

Factors Limiting Controlling of an Inverted Pendulum

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Abstract: The aim of this paper is to show the limitation during an inverted pendulum control process. Assume the control signal and its derivate are limited. The goal is to find the maximum permissible value of the θ angle and state if this value can be determined only by symbolical calculation by using Maple software. This maximum value must guarantee the stability of whole system and the quality of the transient process. The nonlinear mathematical model of the inverted pendulum implemented in Simulink is utilized for result verification. A detailed description of these limitations is important for the application of advanced control methods based on expert knowledge to aircraft equipped with a thrust vectoring nozzles system.

Keywords: inverted pendulum; transfer function; nonlinear analyses; maple

1 Introduction

An inverted pendulum is an inherently unstable system. This system approximates the dynamics of a rocket immediately after lift-off, or dynamics of a thrust vectored aircraft in unstable flight regimes in negligible small dynamic pressure conditions [2]. Assume the force for the inverted pendulum control represents the force generated by a thrust vectoring nozzles system. The nozzle deflection is limited up to ± 20 deg, the rate of deflection is limited up to ± 60 deg/sec and the nozzle dynamics is described by 2nd order transfer function, similarly as in the publication [1]:

$$\frac{400}{s^2 + 40s + 400} \quad (1)$$

The dynamics of the pendulum is given by following nonlinear differential equations system [7]:

$$(M + m) \frac{d^2 x}{dt^2} + ml \frac{d^2 \theta}{dt^2} \cos \theta - ml \left(\frac{d\theta}{dt} \right)^2 \sin \theta = u \quad (2)$$

$$(J + ml^2) \frac{d^2 \theta}{dt^2} = -ml \frac{d^2 x}{dt^2} \cos \theta + mgl \sin \theta \quad (3)$$

where M – cart mass (in this case it can be neglected), m – pendulum mass ($m=15180 \text{ kg}$), l – length to the pendulum centre of gravity ($l=5,4 \text{ m}$), J – moment of inertia of the pendulum ($J=4.2138 \cdot 10^5 \text{ kg} \cdot \text{m}^2$), g – gravity ($g=9.81 \text{ m} \cdot \text{s}^{-2}$), θ – the angle between pendulum and vertical axes [3].

The θ angle transfer function can be obtained after linearization of the system described by equations (2), (3):

$$\frac{\theta(s)}{U(s)} = \frac{K}{s^2 + \omega_0^2} = \frac{-\frac{l}{J}}{s^2 - g \frac{ml}{J}} = \frac{-1,2815 \cdot 10^{-5}}{s^2 - 1,90836} \quad (4)$$

The algorithm for pendulum control is given by following control law:

$$F(s) = sD\theta(s) + P\theta(s) + \frac{I}{s} [\theta(s) - \theta_z(s)] \quad (5)$$

where $F(s)=U(s)$ – inverted pendulum control signal, P – proportional coefficient, I – integral coefficient, D – derivative coefficient. The control system structure is depicted in Figure 1.

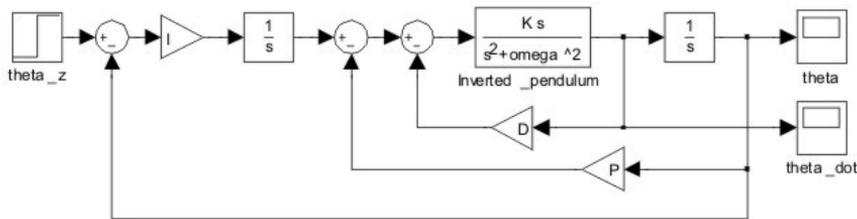


Figure 1

Control system structure with inverted pendulum transfer function

The final transfer function of the system shown in Figure 1 is:

$$\frac{KI}{s^3 + KDs^2 + (KP + \omega_0^2)s + KI} \quad (6)$$

2 Control Signal Limitation in Steady State

Utilize the equation (3) for maximal θ angle computation. Condition $\theta=const$ is valid for steady state. If $\theta=const$, its derivate is zero and its second derivate is also zero. The following equation can be obtained by solving equation (3):

$$ml \frac{d^2 x}{dt^2} \cos \theta = mgl \sin \theta \quad (7)$$

Expression $m \frac{d^2 x}{dt^2}$ represents the control signal, the maximum value of which is given by: $F_{max}=T \sin \varphi$ [6], where φ – the angle of deflection of vectored nozzle. Assume the thrust and aircraft's weight are equal ($T=G=mg$). It is possible to transform equation (7) to get the following equation:

$$mg \sin \varphi \cos \theta = mg \sin \theta \quad (8)$$

Divide equation (8) by expression $\cos \theta$:

$$\sin \varphi = \frac{\sin \theta}{\cos \theta} = \operatorname{tg} \theta \quad (9)$$

The condition (10) for maximum value of θ angle in steady state has been obtained by solving equation (9):

$$\theta_{\max} = \operatorname{arctg}(\sin \varphi_{\max}) \quad (10)$$

3 Limitation during Transient Process

Inverted pendulum control signal is denoted as $Z(s)$ and is depicted in the structure shown in Figure 2. This structure can be utilized for $Z(s)$ transfer function calculation.

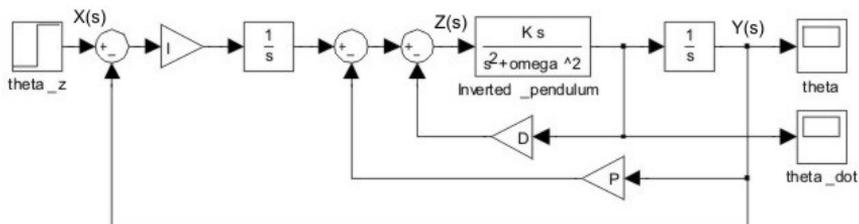


Figure 2
Control system structure with signal's description

The following equation is valid for $Z(s)$:

$$\frac{I}{s} \left[X(s) - \frac{K}{s^2 + \omega_0^2} Z(s) \right] - \frac{KP}{s^2 + \omega_0^2} Z(s) - \frac{KDs}{s^2 + \omega_0^2} Z(s) = Z(s) \quad (11)$$

Solve the equation (11) and place the expression involving $Z(s)$ on the right side:

$$\frac{I}{s} X(s) = Z(s) + \frac{KI}{s(s^2 + \omega_0^2)} Z(s) + \frac{KP}{s^2 + \omega_0^2} Z(s) + \frac{KDs}{s^2 + \omega_0^2} Z(s) \quad (12)$$

$Z(s)$ transfer function can be calculated from the previous equation:

$$\frac{Z(s)}{X(s)} = \frac{I(s^2 + \omega_0^2)}{s^3 + KDs^2 + (KP + \omega_0^2)s + KI} \quad (13)$$

Denominators of transfer functions (6) and (13) are equal and represent the poles of the transfer function and their values guarantee whole system stability and transient process quality. Because the proportional, derivative and integral coefficients influence the poles' placement, it is necessary to select optimal values. The 3rd order polynomials in denominator of transfer functions (6) and (13) are the same. Assume that the 3rd order polynomial has one real root and two complex conjugate roots:

$$(s^2 + 2\xi\omega_z s + \omega_z^2)(s + \omega_z) \quad (14)$$

where ω_z is the desired natural frequency of the system and ξ is the desired system damping. Apply convolution operations to compute the product of polynomial in equation (14) to obtain the generalized 3rd order polynomial form:

$$s^3 + (2\xi + 1)\omega_z s^2 + (2\xi + 1)\omega_z^2 s + \omega_z^3 \quad (15)$$

Substitute the denominator of transfer function (13) by the generalized 3rd order polynomial given by equation (15):

$$\frac{Z(s)}{X(s)} = \frac{I(s^2 + \omega_0^2)}{s^3 + (2\xi + 1)\omega_z s^2 + (2\xi + 1)\omega_z^2 s + \omega_z^3} \quad (16)$$

Transfer function (16) must be transformed into time domain by applying the inverse Laplace transformation for maximum positive and negative values determination. Maple software is used to provide this transformation [4]. It is possible to find a time function of equation (16), but this function is complicated for further symbolical analyses. State the damping value of the system as $\xi=1$ and substitute this value into equation (16):

$$\frac{Z(s)}{X(s)} = \frac{I(s^2 + \omega_0^2)}{s^3 + 3\omega_z s^2 + 3\omega_z^2 s + \omega_z^3} \quad (17)$$

Polynomial of equation (17) has a triple root and is relatively simple for further symbolical analyses and represents the ideal transient process with acceptable quality. The optimal coefficient of the PID regulator can be found by comparing denominators of transfer functions (6), (17):

$$P = -\left(3\omega_z^2 \frac{J}{l} + mg\right) \quad [kgms^{-2}] \quad (18)$$

$$I = -\frac{J}{l} \omega_z^3 \quad [kgms^{-3}] \quad (19)$$

$$D = -3\omega_z \frac{J}{l} \quad [kgms^{-1}] \quad (20)$$

The derivative of the control signal in time domain can be obtained by applying inverse Laplace transform to equation (17):

$$z'(t) = \frac{1}{2} I e^{-\omega_z t} \left[t^2 (\omega_z^2 + \omega_0^2) - 4\omega_z t + 2 \right] \quad (21)$$

Control signal step response in 's' domain is given by following equation:

$$\frac{Z(s)}{X(s)} = \frac{I(s^2 + \omega_0^2)}{s(s^3 + 3\omega_z s^2 + 3\omega_z^2 s + \omega_z^3)} \quad (22)$$

The control signal in time domain can be obtained again by applying the inverse Laplace transform to equation (22) and is described by the following equation:

$$z(t) = \frac{I}{\omega_z^3} \left\{ \omega_0^2 - \left[\frac{t^2}{2} (\omega_z^4 + \omega_z^2 \omega_0^2) + t (\omega_z \omega_0^2 - \omega_z^2) + \omega_0^2 \right] \right\} \quad (23)$$

In Figure 3 is shown the inverted pendulum's control signal step response given by equation (23). Value ω_0^2 is given in transfer function (4) and desired natural frequency value has been selected ($\omega_z=2$).

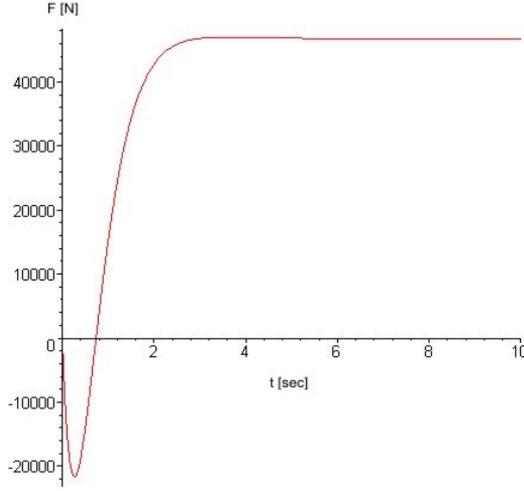


Figure 3
Control signal time response

The extreme value theorem states that if a function f is defined on a closed interval $[a,b]$ (or any closed and bounded set) and is continuous, then the function attains its maximum, i.e. there exists $c \in [a,b]$ with $f(c) \geq f(x)$ for all $x \in [a,b]$. The same is true for the minimum of f . The derivative of function f in c is zero. Equation (21) represents the derivative of the control signal. The equation has two roots:

$$t_{1,2} = \frac{2\omega_z \pm \sqrt{2(\omega_z^2 - \omega_0^2)}}{\omega_z^2 + \omega_0^2} \quad (24)$$

The function described by equation (23) reaches its maximum positive value in time t_1 ($t_1=3.56$ s) and its maximum negative value in time t_2 ($t_2=0.27$ s). It can be observed in Figure 3. Equations (25) and (26) represent maximum positive and negative values of control signal and are gained by substituting (24) into (23):

$$F_{\max,t_1} = \frac{-I}{\omega_z^3} \left[\left(\omega_0^2 + \omega_z^2 + \omega_z \sqrt{2\omega_z^2 - 2\omega_0^2} \right) e^{\Omega_1} - \omega_0^2 \right] \quad (25)$$

$$F_{\max,t_2} = \frac{I}{\omega_z^3} \left[\omega_0^2 - \left(\omega_0^2 + \omega_z^2 - \omega_z \sqrt{2\omega_z^2 - 2\omega_0^2} \right) e^{\Omega_2} \right] \quad (26),$$

where Ω_1 and Ω_2 are given by equation (27), (28) respectively:

$$\Omega_1 = - \frac{\omega_z \left(2\omega_z + \sqrt{2\omega_z^2 + 2\omega_0^2} \right)}{\omega_z^2 + \omega_0^2} \quad (27)$$

$$\Omega_2 = \frac{\omega_z \left(-2\omega_z + \sqrt{2\omega_z^2 - 2\omega_0^2} \right)}{\omega_z^2 + \omega_0^2} \quad (28)$$

Maximum force is transformed into maximum angle by using following assumption:

$$T_{\max} \sin \varphi_{\max} = \theta_{\max} F_{\max} \Rightarrow \theta_{\max} = \frac{T_{\max} \sin \varphi_{\max}}{F_{\max}} \quad (29)$$

The maximum θ angle value for the desired frequency ω_z can be calculated by utilizing equation (29). θ_{\max} values are depicted in Figure 4.

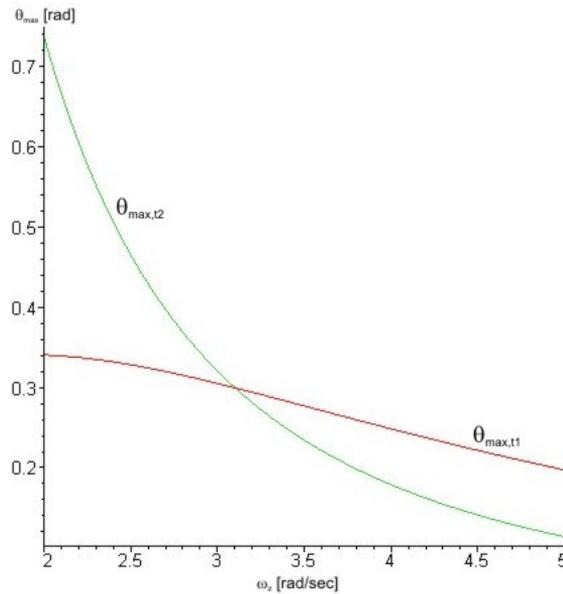


Figure 4

Maximum θ angle values depicted as a function of the desired natural frequency value

4 Limitation Given by Vectoring Nozzle Deflection Rate

The derivative of the control signal given by equation (21) is depicted in Figure 5.

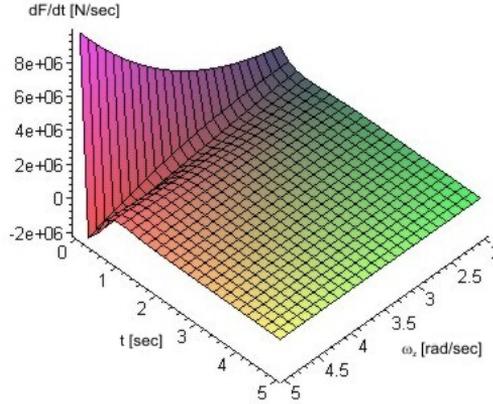


Figure 5

Control signal derivation depicted in 3-dimensional graph for ω_z values in region from 2 to 5 rad/sec

It can be observed in Figure 5, that the control signal derivative reaches its maximum value in time $t=0$. The following expression can be utilized for maximum θ angle computation:

$$\theta_{\max} = \frac{\frac{dF_{\max}}{dt}}{z'(0)} \quad (30)$$

The maximum derivative of the control signal can be determined by applying the derivative operation to the right side of equation (29):

$$\frac{dF_{\max}}{dt} = \frac{d}{dt}(T_{\max} \sin \varphi) = T_{\max} \frac{d\varphi}{dt} \cos \varphi \quad (31)$$

Assume in time $t=0$ the nozzle deflection is zero so $\cos \varphi=1$. The maximum nozzle deflection rate is 60 deg/sec (approximately $\pi/3$ rad/sec) and the maximum thrust is supposed to be constant ($T_{\max}=148916N$):

$$T_{\max} \frac{d\varphi}{dt} \cos \varphi \approx T_{\max} \frac{d\varphi}{dt} = 155945 \text{ N / s} \quad (32)$$

5 Nonlinear Analyses

The structure of the system used for nonlinear analysis is shown in Figure 6 and consists of two main blocks. Block 'vectored_nozzles' describes the vectored nozzles together with their dynamics and limitations mentioned in the introduction of this paper. The nonlinear mathematical model of inverted pendulum given by equations (33) and (34) is implemented in block 'Inverted pendulum'.

$$\left[(M + m) - \frac{(ml)^2 \cos^2 \theta}{J + ml^2} \right] \frac{d^2 x}{dt^2} = ml \left(\frac{d\theta}{dt} \right)^2 - \frac{(ml)^2 g \sin \theta \cos \theta}{J + ml^2} + u \quad (33)$$

$$\left[(J + ml^2) - \frac{(ml)^2 \cos^2 \theta}{M + m} \right] \frac{d^2 \theta}{dt^2} = ml g \sin \theta - \frac{ml \cos \theta}{M + m} \left[ml \left(\frac{d\theta}{dt} \right)^2 + u \right] \quad (34)$$

These equations are based on equations (2) and (3) that have to be rewritten for algebraic loop elimination [2]:

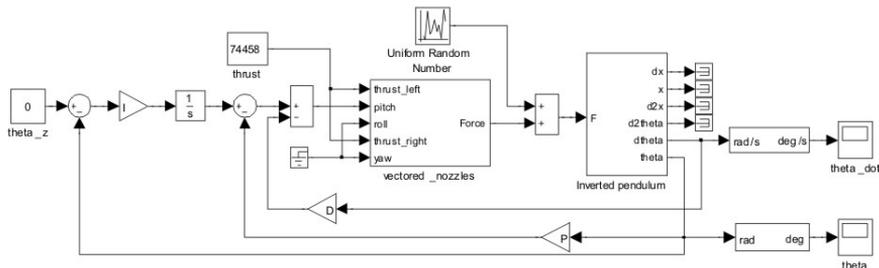


Figure 6
Structure used for nonlinear analyses

It can be seen in Figure 6 that the force generated by the vectoring system is controlled by the pitch command. The coefficients of the PID regulator can be calculated by dividing equations (18), (19) and (20) by the maximum thrust value ($T_{max}=148916N$). It is possible to consider this simplification only for small angle (up to 20 deg). The following m-file was utilized for coefficients' calculation:

```
m=15180;%[m] mass of the aircraft
J=4.2138e5;%[kg*m^2] moment of inertia
l=5.4;%[m] CG position
T=148916;%[N] thrust
g=9.81;%[m*s^2] gravity
omega=2;%desired natural frequency value
P=-(3*omega^2*(J/l)+m*g)/T;%proportional coefficient
I=-(J*omega^3)/(l*T);%integral coefficient
D=-(3*omega*J)/(l*T);%derivate coefficient
```

The natural frequency desired value is shown in the first column of Table 1. The values in 2nd and 3rd column are depicted in Figure 4. The values in the 4th column are the minimum of the values calculated according to equations (10), (25) and (26). The values obtained from nonlinear analyses when the rate limitation and nozzle dynamics has not been assumed are in the 5th column. In the 6th column are values computed calculated according to equation (30) in Chapter 4. The values obtained from nonlinear analyses with rate limitation and nozzle dynamics are in the last column of Table 1.

Table 1

ω_z	$\theta_{\max1}$	$\theta_{\max2}$	$\theta(2,3)$	θ_{\max}	$\theta(4)$	θ_{\max}
2	0.341	0.737	0.3295	0.324	0.2498	0.327
2.5	0.329	0.465	0.328	0.313(0.316)	0.128	0.279 (0.297)
3	0.305	0.321	0.305	0.298(0.303)	0.074	0.144 (0.178)
3.5	0.277	0.234	0.234	0.276(0.287)	0.047	0.081(0.093)
4	0.248	0.179	0.179	0.245(0.261)	0.031	0.042 (0.058)
4.5	0.221	0.141	0.141	0.211(0.231)	0.022	0.037
5	0.197	0.114	0.114	0.181(0.203)	0.016	0.023

The values obtained from nonlinear simulation are in the 5th and 7th column of Table 1. Both values delineate maximum value of given θ angle of the system in stable conditions but transient process for the first values is with acceptable quality (Figure 7) and transient process for the values in brackets is with poor quality (Figure 8). The desired frequency for both responses was $\omega_z=4$.

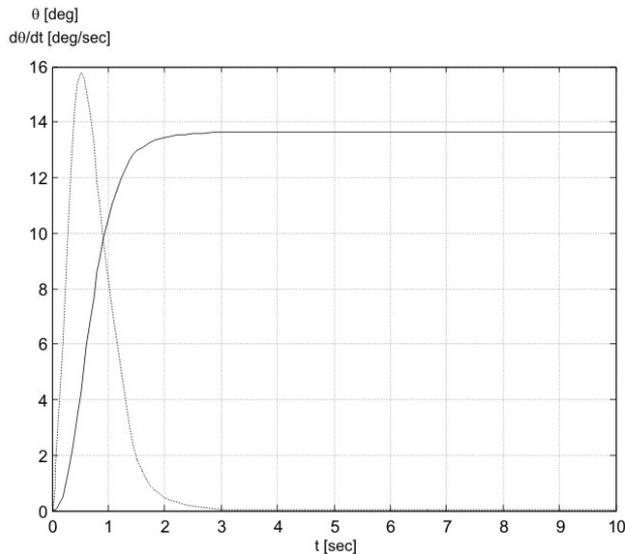


Figure 7

θ angle (solid line) and rate (dotted line) time response with acceptable quality of the transient process

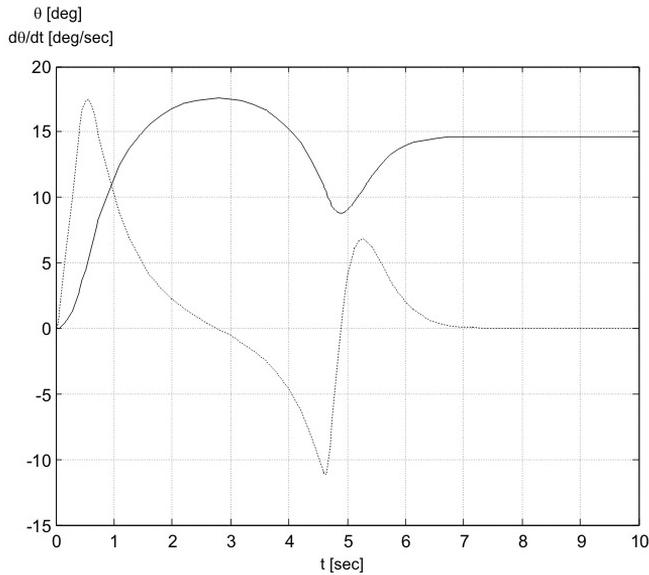


Figure 8

θ angle (solid line) and rate (dotted line) time response with poor quality of the transient process

Time response in Figure 8 converges, but the observable oscillations increase the settling time.

From Table 1, it is visible that the calculated values approximately describe the limiting conditions. This behaviour of the system is explained in Figure 9b where the time response of the control signal is depicted. It can be observed that the control signal reaches its maximum value. The oscillations are observed exactly in time when control signal reaches its maximum value.

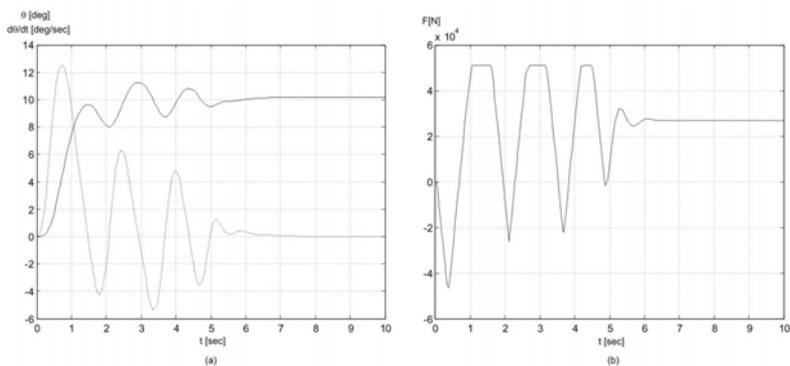


Figure 9

θ angle (solid line), rate (dotted line) and control signal time response

If the given θ angle value does not exceed the limiting conditions, calculated according to the procedure shown in Chapters 2 and 3, the system's time response will certainly be stable with required transient process quality. It is necessary to perform an experiment (e.g. nonlinear simulation) for accurate marginal θ angle value determination. The dynamic properties of the nozzle are much more limiting for the higher desired natural frequency (approximately from $\omega_z=4$). It can be seen by comparing limiting θ angle values in the 5th and in the last column of Table 1.

Conclusions

The possibility to symbolically calculate limitation by using linear analyses and Maple software was shown in this paper. This procedure is appropriate for relatively simple transfer functions and the calculated results only approximately describe the limiting conditions, but do guarantee the system stability. It was shown that it is necessary to perform an experiment for marginal value determination. The obtained values provide a better idea about inverted pendulum dynamics and all factors considered for successful realization of the control system. These facts are also expected to be utilized during application of advanced control methods based on expert knowledge into the inherently unstable systems.

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