A recursive solution concept for multichoice games

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Abstract: We propose a new axiomatization of the Shapley value for cooperative games, where symmetry and efficiency can be discarded and replaced with new natural axioms. From any game, an excluded-player game is built by discarding all coalitions that contain a fixed player. Then it is shown that the Shapley value is the unique value satisfying the linearity axiom, the nullity axiom, the excluded-null-player axiom, and the equity axiom. In the second part, by generalizing the above material, the Shapley value for multichoice games is worked out.

Keywords: Shapley value, multichoice games, equity, generalized nullity axiom

1 Introduction

The value or solution concept of a game is a key concept in cooperative game theory, since it defines a rational imputation given to the players if they join the grand coalition. In this respect, the Shapley value remains the best known solution concept [11], and it has been axiomatized by many authors in various ways (see especially Weber [13], or the survey by Monderer and Samet [8]).

If the definition and axiomatization of the Shapley value is well established for classical cooperative TU-games, the situation is less clear when considering variants of classical TU-games, like multichoice games [7], games in partition function form [12], etc. In this paper, we focus on multichoice games, where players are allowed to have several (and totally ordered) levels of participation. Hence, a solution for multichoice games assigns a numerical value to each possible participation level and to each player. The original proposal of Hsiao and Raghavan [7] for the Shapley value has been, up to our knowledge, scarcely used due to its complexity. Another proposal is due to Faigle and Kern [5], and compared to the former one by Branzei et al. [3], and also by the authors [6]. The value proposed by Faigle and Kern, although elegant but still with a very high computational complexity, is more rooted in combinatorics than in game theory, and takes as a basis the expression of the Shapley value using maximal chains in the lattice of coalitions. In [6], we proposed an alternative view having the same complexity than the usual Shapley value for classical TU-games. It turned out that our value is identical to the egalitarian value proposed by Peters and Zank [10], although they use different axioms and impose some restrictions (namely, all players should have the same set of participation levels).

Although close to the axiomatization proposed by Weber for classical TU-games, our axiomatization in [6] suffered from a complex symmetry axiom, hard to interpret, the fundamental problem there being that the classical notion of symmetry among players cannot hold since two different players may have a different set of participation levels (note that this difficulty was avoided by Peters and Zank, since they considered multichoice games with all players having the same set of participation levels).

In this paper, we propose a new axiomatization for the so-called egalitarian value, which is based essentially on carriers and on a recursive scheme, and which does not make use of a symmetry axiom. In Section 3, we present the main ideas applied on classical TU-games, and we come up with a very simple and natural axiomatization using linearity, a nullity axiom which uses also carriers, and an equity axiom stating that the sharing should be uniform and efficient for the unanimity game based on the grand coalition (this is in fact a very weak version of the efficiency axiom). In Section 4, the same process is applied to multichoice games. An additional axiom (called decreased level axiom) is used, to take into account the case where a player does not participate at the highest level.

In the sequel, $\mathbb{N}$ refers to the set of positive integers. In order to avoid a heavy notation, we will often omit braces for subsets, by writing i instead of $\{i\}$ or 123 for $\{1, 2, 3\}$. Furthermore, cardinalities of subsets $S, T, \dots$ will be denoted by the corresponding lower case letters $s, t, \dots$

2 Mathematical background

We begin by recalling necessary material on lattices (a good introduction on lattices can be found in [4]), in a finite setting. A *lattice* is a set L endowed with a partial order $\leq$ such that for any $x, y \in L$ their least upper bound $x \vee y$ and greatest lower bound $x \wedge y$ always exist. For finite lattices, the greatest element of L (denoted $\top$) and least element $\perp$ always exist. x *covers* y (denoted $x \succ y$) if $x > y$ and there is no z such that $x > z > y$. A *ranked lattice* is a pair (L, r), where L is a lattice and the rank function $r : L \to \mathbb{N}$ satisfies the property that $r(y) = r(x)+1$ whenever y covers x in L. The lattice is *distributive* if $\vee, \wedge$ obey distributivity. An element $j \in L$ is *join-irreducible* if it cannot be expressed as a supremum of other elements. Equivalently j is join-irreducible if it covers only one element. The set of all join-irreducible elements of L is denoted $\mathcal{J}(L)$.

An important property is that in a distributive lattice, any element x can be written as an irredundant supremum of join-irreducible elements in a unique way (this is called the *minimal decomposition* of x). We denote by $\eta^*(x)$ the set of join-irreducible elements in the minimal decomposition of x, and we denote by $\eta(x)$ the *normal decomposition* of x, defined as the set of join-irreducible elements smaller or equal to x, i.e., $\eta(x) := \{j \in \mathcal{J}(L) \mid j \leq x\}$. Let us rephrase differently the above result. We say that $Q \subseteq L$ is a *downset* of L if $x \in Q$ and $y \leq x$ imply $y \in Q$. For any subset P of L, we denote by $\mathcal{O}(P)$ the set of all downsets of P. Then, by Birkhoff's theorem [2], the mapping η is an isomorphism of L onto $\mathcal{O}(\mathcal{J}(L))$.

Given lattices $(L_1, \leq_1), \dots, (L_n, \leq_n)$, the product lattice $L = L_1 \times \cdots \times L_n$ is endowed with the product order $\leq$ of $\leq_1, \dots, \leq_n$ in the usual sense. Elements of L can be written in their vector form $(x_1, \dots, x_n)$. The set L_{-i} denotes $\prod_{k \neq i} L_k$ if $n > 1$, and the singleton set $\{()\}$ otherwise. By this way, for any vector x, $\big((), x\big)$ simply denotes x. All join-irreducible elements of L are of the form $(\perp_1, \dots, \perp_{i-1}, j_i, \perp_{i+1}, \dots, \perp_n)$, for some i and some join-irreducible element j_i of L_i. A *vertex* of L is any element whose components are either top or bottom. We denote $\Gamma(L)$ the set of vertices of L.

3 A new axiomatization of the Shapley value for classical cooperative games

In the whole paper, we consider an infinite denumerable set Ω, the universe of players. As usual, a *game* on Ω is a set function $v : 2^{\Omega} \to \mathbb{R}$ such that $v(\emptyset) = 0$, which assigns to each *coalition* $S \subseteq \Omega$ its *worth* $v(S)$. We denote by 2^{Ω} (power set of Ω) the set of coalitions. In this section, we focus on the particular case of *classical cooperative games*, that is to say, each player has the only choice to cooperate or not.

A set $N \subseteq \Omega$ is said to be a *carrier* of a game v when for all $S \subseteq \Omega$, $v(S) = v(N \cap S)$. Thus a game v with carrier $N \subseteq \Omega$ is completely defined by the knowledge of the coefficients $\{v(S)\}_{S \subseteq N}$ and the players outside N have no influence on the game since they do not contribute to any coalition. In this paper, we restrict our attention to finite games, that is to say, games that posses a finite carrier N with n elements. We denote by $\mathcal{G}(N)$ the set of games with the finite carrier N. For the sake of clarity, and to avoid any ambiguity, the domain of $v \in \mathcal{G}(N)$ will be restricted to the elements of 2^N. $\mathcal{G}$ denotes the set of all finite games:

$$\mathcal{G} := \{\mathcal{G}(N) \mid N \subseteq \Omega, n \in \mathbb{N}\}.$$

Identity games of $\mathcal{G}(N)$ are particular games defined by

$$\forall S \subseteq N \setminus \{\emptyset\}, \quad \delta_S(T) := \begin{cases} 1 \text{ if } T = S, \\ 0 \text{ otherwise.} \end{cases}$$

A *value* on $\mathcal{G}(N)$ is a function $\Phi : \mathcal{G}(N) \times N \to \mathbb{R}$ that assigns to every player i in a game $v \in \mathcal{G}(N)$ his prospect $\Phi(v, i)$ for playing the game. For instance, the Shapley value [11] for cooperative games Φ_{Sh} is defined by

$$\forall v \in \mathcal{G}(N), \forall i \in N,$$
$$\Phi_{Sh}(v, i) := \sum_{S \subseteq N \setminus i} \frac{s!\,(n-s-1)!}{n!}\,(v(S \cup i) - v(S)). \tag{1}$$

The axiomatization is well-known. Φ_{Sh} is the sole value given on $\mathcal{G}(N)$ satisfying (see also Weber [13]):

Linearity (L): for any $i \in N$, $\Phi(v, i)$ is linear w.r.t the variable v.

Player $i \in N$ is said to be *null* for v if $\forall S \subseteq N \setminus i$, $v(S \cup i) = v(S)$.

Nullity (N): for any game $v \in \mathcal{G}(N)$ and any $i \in N$ null for v, $\Phi(v, i) = 0$.

For any permutation σ on N, we denote $v \circ \sigma$ the game defined by $v \circ \sigma(S) := v(\sigma(S)), \ \forall S \in 2^N$.

Symmetry (S): for any permutation σ on N, any game $v \in \mathcal{G}(N)$ and any $i \in N$, $\Phi(v, \sigma(i)) = \Phi(v \circ \sigma, i)$.

This means that Φ must not depend on the labelling of the players.

Efficiency (E): for any game $v \in \mathcal{G}(N)$, $\sum_{i \in N} \Phi(v, i) = v(N)$.

That is to say, the values of the players must be shared in proportion of the overall resources $v(N)$.

We now introduce a new axiomatization of the Shapley value for classical cooperative games. For any game $v \in \mathcal{G}(N)$ and any coalition $S \in 2^N$, we denote by $v^S \in \mathcal{G}(S)$ the *restricted game v to the power set of S*. For $i \in N$, v^{-i} denotes the restricted game $v^{N \setminus i}$. Let us consider the following axioms for values on $\mathcal{G}$.

Excluded-null-player (ENP): for any finite set $N \subseteq \Omega$ and any game $v \in \mathcal{G}(N)$, if $i \in N$ is null for v,

$$\forall j \in N \setminus i, \quad \Phi(v, j) = \Phi(v^{-i}, j).$$

This simply means that if a null player leaves the game, then other players should keep the same value in the associated restricted game. Note that this axiom completes in a certain sense the above axiom **(N)** since the former deals with null players whereas the latter addresses the others. Therefore, one can merge **(N)** and **(ENP)**:

Generalized nullity (GN): for any finite set $N \subseteq \Omega$ and any game $v \in \mathcal{G}(N)$, if $i \in N$ is null for v,

$$\begin{cases} \Phi(v, i) = 0, \\ \Phi(v, j) = \Phi(v^{-i}, j), \text{ for any player } j \in N \setminus i. \end{cases}$$

We define the particular *unanimity game* of $\mathcal{G}(N)$ by $u_N(S) := \begin{cases} 1, & \text{if } S = N, \\ 0, & \text{otherwise.} \end{cases}$

Equity (Eq): for any finite set $N \subseteq \Omega$, for any player $i \in N$,

$$\Phi(u_N, i) = \frac{1}{n}.$$

This natural axiom simply states that in the particular game where the grand coalition is the unique to produce a unitary worth (all others giving nothing), all players should share the same fraction of this unit.

Theorem 3.1 *Φ_{Sh} is the sole value on $\mathcal{G}$ satisfying axioms **(L)**, **(GN)** and **(Eq)**.*

Note that since the result is given over $\mathcal{G}$, axioms **(L)** and **(N)** should be adjusted in accordance with the arbitrariness of the choice of N. Actually, it is sufficient to specify for these axioms "for any finite set $N \subseteq \Omega$, for any game $v \in \mathcal{G}(N)$".

An important remark is that this new axiomatization has the advantage of characterizing Φ_{Sh} for all games of $\mathcal{G}$, and not only for the games of $\mathcal{G}(N)$, where N is a fixed finite set. This is due to the recursive nature of the axiom **(ENP)**.

We present now another axiomatization of Φ_{Sh}, where the generalized nullity axiom is outlined in another way.

Definition 3.2 *Let $v \in \mathcal{G}(N)$ be any finite game. We call* support *of v, denoted by $\mathfrak{S}(v)$, the minimal carrier of v, that is,*

$$\mathfrak{S}(v) := \bigcap_{C \text{ is a carrier of } v} \{C \in 2^N\}.$$

Actually, a *carrier axiom* has been introduced for the first time by Myerson [9], saying that, if C is a carrier for the game v, then the worth $v(C)$ should be shared only among the members of the carrier. It is shown that this axiom is equivalent to the conjunction of the above axioms **(N)** and **(E)**. With regard to our work, we focus our attention on the support of the game and give an axiom for players in accordance with their membership of the support of the game. If there is no ambiguity, we denote by $v^{\mathfrak{S}}$ the restricted game $v^{\mathfrak{S}(v)}$.

> **Restricted-support games (RS)**: for any finite set $N \subseteq \Omega$, any game $v \in \mathcal{G}(N)$, and any player $i \in N$,
>
> $$\Phi(v,i) = \begin{cases} \Phi(v^{\mathfrak{S}}, i) \text{ if } i \in \mathfrak{S}(v), \\ 0 \text{ otherwise.} \end{cases}$$

Corollary 3.3 *Φ_{Sh} is the sole value on $\mathcal{G}$ satisfying axioms **(L)**, **(RS)** and **(Eq)**.*

To show this result, we propose an alternative characterization of the support of a game:

Lemma 3.4 *Let $v \in \mathcal{G}(N)$ be any game. Then $\mathfrak{S}(v)$ is the set of players which are not null for v.*

4 The Shapley value of multichoice games

In previous section, the lattice representing actions of players was $L := \{0,1\}^{\Omega}$, 0 (resp. 1) denoting absence (resp. presence) of a player. Now, for every player i belonging to a finite carrier of players N, it is assumed that she may act at a level of participation $k \in L_i$ to the game. Actually, $L_i := \{0,1,2,\dots,\top_i\}$ is a linear lattice, where 0 means absence of participation and $\top_i$ represents the maximal participation to the game. Thus $L = L_1 \times \cdots \times L_n$ is the set of all possible joint actions of players of N. We denote by $\mathcal{L}(N)$ the set of all cartesian products of finite linear lattices over N, and by $\mathcal{L}$, the union of all these ones for every finite set N:

$$\mathcal{L}(N) := \{\prod_{i=1}^{n} L_i \mid \top_1,\dots,\top_n \in \mathbb{N}\},$$
$$\mathcal{L} := \{\mathcal{L}(N) \mid N \subseteq \Omega, n \in \mathbb{N}\}.$$

Note that it shall be useful for the sequel to introduce the following binary relation over $\mathcal{L}$ defined for all $L \in \mathcal{L}(N), L' \in \mathcal{L}(N')$, by

$$L \;\mathcal{R}\; L' \text{ iff } \begin{cases} n = n', \\ (\top'_1,\dots,\top'_n) \text{ is a permutation of } (\top_1,\dots,\top_n). \end{cases}$$

This relation is obviously an equivalence relation. We denote by $\overline{\mathcal{L}}$ the quotient set $\mathcal{L}/_{\mathcal{R}}$.

Thus, it turns out that $\overline{\mathcal{L}}$ is isomorphic to the set of the partitions of positive integers, where a *partition* of a positive integer m is a finite nonincreasing sequence[1] of positive integers $(\lambda_1,\dots,\lambda_n)$ such that $\sum_{i=1}^{n} \lambda_i = m$ (see [1]). The λ_i's, corresponding to the maximal levels of participation of players, are called the *parts* of the associated partition. With a slight abuse of notation, we may assimilate $\overline{\mathcal{L}}$ to the set of partitions of positive integers. For any $\lambda := (\lambda_1,\dots,\lambda_n) \in \overline{\mathcal{L}}$, $|\lambda|$ is the sum of the λ_i's, i.e., the unique integer whose partition is given by λ. Also, let us endow $\overline{\mathcal{L}}$ with the following order. For all $\lambda := (\lambda_1,\dots,\lambda_n) \in \overline{\mathcal{L}}, \lambda' := (\lambda'_1,\dots,\lambda'_{n'}) \in \overline{\mathcal{L}}$,

$$\lambda' \leq \lambda \text{ iff } \begin{cases} n' \leq n, \\ \forall i \in \{1,\dots,n'\}, \lambda'_i \leq \lambda_i \end{cases}.$$

For instance, we have $(2,1,1) \leq (4,3,2,1)$. Note that $\lambda := (1)$ is the bottom of $(\overline{\mathcal{L}}, \leq)$.

Proposition 4.1 *$(\overline{\mathcal{L}}, \leq)$ is a ranked lattice, whose rank function is given by $r(\lambda) = |\lambda|$, $\forall \lambda \in \overline{\mathcal{L}}$.*

For $L \in \mathcal{L}$, $\mathcal{G}(L)$ denotes the set of functions defined on L which vanish at $\perp := (0,\dots,0)$: this corresponds to *multichoice games* as introduced by Hsiao and Raghavan [7], where each player has a set of possible ordered actions. For the sake of commodity, we will assimilate any element L of $\mathcal{L}$ with its

[1] In the sequel, elements of $\overline{\mathcal{L}}$ are assumed to be given under this form.

representative element in $\overline{\mathcal{L}}$. In this way, for any $\lambda := (\lambda_1, \dots, \lambda_n) \in \overline{\mathcal{L}}$, $v \in \mathcal{G}(\lambda)$ means that v is any game with n players such that their maximal participation levels are given *up to the order of players* by $\lambda_1, \dots, \lambda_n$. We denote by $\mathcal{G}^{\mathcal{M}}$ the set of all multichoice games, that is to say,

$$\mathcal{G}^{\mathcal{M}} := \{\mathcal{G}(L) \mid L \in \mathcal{L}\}.$$

The set $\mathcal{J}(L)$ of join-irreducible elements of L is $\{(0_{-i}, k_i) \mid i \in N, k \in L_i \setminus \{0\}\}$, using our notation for compound vectors (see Section 2); hence each join-irreducible element $(0_{-i}, k_i)$, which we will often denote by k_i if no ambiguity occurs, corresponds to a single player playing at a given level. Thus a value on $\mathcal{G}(L)$ is a function $\Phi : \mathcal{G}(L) \times \mathcal{J}(L) \to \mathbb{R}$ that assigns to every player i playing at the level k in a game $v \in \mathcal{G}(L)$ his prospect $\Phi(v, k_i)$. Our aim is to define the Shapley value $\Phi(v, k_i)$ for each join-irreducible element k_i.

Our approach will take here a similar way, such as the axiomatization given for classical cooperative games. Note that an axiomatization of the Shapley value for multichoice games has already been done in [6] and [10]. The computed formula is the same. However, the former uses a symmetry axiom which is not really natural, whereas the latter is less intuitive and requires more material. Another important difference in [10] is that the extended Shapley value is only given for multichoice games where the number of possible actions is the same for all players. Moreover, none are given in a simple recursive way on the whole set $\mathcal{G}^{\mathcal{M}}$.

Let us first give the following axioms generalizing the ones given for classical games.

> **Linearity ($\mathbf{L}^{\mathcal{M}}$)**: for any $L \in \mathcal{L}$, for all join-irreducible $k_i \in \mathcal{J}(L)$, $\Phi(v, k_i)$ is linear on the set of games $\mathcal{G}(L)$, which directly implies
>
> $$\Phi(v, k_i) = \sum_{x \in L} p_x^{k_i}\, v(x), \quad \text{with } p_x^{k_i} \in \mathbb{R}.$$

For some $k \in L_i, k \neq 0$, player i is said to be *k-null* (or simply k_i is *null*) for $v \in \mathcal{G}(L)$ if $v(x, k_i) = v(x, (k-1)_i)$, $\forall x \in L_{-i}$. If $\top_i$ is null for v and $\top_i = 1$, player i is simply said to be null for v.

> **Nullity ($\mathbf{N}^{\mathcal{M}}$)**: for any $L \in \mathcal{L}$, for any game $v \in \mathcal{G}(L)$, for any player i who is k-null for v,
>
> $$\Phi(v, k_i) = 0.$$

For some $i \in N$, and $v \in \mathcal{G}(L)$, if $\top_i \neq 1$, we define by $v^{-\top_i}$ the restriction of v to the product $L_{-i} \times (L_i \setminus \top_i)$. Moreover, v^{-i} denotes the mapping defined over $L_{-i} : x \mapsto v(x, 0_i)$.

> **Excluded-null-player ($\mathbf{ENP}^{\mathcal{M}}$)**: for any $L \in \mathcal{L}$, for any game $v \in \mathcal{G}(L)$, for any player $i \in N$ such that $\top_i = 1$, if i is null for v,
>
> $$\forall j \in N \setminus i, \quad \Phi(v, \top_j) = \Phi(v^{-i}, \top_j).$$

> **Decreased-level ($\mathbf{DL}^{\mathcal{M}}$)**: for any $L \in \mathcal{L}$, for any game $v \in \mathcal{G}(L)$, for any player $i \in N$ such that $\top_i \neq 1$, if $\top_i$ is null for v,

(i) $\forall k \in L_i \setminus \{0, \top_i\}, \quad \Phi(v, k_i) = \Phi(v^{-\top_i}, k_i)$.
(ii) $\forall j \in N \setminus i, \quad \Phi(v, \top_j) = \Phi(v^{-\top_i}, \top_j)$.

Likewise the previous section, **(N$^{\mathcal{M}}$)**, **(ENP$^{\mathcal{M}}$)** and **(DL$^{\mathcal{M}}$)** may be merged in the following axiom:

Generalized nullity (GN$^{\mathcal{M}}$): for any $L \in \mathcal{L}$, for any game $v \in \mathcal{G}(L)$, for any player i which is k-null for v, any player $j \in N$ and any level $l \in \{1, \dots, \top_j\}$,

$$\Phi(v, l_j) = \begin{cases} 0 \text{ if } j = i \text{ and } l = k, \\ \Phi(v^{-i}, l_j) \text{ if } j \neq i \text{ and } k = \top_i = 1, \\ \Phi(v^{-\top_i}, l_j) \text{ if } j \neq i \text{ and } k = \top_i \neq 1. \end{cases}$$

Note that this axiom is stronger than the simple concatenation of **(N$^{\mathcal{M}}$)**, **(ENP$^{\mathcal{M}}$)** and **(DL$^{\mathcal{M}}$)**. Thus its validity is easily verifiable by checking the formulae are true.
For any $L \in \mathcal{L}$, we define the particular *unanimity game* of $\mathcal{G}(L)$ by
$$u_\top(x) := \begin{cases} 1, & \text{if } x = \top, \\ 0, & \text{otherwise.} \end{cases}$$

Equity (Eq$^{\mathcal{M}}$): for any $L \in \mathcal{L}$, for any player $i \in N$,

$$\Phi(u_\top, \top_i) = \frac{1}{n}.$$

Theorem 4.2 *Under axioms* **(L$^{\mathcal{M}}$)**, **(GN$^{\mathcal{M}}$)**, *and* **(Eq$^{\mathcal{M}}$)**,
Φ *is given on* $\mathcal{G}^{\mathcal{M}}$ *by:*

$$\Phi(v, k_i) = \sum_{x \in \Gamma(L_{-i})} \frac{h(x)!\,(n - h(x) - 1)!}{n!} \times [v(x, k_i) - v(x, (k-1)_i)], \tag{2}$$

for any finite set $N \subseteq \Omega, \forall L \in \mathcal{L}(N), \forall v \in \mathcal{G}(L)$, $\forall k_i \in \mathcal{J}(L)$, *and where* $h(x) := |\{j \in N \setminus i \mid x_j = \top_j\}|$.

Sketch of the proof

It is quite easy to show that the formula satisfies the axioms.

Conversely, we have to show that the formula is uniquely determined by the axioms. First, under **(L$^{\mathcal{M}}$)** and **(N$^{\mathcal{M}}$)**, Φ is given by

$$\Phi(v, k_i) = \sum_{x \in L_{-i}} p_x^{k_i}(L)\,[v(x, k_i) - v(x, (k-1)_i)], \tag{3}$$

for any finite set $N \subseteq \Omega, \forall L \in \mathcal{L}(N), \forall v \in \mathcal{G}(L), \forall k_i \in \mathcal{J}(L)$,

with $p_x^{k_i}(L) \in \mathbb{R}$.

Now, the coefficients $p_x^{k_i}(L)$'s of (3) are computed by a basic *transfinite induction*, which is an extension of mathematical induction on sets endowed with a wellfounded relation. A binary relation R is *wellfounded* on a set E if every nonempty subset of E has an R-minimal element; that is, for every nonempty subset X of E, there is an element m of X such that for every element x of X, the pair (x,m) is not in R. Considering the strict order $<$ associated to $\leq$, it is easy to see that $<$ is wellfounded on $\overline{\mathcal{L}}$. Thus, the inductive step rests on showing the formula over $\mathcal{G}(\lambda)$ if it is true for games defined over all predecessors of λ in $(\overline{\mathcal{L}},\leq)$. Consequently, if the formula is also satisfied on $\mathcal{G}\big((1)\big)$, then the induction hypothesis applies and the result is satisfied for any game of $\mathcal{G}^{\mathcal{M}}$.

The case $\lambda=(1)$ corresponds to classical cooperative games with one player, which correponds to Theorem 3.1. Indeed, in this case, $\mathcal{J}(L)$ has only one element one can denote by 1_1 (which is also one of the only two elements of L), for which (3) under **(Eq$^{\mathcal{M}}$)** writes $\Phi(v,1_1)=v(1_1)$.

For any $\lambda:=(\lambda_1,\dots,\lambda_n)\in\overline{\mathcal{L}}\setminus\{(1)\}$, let us assume that (2) holds for all games of $\mathcal{G}(\lambda')$ such that $\lambda'\prec\lambda$. We now show that under **(ENP$^{\mathcal{M}}$)**, **(DL$^{\mathcal{M}}$)** and **(Eq$^{\mathcal{M}}$)**, the unicity of all coefficients in (3) is given for any game $v\in\mathcal{G}(\lambda)$. This being done, as it has been checked that (2) satisfies the axioms, the result will be proved. Let N be any set of players of cardinality n, and L be any linear lattice such that maximum levels $\top_1,\dots,\top_n$, in any order, are given by λ.

- We first show the unicity of the $\Phi(v,k_i)$'s, for any player $i\in N$ such that $\top_i\neq 1$, and any level $k<\top_i$. Assuming that $\top_i$ is null for a particular game of $\mathcal{G}(L)$, and denoting by L' the lattice $L_{-i}\times(L_i\setminus\top_i)$, axiom **(DL$^{\mathcal{M}}$)-(i)** is used in order to identify the coefficients $p_x^{k_i}(L)$'s with the $p_x^{k_i}(L')$'s. However, since associated partition of L' is one of the predecessors of λ, thus all $p_x^{k_i}(L')$ are known by assumption. Thus the $p_x^{k_i}(L)$'s and then $\Phi(v,k_i)$'s in this situation are given.

- Then, let $i\in N$ be any player and $j\in N\setminus i$ such that $\top_j\neq 1$. Then for another particular game for which $\top_j$ is null, **(DL$^{\mathcal{M}}$)-(ii)** is used to identify the $p_x^{\top_i}(L)$'s with the $p_x^{\top_i}(L')$'s, where $L':=L_{-j}\times(L_j\setminus\top_j)$. Consequently, we have proved the unicity of coefficients $p_x^{\top_i}(L)$ for all $i\in N$, and for all $x\in L_{-i}$ such that $\exists j\in N\setminus i, x_j\neq\top_j,\top_j-1$.

- Lastly, it remains to show the unicity of the $p_x^{\top_i}(L)$'s, where $i\in N$ and $x\in L_{-i}$ such that $\forall j\in N\setminus i,\ x_j\in\{\top_j,\top_j-1\}$. This in view, we consider the partition $\big\{C_{i,m}\big\}_{i\in N;0\leq m\leq n-1}$ of these indices, where $C_{i,m}$ denotes the set of elements of L_{-i} whose m coordinates x_j are $\top_j-1$ and the others are $\top_j$. For any $i\in N$, we show the unicity of the $p_x^{\top_i}(L)$'s by induction on m. For $x\in C_{i,0}$, that is to say, $x=\top_{-i}:=(\top_1,\dots,\top_{i-1},\top_{i+1},\dots,\top_n)$, $p_x^{\top_i}(L)$ is given by **(Eq$^{\mathcal{M}}$)**:

  $$\Phi(u_\top,\top_i)=p_{\top_{-i}}^{\top_i}(L)=\frac{1}{n}.$$

  Now, the unicity of the $p_x^{\top_i}(L)$'s for $x\in C_{i,m}$, $1\leq m\leq n-1$ is shown by induction on m: assuming that all $p_x^{\top_i}(L)$'s are given for all elements of $C_{i,m}$ (m being fixed in $\{0,\dots,n-2\}$), every $x\in C_{i,m+1}$ is considered and associated to any $j_0 \in N \setminus i$ such that $x_{j_0} = \top_{j_0} - 1$. Now, two situations may arise: either $\top_{j_0} \neq 1$ or $\top_{j_0} = 1$. In the first case, the approach is the same as in the previous item, where $j := j_0$: by identification of coefficients in (3) with coefficients given by **(DL$^{\mathcal{M}}$)-(ii)**, we show that $p_x^{\top_i}(L) = p_x^{\top_i}(L') - p_{x'}^{\top_i}(L)$, where $x' \in C_{i,m}$, and is defined $x'_j := \begin{cases} \top_{j_0} \text{ if } j = j_0, \\ x_j \text{ otherwise} \end{cases}$ ($p_{x'}^{\top_i}(L)$ is given by hypothesis in the current induction, and $p_x^{\top_i}(L')$ is given by hypothesis in the backward transfinite induction). Finally, if $\top_{j_0} = 1$, for any game $v \in \mathcal{G}(L)$ for which j_0 is null, **(ENP$^{\mathcal{M}}$)** is used to compute $p_x^{\top_i}(L)$ in terms of $p_x^{\top_i}(L')$ and $p_{x'}^{\top_i}(L)$ (the formula is the same as above), where this time $L' := L_{-j_0}$. Note that even if $m+1$ choices of j_0 are possible, one cannot guarantee the existence of such an index such that $\top_{j_0} = 1$ for all $i \in N$, or such that $\top_{j_0} \neq 1$ for all $i \in N$. As a consequence, axioms **(DL$^{\mathcal{M}}$)-(ii)** and **(ENP$^{\mathcal{M}}$)** are both necessary.

This ends the proof of the current inductive step: $\forall i \in N$, all $p_x^{\top_i}(L)$'s are given for any $x \in L_{-i}$ such that $\forall j \in N \setminus i$, $x_j \in \{\top_j, \top_j - 1\}$.

Consequently, for all linear lattice L associated to λ, $\forall k_i \in \mathcal{J}(L)$, $\forall x \in L_{-i}$, all $p_x^{k_i}(L)$'s are given, which also completes the inductive step of the transfinite induction. ■

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