begin{aligned} x^{(n)} &= f\left(x, \dot{x}, \ldots, x^{(n-1)}\right) + g\left(x, \dot{x}, \ldots, x^{(n-1)}\right)u + d \\ y &= x \end{aligned} \tag{2}$$

where $\underline{x} = \left[x_1, x_2, \ldots, x_n\right]^T$ represents the state vector, u is the control input, y and d denote the system output and the external disturbance, respectively. All elements of the state vector $\underline{x}$ are assumed to be available and the external disturbance d is assumed to be bounded but unknown or uncertain. At the begining $f(\underline{x})$ and $g(\underline{x})$ are assumed to be smooth and $g(\underline{x}) \neq 0$ for $\underline{x}$ in certain controllability region $U_c \subset R^n$. Without loss of generality we suppose that $g(\underline{x}) > 0$, but the analysis throughout this paper can be easily tailored to systems with $g(\underline{x}) < 0$. Differentiating the output y with respect to time for n times gives the following input/output form

$$y^{(n)} = f(\underline{x}) + g(\underline{x})u + d \tag{3}$$

Note that the above system has a relative degree of n.

Remark 1. For more general nonlinear system

$$\begin{aligned} \dot{\underline{z}} &= F(\underline{z}) + G(\underline{z})u + d' \\ y &= H(\underline{z}) \end{aligned} \tag{4}$$

where $\underline{z} \in R^n$, $u, v \in R$, $F(\underline{z})$, $G(\underline{z})$ and $H(\underline{z})$ are smooth functions, we say that the system has a relative degree of m if m is the smallest integer such that $L_G L_F^{m-1} H \neq 0$.

We obtain [2]

$$\begin{aligned} y^{(m)} &= L_F^m H + L_G L_F^{m-1} Hu + L_{F+Gu+d'}^{m-1} L_{d'} H \\ &\quad + \sum_{k=1}^{m-1} L_{F+Gu+d'}^{k-1} L_{d'} L_F^{m-k} H \end{aligned} \tag{5}$$

where $L_F(\cdot)$, and $L_G(\cdot)$ denote the Lie derivatives with respect to F and G, respectively. If we let $y = x_1$, then (5) can be rewritten as the input/output form of (3).

If $f(\underline{x})$ and $g(\underline{x})$ are known, a nonlinear tracking control can be obtained. Let y_r be the desired continuous differentiable uniformly bounded trajectory and let

$$e = y - y_r \tag{6}$$

be the tracking error. Then employing the technique of feedback linearization [2] the following suitable control law can be derived to achieve the tracking control goal

$$u = \frac{1}{g(\underline{x})}\left[-f(\underline{x}) + u_a + \nu\right] \tag{7}$$

where u_a is an auxiliary control variable [13, optimal control] yet to be specified and

$$\nu = y_r^{(n)} + k_1\left(y_r^{(n-1)} - y^{(n-1)}\right) + \ldots + k_n\left(y_r - y\right) \tag{8}$$

Note that the coeficients $k_1, \ldots, k_n$ are positive constants to be assigned such that the polynomial $s^n + k_1 s^{n-1} + \ldots + k_n$ is Hurwitz. As a result, the system error dynamic has the following input/output form

$$e^{(n)} + k_1 e^{(n-1)} + \ldots + k_n e = u_a + d \tag{9}$$

which can be represented in space form as

$$\dot{\underline{e}} = \mathbf{\Lambda}_c \underline{e} + \underline{b}_c\left(u_a + d\right) \tag{10}$$

where

$$\mathbf{\Lambda}_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -k_n & -k_{n-1} & & \cdots & & -k_1 \end{bmatrix} \tag{11}$$

$$\underline{b}_c = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T \tag{12}$$

$$\underline{e} = \begin{bmatrix} e & \cdots & e^{(n-2)} & e^{(n-1)} \end{bmatrix}^T \tag{13}$$

The above mentioned design method is useful only if $f(\underline{x})$ and $g(\underline{x})$ are known exactly. If $f(\underline{x})$ and $g(\underline{x})$ are unknown, then adaptive strategies must be employed. Let us now discuss a fuzzy system based adaptive algorithm.

Fist, we employ two fuzzy systems $\hat{f}(\underline{x} \mid \underline{\theta}_f)$ and $\hat{g}(\underline{x} \mid \underline{\theta}_g)$ [15] to approximate (or model) the nonlinear functions $f(\underline{x})$ and $g(\underline{x})$ of the system (1).

In this article is used the set of fuzzy systems with singleton fuzzifier, product inference, centroid defuzzifier, triangular antecendent membership function and singleton consequent membership function with n inputs of $x_i \in \left[c_{x_i} - k_{x_i}, c_{x_i} + k_{x_i}\right]$ for $i = 1, \ldots, n$ and $\bar{u} \in [0,1]$ as the normalized output. The generalized expresion of the class of the fuzzy controllers can be written as

$$\bar{u} = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} N_{i_1 \cdots i_n} x_1^{i_1-1} \cdots x_n^{i_n-1} \tag{14}$$

$$N_{i_1 \cdots i_n} = \frac{\left[\sum_{j_1=1}^{2} \cdots \sum_{j_n=1}^{2} R_{j_1 \cdots j_n} K_{j_1 \cdots j_n} C_{j_1 \cdots j_n}\right]}{2^n \prod_{i=1}^{n} k_{x_i}} \tag{15}$$

$$C_{j_1 \cdots j_n} = \left[\frac{(-1)^{j_1}}{k_{x_1} - (-1)^{j_1} c_{x_1}}\right]^{i_1-1} \cdots \left[\frac{(-1)^{j_n}}{k_{x_n} - (-1)^{j_n} c_{x_n}}\right]^{i_1-1} \tag{16}$$

$$K_{j_1 \cdots j_n} = \left[k_{x_1} - (-1)^{j_1} c_{x_1}\right] \cdots \left[k_{x_n} - (-1)^{j_n} c_{x_n}\right] \tag{17}$$

On the other hand given the coeficients of the explicit form $N_{i_1 \cdots i_n}$ we can reconstruct the rule base from the generalized expression of the class of fuzzy systems [16] by using the following theorem.

Theorem 1: For a class of FLS with singleton fuzzifier, product inference, centroid defuzzifier, triangular antecendent membership function and singleton consequent membership function, i.e. given the coefficients of the explicit form, i.e. $N_{i_1 \cdots i_n}$, the control function can be expressed in terms of fuzzy rules as

$$R_{j_1 \cdots j_n} = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} N_{i_1 \cdots i_n} D_{j_1 \cdots j_n} \tag{18}$$

with

$$D_{j_1 \cdots j_n} = \left[c_{x_1} + (-1)^{j_1} k_{x_1}\right]^{i_1-1} \cdots \left[c_{x_n} + (-1)^{j_n} k_{x_n}\right]^{i_n-1} \tag{19}$$

Proof: The proof is found by directly expanding terms and comparing coefficients. For details, please refer [16].

Therefore, one can express an equation in the form of generalized multilinear equations, such as polynomials, exactly as a rule base of FLS. Theorem 1 is useful in cases where the implementation of an FLS performs inference on a given fuzzy rule base but without any numerical computation capability.

Now, we can express the fuzzy controller in the form of fuzzy IF-THEN rules.

1) For the nonlinear-cancellation fuzzy controller of $f(\underline{x})$

RULE i: IF x_1 is $A_1^{x_1}$ and ... and x_n is $A_1^{x_n}$, THEN $\bar{u}_f = R_i^f$

2) For the nonlinear-cancellation fuzzy controller of $g(\underline{x})$

RULE i: IF x_1 is $A_1^{x_1}$ and ... and x_n is $A_1^{x_n}$, THEN $\bar{u}_g = R_i^g$

The generalized expression of the class of the fuzzy approximators for nonlinear term cancelation with input x can be written as controller for pole-placement

$$\bar{u}_f = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} N_{i_1 \cdots i_n}^{f} x_1^{i_1-1} \cdots x_n^{i_n-1} \tag{20}$$

$$\bar{u}_g = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} N_{i_1 \cdots i_n}^{g} x_1^{i_1-1} \cdots x_n^{i_n-1} \tag{21}$$

So terms for $\hat{f}(\underline{x} \mid \underline{\theta}_f)$ and $\hat{g}(\underline{x} \mid \underline{\theta}_g)$ can be written as

$$\hat{f}(\underline{x} \mid \underline{\theta}_f) = \underline{\theta}_f^T \underline{\omega}_x \tag{22}$$

with $\underline{\theta}_f^T = \left(\underline{k}_{f_b}^T, \underline{k}_{f_c}^T\right)$

and $\underline{\omega}_x^T = \left(\underline{x}^T, \underline{x}_c^T\right)$

and

$$\hat{g}(\underline{x} \mid \underline{\theta}_g) = \underline{\theta}_g^T \underline{\omega}_x \tag{23}$$

with $\underline{\theta}_g^T = \left(\underline{k}_{g_b}^T, \underline{k}_{g_c}^T\right)$

and $\underline{\omega}_x^T = \left(\underline{x}^T, \underline{x}_c^T\right)$

with $\underline{k}_{f_b}^T = \left[k_1^f, \cdots, k_n^f\right]$ and $\underline{k}_{g_b}^T = \left[k_1^g, \cdots, k_n^g\right]$

where

$$k_1^f = 2N_{211\ldots111}^f \qquad k_1^g = 2N_{211\ldots111}^g$$
$$k_2^f = 2N_{121\ldots111}^f \qquad k_2^g = 2N_{121\ldots111}^g$$
$$k_{n-1}^f = 2N_{111\ldots121}^f \qquad k_{n-1}^g = 2N_{111\ldots121}^g$$
$$k_n^f = 2N_{111\ldots112}^f \qquad k_n^g = 2N_{111\ldots112}^g$$

The composite state vector $\underline{x}_c$ and the associated parameter vectors $\underline{k}_{f_c}$, $\underline{k}_{g_c}$ are defined as

$$\underline{x}_c^T = \left(rx_1x_2\ldots x_n, rx_1x_2\ldots x_{n-1}, \cdots, x_{n-1}x_n, 1\right) \tag{24}$$

$$\underline{k}_{f_c}^T = \left(k_{n+1}^f, k_{n+2}^f, \cdots, k_{n+n_c-1}^f, k_{n+n_c}^f\right) \tag{25}$$

$$\underline{k}_{g_c}^T = \left(k_{n+1}^g, k_{n+2}^g, \cdots, k_{n+n_c-1}^g, k_{n+n_c}^g\right) \tag{26}$$

where

$$k_{n+1}^f = 2N_{222\ldots222}^f \qquad k_{n+1}^g = 2N_{222\ldots222}^g$$
$$k_{n+2}^f = 2N_{222\ldots221}^f \qquad k_{n+2}^g = 2N_{222\ldots221}^g$$
$$k_{n+n_c-1}^f = 2N_{111\ldots122}^f \qquad k_{n+n_c-1}^g = 2N_{111\ldots122}^g$$
$$k_{n+n_c}^f = 2N_{111\ldots111}^f \qquad k_{n+n_c}^g = 2N_{111\ldots111}^g$$

with $n_c = 2^{n+1} - (n+1)$.

Let

$$\underline{\theta}_f^* = \arg\min_{\underline{\theta}_f}\max_{\underline{x}}\left|\hat{f}\left(\underline{x}, \underline{\theta}_f\right)\right| \tag{27}$$

$$\underline{\theta}_g^* = \arg\min_{\underline{\theta}_g}\max_{\underline{x}}\left|\hat{g}\left(\underline{x}, \underline{\theta}_g\right)\right| \tag{28}$$

be the best parameter approximation of $\underline{\theta}_f$ and $\underline{\theta}_g$, respectively, and let

$$\underline{\phi}_f = \underline{\theta}_f - \underline{\theta}_f^*,\ \underline{\phi}_g = \underline{\theta}_g - \underline{\theta}_g^* \tag{29}$$

be the corresponding parameter estimation errors. Then using the certainty equivalence principle [5] the following fuzzy adaptive control law is derived

$$u = \frac{1}{\hat{g}\left(\underline{x}, \underline{\theta}_g\right)}\left[-\hat{f}\left(\underline{x}, \underline{\theta}_f\right) + u_a + \nu\right] \tag{30}$$

Applying this control law to the system (1) yields

$$\begin{aligned} y^{(n)} &= f\left(\underline{x}\right) + g\left(\underline{x}\right)u + d \\ &= f\left(\underline{x}\right) + g\left(\underline{x}\right)u - \hat{g}\left(\underline{x}, \underline{\theta}_g\right)u + \hat{g}\left(\underline{x}, \underline{\theta}_g\right)u + d \\ &= \left(f\left(\underline{x}\right) - \hat{f}\left(\underline{x}, \underline{\theta}_f\right)\right) + \left(g\left(\underline{x}\right) - \hat{g}\left(\underline{x}, \underline{\theta}_g\right)\right)u + u_a + \nu + d \end{aligned} \tag{31}$$

By means of the best approximation (using the universal approximation theorem [6], [7], [17]), the above equation can be rewritten as

$$\begin{aligned} \underline{\dot{e}} = \mathbf{\Lambda}_c \underline{e} + \underline{b}_c &\left[\left(\hat{f}\left(\underline{x}\middle|\underline{\theta}_f\right) - \hat{f}\left(\underline{x}\middle|\underline{\theta}_f^*\right)\right)\right. \\ &\left. + \left(\hat{g}\left(\underline{x}\middle|\underline{\theta}_g\right) - \hat{g}\left(\underline{x}\middle|\underline{\theta}_g^*\right)\right)u\right] + \underline{b}_c\left[u_a + w\right] \end{aligned} \tag{32}$$

where

$$w = \left(f\left(\underline{x}\right) - \hat{f}\left(\underline{x}\middle|\underline{\theta}_f^*\right)\right) + \left(g\left(\underline{x}\right) - \hat{g}\left(\underline{x}\middle|\underline{\theta}_g^*\right)\right)u + d \tag{33}$$

In order to track the desired signal y_r, the fuzzy systems $\hat{f}\left(\underline{x} \mid \underline{\theta}_f\right)$ and $\hat{g}\left(\underline{x} \mid \underline{\theta}_g\right)$ should be trained to achieve $\hat{f}\left(\underline{x} \mid \underline{\theta}_f^*\right)$ and $\hat{g}\left(\underline{x} \mid \underline{\theta}_g^*\right)$ respectively, so that the term

$$\left[\left(\hat{f}\left(\underline{x}\middle|\underline{\theta}_f^*\right) - \hat{f}\left(\underline{x}\middle|\underline{\theta}_f\right)\right) + \left(\hat{g}\left(\underline{x}\middle|\underline{\theta}_g^*\right) - \hat{g}\left(\underline{x}\middle|\underline{\theta}_g\right)\right)u\right] = 0 \tag{34}$$

The effect of w, denoting the sum of the approximation errors and external disturbances in the above error dynamics equation, is crucial and will be attenuated by u_a. Fortunately, the H_∞ control design approach [12] can be efficiently employed to attenuate the effect of w in the error dynamic system (32). Our solution utilizes the concept of H_∞ tracking performance to deal with the robust adaptive tracking control problem. Then, the problem we are investigating becomes that of finding an adaptive scheme for u_a, $\underline{\theta}_f$ and $\underline{\theta}_g$ to achieve the following H_∞ tracking performance [12], [18]

$$\int_0^T \underline{e}^T \mathbf{Q}\underline{e}dt \le \underline{e}^T(0)\mathbf{P}\underline{e}(0) + \frac{1}{\gamma_f}\underline{\phi}_f^T(0)\underline{\phi}_f(0) + \frac{1}{\gamma_g}\underline{\phi}_g^T(0)\underline{\phi}_g(0) + \rho^2\int_0^T w^T w dt \tag{35}$$

for appropriate positive definite weighting matrices $\mathbf{Q} = \mathbf{Q}^T$, $\mathbf{P} = \mathbf{P}^T$, positive weighting factors γ_f and γ_g, prescribed attenuation level ρ and time T. In the inequality (35), T is the terminal time of the control effort and can take any finite or infinite value. The initial errors $\underline{e}(0)$, $\underline{\phi}_f(0)$ and $\underline{\phi}_g(0)$ are considered to be free of the disturbances which can influence the tracking error $\underline{e}$. The physical meaning of (35) is that the effect of w on the tracking error $\underline{e}$ is attenuated by a factor ρ from an energy point of view. In general ρ is a small value less then 1.

Remark 2. From the above analysis, we note the following

- In the case of $\rho \to \infty$, (35) becomes the H_2 tracking performance without consideration of disturbance attenuation [12].
- The weighting factors γ_f and γ_g are called the adaptive gains of $\underline{\theta}_f$ and $\underline{\theta}_g$ update algorithms, respectively. It can be seen from (35), that the larger the value of γ_f, the smaller the effect of $\underline{\phi}_f(0)$ on the tracking error $\underline{e}$. Similar argument for $\underline{\phi}_g(0)$ can also be made. However, it is easy to see that large values of γ_f or γ_g will cause $\underline{\theta}_f$ and $\underline{\theta}_g$ to change rapidly. This may be harmful to the system.

3 Adaptive Fuzzy Control

The following theorem gives the solution of the adaptive H_∞ tracking problem for the SISO nonlinear system (1).

Theorem 2. Consider the nonlinear system (1) with unknown or uncertain $f(\underline{x})$ and $g(\underline{x})$. If the following adaptive fuzzy control law is adopted

$$u = \frac{1}{\hat{g}(\underline{x}|\underline{\theta}_g)}\left[-\hat{f}(\underline{x}|\underline{\theta}_f) + u_a + \nu\right] \tag{36}$$

with

$$u_a = -\frac{1}{r}\underline{b}_c^T \mathbf{P}\underline{e} \tag{37}$$

$$\dot{\underline{\theta}}_f = \gamma_f \underline{e}^T \mathbf{P}\underline{b}_c \underline{\omega}_x \tag{38}$$

$$\dot{\underline{\theta}}_g = \gamma_g \underline{e}^T \mathbf{P}\underline{b}_c \underline{\omega}_x u \tag{39}$$

where the signal ν is given by (8), r is a positive scalar, the fuzzy systems $\hat{f}\left(\underline{x} \mid \underline{\theta}_f\right)$ and $\hat{g}\left(\underline{x} \mid \underline{\theta}_g\right)$ are defined by (22), (23) and the positive definite matrix $\mathbf{P} = \mathbf{P}^T$ is the solution of the Riccati-like equation

$$\mathbf{\Lambda}_c \mathbf{P}^T + \mathbf{P}\mathbf{\Lambda}_c + \mathbf{Q} - \frac{2}{r}\mathbf{P}\underline{b}_c \underline{b}_c^T \mathbf{P} + \frac{1}{\rho^2}\mathbf{P}\underline{b}_c \underline{b}_c^T \mathbf{P} = 0 \tag{40}$$

then the H_∞ tracking performance in (35) is achieved for a prescribed attenuation level ρ.

Proof. Consider the Lyapunov function in the form

$$V = \frac{1}{2}\underline{e}^T \mathbf{P}\underline{e} + \frac{1}{2\gamma_f}\underline{\phi}_f^T \underline{\phi}_f + \frac{1}{2\gamma_g}\underline{\phi}_g^T \underline{\phi}_g \tag{41}$$

Taking the time derivative of V along the trajectory of the error dynamic (8), we have

$$\begin{aligned}
\dot{V} &= \frac{1}{2}\dot{\underline{e}}^T \mathbf{P}\underline{e} + \frac{1}{2}\underline{e}^T \mathbf{P}\dot{\underline{e}} + \frac{1}{2\gamma_f}\dot{\underline{\phi}}_f^T \underline{\phi}_f + \frac{1}{2\gamma_f}\underline{\phi}_f^T \dot{\underline{\phi}}_f \\
&\quad + \frac{1}{2\gamma_g}\dot{\underline{\phi}}_g^T \underline{\phi}_g + \frac{1}{2\gamma_g}\underline{\phi}_g^T \dot{\underline{\phi}}_g \\
&= \frac{1}{2}\underline{e}^T \mathbf{\Lambda}_c^T \mathbf{P}\underline{e} + \frac{1}{2}\underline{e}^T \mathbf{P}\mathbf{\Lambda}_c \underline{e} + \frac{1}{2}\underline{e}^T \mathbf{P}\underline{b}_c u_a + \frac{1}{2}u_a^T \underline{b}_c^T \mathbf{P}\underline{e} \\
&\quad + \frac{1}{2}\left[\left(\hat{f}\left(\underline{x}\middle|\underline{\theta}_f^*\right) - \hat{f}\left(\underline{x}\middle|\underline{\theta}_f\right)\right)^T + \left(\hat{g}\left(\underline{x}\middle|\underline{\theta}_g^*\right) - \hat{g}\left(\underline{x}\middle|\underline{\theta}_g\right)\right)^T u\right]\underline{b}_c^T \mathbf{P}\underline{e} \\
&\quad + \frac{1}{2}\underline{e}^T \mathbf{P}\underline{b}_c \left[\left(\hat{f}\left(\underline{x}\middle|\underline{\theta}_f^*\right) - \hat{f}\left(\underline{x}\middle|\underline{\theta}_f\right)\right) + \left(\hat{g}\left(\underline{x}\middle|\underline{\theta}_g^*\right) - \hat{g}\left(\underline{x}\middle|\underline{\theta}_g\right)\right)u\right] \\
&\quad + \frac{1}{2}\underline{e}^T \mathbf{P}\underline{b}_c w + \frac{1}{2}w^T \underline{b}_c^T \mathbf{P}\underline{e} + \frac{1}{2\gamma_f}\dot{\underline{\phi}}_f^T \underline{\phi}_f + \frac{1}{2\gamma_f}\underline{\phi}_f^T \dot{\underline{\phi}}_f \\
&\quad + \frac{1}{2\gamma_g}\dot{\underline{\phi}}_g^T \underline{\phi}_g + \frac{1}{2\gamma_g}\underline{\phi}_g^T \dot{\underline{\phi}}_g
\end{aligned} \tag{42}$$

Using (22), (23), (28), (39) and the fact that

$$\dot{\underline{\phi}}_f = \dot{\underline{\theta}}_f, \ \dot{\underline{\phi}}_g = \dot{\underline{\theta}}_g \tag{43}$$

we obtain

$$\begin{aligned}\dot{V} = {} & \frac{1}{2}\underline{e}^T\left[\mathbf{\Lambda}_c^T\mathbf{P} + \mathbf{P}\mathbf{\Lambda}_c - \frac{2}{r}\mathbf{P}\underline{b}_c\underline{b}_c^T\mathbf{P}\right]\underline{e} \\ & -\left[\underline{\phi}_f^T\underline{\omega}_x + u\underline{\phi}_g^T\underline{\omega}_x\right]\underline{b}_c^T\mathbf{P}\underline{e} \\ & +\frac{1}{\gamma_f}\underline{\phi}_f^T\dot{\underline{\phi}}_f + \frac{1}{\gamma_g}\underline{\phi}_g^T\dot{\underline{\phi}}_g + \frac{1}{2}w^T\underline{b}_c^T\mathbf{P}\underline{e} + \frac{1}{2}\underline{e}^T\mathbf{P}\underline{b}_c w\end{aligned} \tag{44}$$

Introducing (40) into (44) implies

$$\begin{aligned}\dot{V} = {} & -\frac{1}{2}\underline{e}^T\mathbf{Q}\underline{e} - \frac{1}{2\rho^2}\underline{e}^T\mathbf{P}\underline{b}_c\underline{b}_c^T\mathbf{P}\underline{e} \\ & -\underline{\phi}_f^T\left(\underline{\omega}_x\underline{b}_c^T\mathbf{P}\underline{e} - \frac{1}{\gamma_f}\dot{\underline{\theta}}_f\right) \\ & -\underline{\phi}_g^T\left(u\underline{\omega}_x\underline{b}_c^T\mathbf{P}\underline{e} - \frac{1}{\gamma_g}\dot{\underline{\theta}}_g\right) \\ & +\frac{1}{2}w^T\underline{b}_c^T\mathbf{P}\underline{e} + \frac{1}{2}\underline{e}^T\mathbf{P}\underline{b}_c w\end{aligned} \tag{45}$$

Using the adaptation laws (38) and (39), equation (45) can be rewritten into the form

$$\begin{aligned}\dot{V} = {} & -\frac{1}{2}\left(\frac{1}{\rho}\underline{b}_c^T\mathbf{P}\underline{e} - \rho w\right)^T\left(\frac{1}{\rho}\underline{b}_c^T\mathbf{P}\underline{e} - \rho w\right) + \frac{1}{2}\rho^2 w^T w \\ & -\frac{1}{2}\underline{e}^T\mathbf{Q}\underline{e} \\ \le {} & -\frac{1}{2}\underline{e}^T\mathbf{Q}\underline{e} + \frac{1}{2}\rho^2 w^T w\end{aligned} \tag{46}$$

Integrating the above equation from 0 to T yields

$$V(T) - V(0) \le -\frac{1}{2}\int_0^T \underline{e}^T\mathbf{Q}\underline{e}dt + \frac{1}{2}\rho^2\int_0^T w^T w dt \tag{47}$$

Since $\mathrm{V}(\mathrm{T}) \geq 0$ inequality (47) implies that

$$\int_0^T \underline{e}^T \mathbf{Q} \underline{e} dt \leq \underline{e}^T(0) \mathbf{P} \underline{e}(0) + \frac{1}{\gamma_f} \underline{\phi}_f^T(0) \underline{\phi}_f(0) + \frac{1}{\gamma_g} \underline{\phi}_g^T(0) \underline{\phi}_g(0) + \rho^2 \int_0^T w^T w dt \tag{48}$$

This is the H_∞ tracking performance of (35).

Q.E.D.

Remark 3. If w is bounded, then the H_∞ tracking performance will be improved as the prescribed attenuation level ρ is decreased.

Remark 4. The Riccati-like equation (40) can be rewritten into the form

$$\mathbf{P}\boldsymbol{\Lambda}_c + \boldsymbol{\Lambda}_c^T \mathbf{P} + \mathbf{P} \underline{b}_c \left(\frac{1}{\rho^2} - \frac{2}{r} \right) \underline{b}_c^T \mathbf{P} + \mathbf{Q} = 0 \tag{49}$$

As it follows from Theorem 1, the sufficient condition for the H_∞ tracking performance existence for the nonlinear system with adaptive fuzzy control law (37)-(39) is that the solution **P** of (40) must be positive definite and symmetric. It can be shown that in order to achieve this requirement the following condition must be satisfied [12]

$$2\rho^2 \geq r \tag{50}$$

i.e., if the inequality (50) is satisfied, then for the nonlinear system (1) the H_∞ tracking performance with the prescribed attenuation level ρ can always be achieved via the adaptive fuzzy control (37)-(39). In general, as ρ is decreased r must be decreased in order to satisfy the inequality (50). However, (37) implies that the control variable u_a must be increased to attenuate w to the desired level ρ. Thus, there is a tradeoff between the H_∞ performance and the control magnitude.

4 Simulation Example

Example 1

The above described adaptive fuzzy control algorithm will now be evaluated using the inverted pendulum system depicted in Fig. 1.

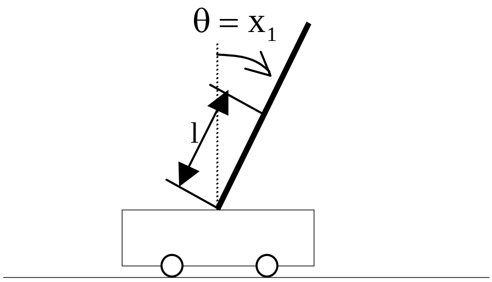

Figure 1
The inverted pendulum system

Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. The dynamic equation of the inverted pendulum is given by [6]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g\sin x_1 - \dfrac{mlx_2^2\cos(x_1)\sin(x_1)}{m_c+m}}{l\left(\dfrac{4}{3} - \dfrac{m\cos^2(x_1)}{m_c+m}\right)} \\ &\quad + \frac{\dfrac{\cos(x_1)}{m_c+m}}{l\left(\dfrac{4}{3} - \dfrac{m\cos^2(x_1)}{m_c+m}\right)} u_c + d \\ y &= x_1 \end{aligned} \tag{51}$$

where g is the acceleration due to gravity, m_c denotes the mass of the cart, m is the mass of the pole, l is the half-length of the pole, the force u_c represents the control signal and d is the external disturbance. In simulations following parameter values are used: $m_c = 1\text{Kg}$, $m = 0.1\text{Kg}$ and $l = 0.5\text{m}$. The reference signal is assumed to be $y_r(t) = (\pi/30)\sin(t)$ and an external disturbance $d(t) = 0.1\sin(t)$.

If we require

$$|\underline{x}| \le \frac{\pi}{6},\ |u| \le 180 \tag{52}$$

and substitute the functions sin(.) and cos(.) by their bounds, we can determine the bounds

$$f^M(x_1, x_2) = 15.78 + 0.366x_2^2 \tag{53}$$

$$g^M(x_1, x_2) = 1.46,\ g_m(x_1, x_2) = 1.12 \tag{54}$$

$k_1 = 2$, $k_2 = 1$ and $\mathbf{Q} = \mathrm{diag}(10,10)$ are set. In order to simplify further calculations $r = 2\rho^2$ is chosen. Then the algebraic Riccati equation solution is $\mathbf{P} = \begin{bmatrix} 15 & 5 \\ 5 & 5 \end{bmatrix}$ and $\lambda_{min}(\mathbf{P}) = 2.93$. Five Gaussian membership functions for both x_1 and x_2 (i=1,2) are selected to cover the whole universe of discourse

$$\mu_{F_i^1}(x_i) = \exp\left(-\left(\frac{x_i - \pi/6}{\pi/24}\right)^2\right) \tag{55}$$

$$\mu_{F_i^2}(x_i) = \exp\left(-\left(\frac{x_i - \pi/12}{\pi/24}\right)^2\right) \tag{56}$$

$$\mu_{F_i^3}(x_i) = \exp\left(-\left(\frac{x_i}{\pi/24}\right)^2\right) \tag{57}$$

$$\mu_{F_i^4}(x_i) = \exp\left(-\left(\frac{x_i + \pi/12}{\pi/24}\right)^2\right) \tag{58}$$

$$\mu_{F_i^5}(x_i) = \exp\left(-\left(\frac{x_i + \pi/6}{\pi/24}\right)^2\right) \tag{59}$$

Using the method of trial and errors $\gamma_f = 50$ and $\gamma_g = 1$ are chosen. The pendulum initial position is chosen as far as possible $\left(\theta(0) = x_1 = \pi/12\right)$ to emphasize the efficiency of our algorithm.

Two cases have been considered in order to show the influence of the linguistic rules incorporation into the control law:

Case one: the initial values of $\underline{\theta}_f$ and $\underline{\theta}_g$ are chosen arbitrarily.

Case two: the initial values of $\underline{\theta}_f$ and $\underline{\theta}_g$ are deduced from the fuzzy rules describing the system dynamic behavior. For example, if we consider the unforced system, i.e. $u_c = 0$, the acceleration is equal to $f(x_1, x_2)$. So intuitively we can state:

"The bigger is x_1, the larger is $f(x_1, x_2)$".

Transforming this fuzzy information into a fuzzy rule we obtain

$R_f^{(1)}$: IF x_1 is F_1^5 and x_2 is F_2^5, THEN $f(x_1, x_2)$ is "Positive Big"

where "Positive Big" is a fuzzy set whose membership function is $\mu_{F_i^l}(x_i)$ given by (55)-(59). The acceleration is proportional to the gravity, i.e. $f(x_1, x_2) \cong \alpha \sin(x_1)$, where α is a constant. As $f(x_1, x_2)$ achieves its maximum at $x_1 = \pi/2$, using (53) we obtain $\alpha \cong 16$. The resulting set of 25 fuzzy rules characterizing $f(x_1, x_2)$ is given in Tab. 1.

Table 1
Linguistic rules for $f(x_1, x_2)$

$f(x_1, x_2)$			x_1				
			F_1^1	F_1^2	F_1^3	F_1^4	F_1^5
			$-\frac{\pi}{6}$	$-\frac{\pi}{12}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$
x_2	F_2^1	$-\frac{\pi}{6}$	-8	-4	0	4	8
	F_2^2	$-\frac{\pi}{12}$	-8	-4	0	4	8
	F_2^3	0	-8	-4	0	4	8
	F_2^4	$\frac{\pi}{12}$	-8	-4	0	4	8
	F_2^5	$\frac{\pi}{6}$	-8	-4	0	4	8

Now the following observation is used to determine the fuzzy rules for $g(x_1, x_2)$:

"The smaller is x_1, the larger is $g(x_1, x_2)$".

Similarly to the case of $f(x_1, x_2)$ and based on the bounds (53)-(54) this observation can be quantified into the 25 fuzzy rules summarized in Tab. 2.

Table 2
Linguistic rules for $g(x_1, x_2)$

$g(x_1, x_2)$			x_1				
			F_1^1	F_1^2	F_1^3	F_1^4	F_1^5
			$-\frac{\pi}{6}$	$-\frac{\pi}{12}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$
x_2	F_2^1	$-\frac{\pi}{6}$	1.26	1.36	1.46	1.36	1.26
	F_2^2	$-\frac{\pi}{12}$	1.26	1.36	1.46	1.36	1.26
	F_2^3	0	1.26	1.36	1.46	1.36	1.26
	F_2^4	$\frac{\pi}{12}$	1.26	1.36	1.46	1.36	1.26
	F_2^5	$\frac{\pi}{6}$	1.26	1.36	1.46	1.36	1.26

To obtain the same tracking performances the attenuation level ρ is equal to 0.2 in the first case and to 0.8 in the second one.

The tracking performance of both cases for a sinusoidal trajectory is illustrated in Fig. 2.

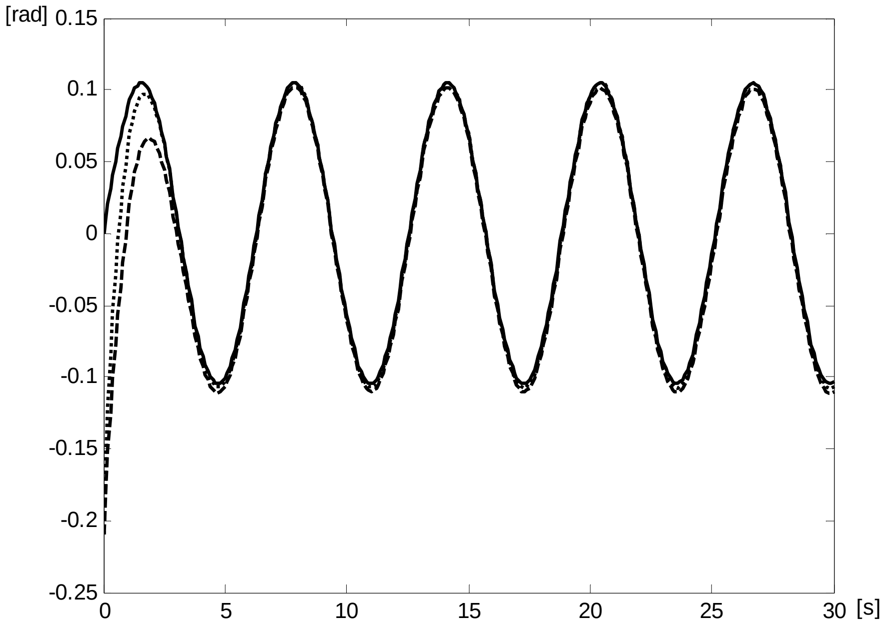

Figure 2

The state x_1 in case 1(dashed line), in case 2 (dotted line) and desired value

$y_r(t)$ (solid line) for $\underline{x}(\underline{0}) = (\pi/12, 0)^T$

Example 2

In this example, we apply the adaptive fuzzy controller to the system

$$y'' + \frac{1}{0.25 + y} y' + 1.7y - 0.5u = 0 \tag{60}$$

Define six fuzzy sets over interval <-10, 10> with labels N3, N2, N1, P1, P2, P3. The membership functions are

$$\mu_{N3}(x) = \frac{1}{1 + e^{5(x+2)}} \tag{61}$$

$$\mu_{N2}(x) = \frac{1}{e^{(x+1.5)^2}} \tag{62}$$

$$\mu_{N1}(x) = \frac{1}{e^{(x+0.5)^2}} \tag{63}$$

$$\mu_{P1}(x) = \frac{1}{e^{(x-0.5)^2}} \tag{64}$$

$$\mu_{P2}(x) = \frac{1}{e^{(x-1.5)^2}} \tag{65}$$

$$\mu_{P3}(x) = \frac{1}{1+e^{-5(x-2)}} \tag{66}$$

The reference model is assumed to be

$$M(s) = \frac{1}{s^2 + 2s + 1} \tag{67}$$

and the reference signal is the series of jumps with variant magnitude.

We choose $\mathbf{P} = \begin{bmatrix} 50 & 30 \\ 30 & 20 \end{bmatrix}$, $k_1 = 2$, $k_2 = 1$, and $\lambda_{min}(\mathbf{P}) = 1.52$. To satisfy the constraint related to $|\underline{x}|$ we choose $\rho = 0.01$.

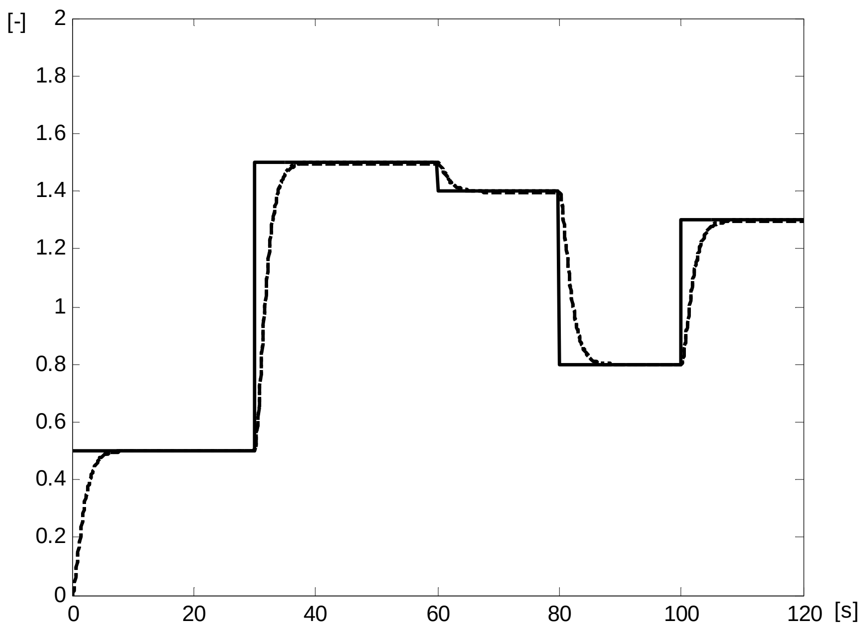

Figure 3
The state x_1 (dashed line), its desired reference model value $y_m(t)$ (dotted solid line) and reference signal (solid line)

At 75th second of simulation the system (60) was switched to another system

$$y''' + 5y'' + \left[\frac{1}{(0.25+y)^2} - 1.7\right]y' + y - 5u = 0 \tag{68}$$

All initial states have been set to zero $y(0) = y'(0) = y''(0) = y'''(0) = 0$.

As it can be seen from Fig. 3, the simulation results confirm good adaptation capability of the proposed control system. The system dynamic changes are in particular manifested by changes of control input signal (Fig. 4).

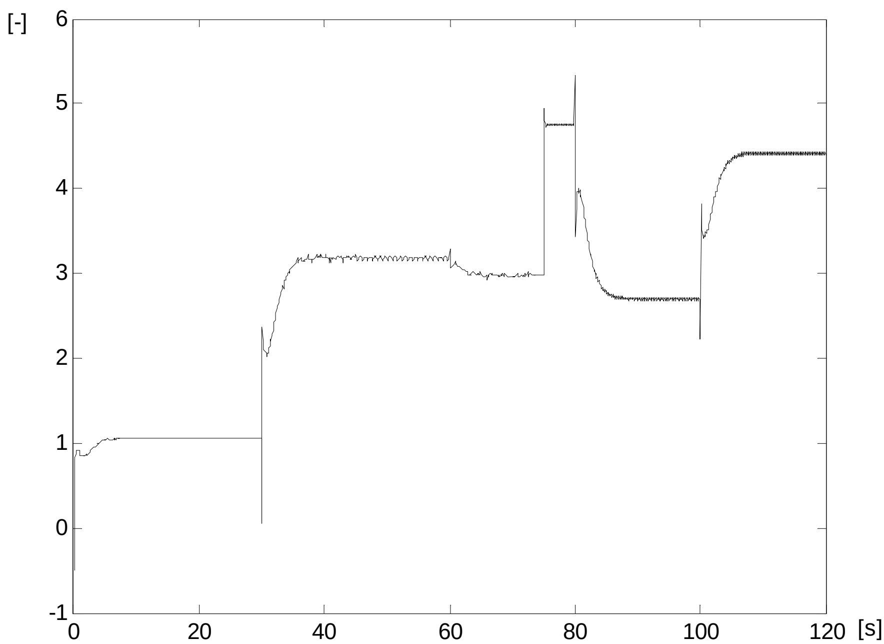

Figure 4
Control signal

Conclusions

In this paper the adaptive fuzzy controller has been proposed for the class of nonlinear systems subject to large uncertainties or to unknown variations in the parameters and the structure of the plant.

The proposed adaptive control scheme has involved both fuzzy systems and H_∞ control. The adaptive fuzzy systems can be considered as a rough tuning control for approximation of the nonlinear system and the H_∞ control can be considered as a fine-tuning control used to filter the approximation errors and external disturbances. Therefore, the proposed adaptive algorithm will be useful for the unknown (or uncertain) nonlinear system control design. The simulation results show that approximation errors and external disturbances can be successfully attenuated using the proposed control design method within a desired attenuation level, i.e. H_∞ tracking performance is achieved.

Further work is under investigation to apply the proposed robust adaptive algorithm to multi input multi output (MIMO) systems.

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