A Dynamic Model of a Small Open Economy Under Flexible Exchange Rates

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Abstract: In the paper a three dimensional dynamic model of a small open economy, describing the development of net real national income, real physical capital stock and the expected exchange rate of the near future, which was introduced by T. Asada in [1], is analysed under flexible exchange rates. We study the question of the existence of business cycles. Sufficient conditions for the existence of a pair of purely imaginary eigenvalues with the third one negative in the linear approximation matrix of the model are found. For the existence of business cycles and their properties the structure of the bifurcation equation of the model is very important. Formulae for the calculation of the bifurcation coefficients in the bifurcation equation of the model are derived. Theorem on the existence of business cycles in a small neighbourhood of the equilibrium point is presented.

Keywords: dynamical model; matrix of linear approximation; eigenvalues; bifurcation equation; business cycle

1 Introduction

Toichiro Asada formulated in [1] a Kaldorian business cycle model in a small open economy. He studied both the system of fixed exchange rates and that of flexible exchange rates with the possibility of capital mobility. In this article we investigate Asada's model which was introduced in [1] under flexible exchange rates. In this case Asada's model has the form

$$\begin{aligned} &\dot{Y} = \alpha\left(C + I + G + J - Y\right), \alpha > 0, \\ &\dot{K} = I, \qquad\qquad (1) \\ &\dot{\pi}^e = \gamma\left(\pi - \pi^e\right), \gamma > 0, \end{aligned}$$

where

$$C = c(Y - T) + C_0, 0 < c < 1, C_0 > 0,$$
$$T = \tau Y - T_0, 0 < \tau < 1, T_0 > 0,$$
$$I = I(Y, K, r), I_Y = \frac{\partial I}{\partial Y} > 0, I_K = \frac{\partial I}{\partial K} < 0, I_r = \frac{\partial I}{\partial r} < 0,$$
$$\frac{M}{p} = L(Y, r), L_Y = \frac{\partial L}{\partial Y} > 0, L_r = \frac{\partial L}{\partial r} < 0, \tag{2}$$
$$J = J(Y, \pi), J_Y = \frac{\partial J}{\partial Y} < 0, J_\pi = \frac{\partial J}{\partial \pi} > 0,$$
$$Q = \beta\left(r - r_f - \frac{\pi^e - \pi}{\pi} \right), \beta > 0,$$
$$A = J + Q,$$
$$A = 0,$$
$$M = \text{constant}, \left(\dot{M} = 0\right).$$

The meanings of the symbols in (1) and (2) are as follows: Y - net real national income, C - real private consumption expenditure, I - net real private investment expenditure on physical capital, G - real government expenditure (fixed), T - real income tax, K - real physical capital stock, M - nominal money stock, p - price level, r - nominal rate of interest of domestic country, r_f - nominal rate of interest of foreign country, π - exchange rate, π^e - expected exchange rate of near future, J - balance of current account (net export) in real terms, Q - balance of capital account in real terms, A - total balance of payments in real terms, α - adjustment speed in goods market, β - degree of capital mobility, α, β, γ - positive parameters, and the meanings of other symbols are as follows, $\dot{Y} = \frac{dY}{dt}$, $\dot{K} = \frac{dK}{dt}$, $\dot{\pi}^e = \frac{d\pi^e}{dt}$, t - time.

In the whole paper we assume as well as Asada in [1] a fixed-price economy, so that p is exogenously given and normalized to the value 1. Asada assumed that the equilibrium on the money market $M = L(Y, r)$ is always preserved, which enables using the Implicit-function theorem to express interest rate r as the function of Y, so

$$r = r(Y),\ r_Y = \frac{\partial r}{\partial Y} > 0.$$

Solving equation $A = J(Y,\pi) + \beta\left[r(Y) - r_f - \frac{\pi^e - \pi}{\pi}\right] = 0$ with respect to π, we have

$$\pi = \pi(Y, \pi^e),\ \pi_Y = \frac{\partial \pi}{\partial Y} = \frac{-J_Y - \beta r_Y}{J_\pi + \frac{\beta\pi^e}{\pi^2}},\ \pi_{\pi^e} = \frac{\partial \pi}{\partial \pi^e} = \frac{\beta}{J_\pi \pi + \frac{\beta \pi^e}{\pi}} > 0.$$

Further we suppose that r_f is also given exogenously because of the assumption of a small open economy. Under these assumptions, taking into account (2) and the explicit expression for r, the model (1) takes the form

$$\begin{aligned} \dot{Y} &= \alpha\left[c(1-\tau)Y + cT_0 + C_0 + G + I(Y,K,r(Y)) + J(Y,\pi(Y,\pi^e)) - Y\right] \\ \dot{K} &= I(Y,K,r(Y)) \\ \dot{\pi}^e &= \gamma\left[\pi(Y,\pi^e) - \pi^e\right] \end{aligned} \tag{3}$$

In the whole paper we suppose that:

(i) the model (3) has a unique equilibrium point $E^* = (Y^*, K^*, \pi^{e^*}),\ Y^* > 0,\ K^* > 0,\ \pi^{e^*} > 0,$ to an arbitrary triple of positive parameters (α, β, γ).

(ii) $0 < I_Y + I_r r_Y < 1 - c(1-\tau) - J_Y$ at the equilibrium point.

(iii) $\pi_{\pi^e} - 1 < 0$ at the equilibrium point.

(iv) The functions in the model (3) have the following properties: the function I is linear in the variable K and r. The function π is linear in the variable π^e. The function J is nonlinear in the variable π. In the variable Y the functions I, J, r, π are nonlinear, and have continuous partial derivatives with respect to Y up to the sixth order in a small neighbourhood of the equilibrium point.

In [1] Asada found sufficient conditions for local stability and instability of the equilibrium point. He studied how changes of the parameter β affect the dynamic characteristics of the model.

We analyse the question of the existence of business cycles analytically. Stable business cycles can arise only in the case when the linear approximation matrix of the model (3) has at the equilibrium point a pair of purely imaginary eigenvalues with the third one negative. In Section 2, Theorem 1 gives sufficient conditions for the existence of a pair of purely imaginary eigenvalues with the third one negative. The bifurcation equation of the model (3) is very important for the existence of business cycles. In Section 3, Theorem 2 gives the formulae for the calculation of the bifurcation coefficients in the bifurcation equation. Theorem 3 speaks about the existence of business cycles in a small neighbourhood of the equilibrium point.

Such an analytical approach was applied to study similar models in [6], [7], [8], [9], [10].

2 The Analysis of the Model (3)

Consider an isolated equilibrium poin $E^* = \left(Y^*, K^*, \pi^{e^*}\right)$, $Y^* > 0$, $K^* > 0$, $\pi^{e^*} > 0$, of the model (3).

After the transformation

$$Y_1 = Y - Y^*, K_1 = K - K^*, \pi_1^e = \pi^e - \pi^{e^*},$$

the equilibrium point $E^* = \left(Y^*, K^*, \pi^{e^*}\right)$ goes into the origin $E_1^* = (0,0,0)$, and the model (3) becomes

$$\begin{aligned}
\dot{Y}_1 &= \alpha\Big[c(1-\tau)\left(Y_1 + Y^*\right) + I\left(Y_1 + Y^*, K_1 + K^*, r\left(Y_1 + Y^*\right)\right) + cT_0 + \\
&\quad + C_0 + G + J\left(Y_1 + Y^*, \pi\left(Y_1 + Y^*, \pi_1^e + \pi^{e^*}\right)\right) - \left(Y_1 + Y^*\right)\Big] \\
\dot{K}_1 &= I\left(Y_1 + Y^*, K_1 + K^*, r\left(Y_1 + Y^*\right)\right) \qquad (4) \\
\dot{\pi}_1^e &= \gamma\Big[\pi\left(Y_1 + Y^*, \pi_1^e + \pi^{e^*}\right) - \left(\pi_1^e + \pi^{e^*}\right)\Big]
\end{aligned}$$

Performing the Taylor expansion of the functions on the right-hand side of this system at the equilibrium point $E_1^* = (0,0,0)$ the model (4) obtains the form

$$\begin{aligned}
\dot{Y}_1 &= \alpha\big[c(1-\tau)Y_1 + I_Y Y_1 + I_K K_1 + I_r r_Y Y_1 + J_Y Y_1 + \\
&\quad + J_\pi \pi_Y Y_1 + J_\pi \pi_{\pi^e} \pi_1^e - Y_1\big] + \widetilde{Y}_1 \\
\dot{K}_1 &= I_Y Y_1 + I_K K_1 + I_r r_Y Y_1 + \widetilde{K}_1 \qquad (5)
\end{aligned}$$

$$\dot{\pi}_1^e = \gamma\left[\pi_Y Y_1 + \pi_{\pi^e}\pi_1^e - \pi_1^e\right] + \tilde{\pi}_1^e,$$

where
$I_Y = \dfrac{\partial I(E^*)}{\partial Y}$, $I_K = \dfrac{\partial I(E^*)}{\partial K}$, $I_r = \dfrac{\partial I(E^*)}{\partial r}$, $r_Y = \dfrac{\partial r(E^*)}{\partial Y}$, $J_Y = \dfrac{\partial J(E^*)}{\partial Y}$,
$J_\pi = \dfrac{\partial J(E^*)}{\partial \pi}$, $\pi_Y = \dfrac{\partial \pi(E^*)}{\partial Y}$, $\pi_{\pi^e} = \dfrac{\partial \pi(E^*)}{\partial \pi^e}$, and the functions $\tilde{Y}_1, \tilde{K}_1, \tilde{\pi}_1^e$ are nonlinear parts of the Taylor expansion.

The linear approximation matrix $\mathbf{A} = \mathbf{A}(\alpha, \beta, \gamma)$ of the system (5) has the following form

$$\mathbf{A} = \begin{pmatrix} \alpha\left[c(1-\tau) + I_Y + I_r r_Y + J_Y + J_\pi \pi_Y - 1\right] & \alpha I_K & \alpha J_\pi \pi_{\pi^e} \\ I_Y + I_r r_Y & I_K & 0 \\ \gamma \pi_Y & 0 & \gamma\left(\pi_{\pi^e} - 1\right) \end{pmatrix}. \tag{6}$$

The characteristic equation of $\mathbf{A}(\alpha, \beta, \gamma)$ is given by

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0, \tag{7}$$

where

$$b_1 = -\operatorname{trace} \mathbf{A}(\alpha, \beta, \gamma) =$$
$$= -\left\{\alpha\left[c(1-\tau) + I_Y + I_r r_Y + J_Y + J_\pi \pi_Y - 1\right] + I_K + \gamma\left(\pi_{\pi^e} - 1\right)\right\}$$

$$b_2 = \text{sum of all principal second – order minors of } \mathbf{A}(\alpha, \beta, \gamma) =$$
$$= \alpha I_K\left[c(1-\tau) + J_Y + J_\pi \pi_Y - 1\right] + \gamma I_K\left(\pi_{\pi^e} - 1\right) +$$
$$+ \alpha\gamma\left\{\left[c(1-\tau) + I_Y + I_r r_Y + J_Y - 1\right]\left(\pi_{\pi^e} - 1\right) - J_\pi \pi_Y\right\}$$

$$b_3 = -\det \mathbf{A}(\alpha, \beta, \gamma) =$$
$$= -\alpha\gamma I_K\left\{\left(\pi_{\pi^e} - 1\right)\left[c(1-\tau) + J_Y - 1\right] - J_\pi \pi_Y\right\}.$$

As we are interested in the existence and stability of limit cycles we need to find such values of parameters α, β, γ at which the equation (7) has a pair of purely imaginary eigenvalues and the third one is negative. We will call such values of parameters α, β, γ the critical values of the model (3). We denote these critical values by $\alpha_0, \beta_0, \gamma_0$. The mentioned types of eigenvalues are ensured by the Liu's conditions [5].

$$b_1 > 0, b_3 > 0, b_1 b_2 - b_3 = 0. \tag{8}$$

For an arbitrary α exists γ such that the first inequality is satisfied under the condition (iii). The second inequality is satisfied under the condition $(\pi_{\pi^e} - 1)[c(1-\tau) + J_Y - 1] > J_\pi \pi_Y$. This condition is satisfied when β is sufficiently small.

The equation $b_1 b_2 - b_3 = 0$ gives

$$\{\alpha[c(1-\tau) + I_Y + I_r r_Y + J_Y + J_\pi \pi_Y - 1] + I_K + \gamma(\pi_{\pi^e} - 1)\}.$$
$$.\{\gamma I_K(\pi_{\pi^e} - 1) + \alpha I_K[c(1-\tau) + J_Y + J_\pi \pi_Y - 1] +$$
$$+ \alpha\gamma[(c(1-\tau) + I_Y + I_r r_Y + J_Y - 1)(\pi_{\pi^e} - 1) - J_\pi \pi_Y]\} -$$
$$- \alpha\gamma I_K\{(\pi_{\pi^e} - 1)[c(1-\tau) + J_Y - 1] - J_\pi \pi_Y\} = 0.$$

Asada indicated in [1] that $b_1 b_2 - b_3 > 0$ for $\pi_Y < 0$ and for $\pi_Y = 0$. The equation $b_1 b_2 - b_3 = 0$ is satisfied under the condition $\pi_Y > 0$. This inequality is satisfied if $\beta < \frac{-J_Y}{r_Y}$.

The equation $b_1 b_2 - b_3 = 0$ can be expressed in the form $f_1(\alpha,\beta)\gamma^2 + f_2(\alpha,\beta)\gamma + f_3(\alpha,\beta) = 0$,

where

$$f_1(\alpha,\beta) = J_\pi^2 \pi^2 [I_K - \alpha(J_Y - R) - \alpha\beta r_Y]$$

$$f_2(\alpha,\beta) = \alpha^2\beta^2 T J_\pi \pi r_Y + \alpha^2 \beta J_\pi \pi (J_Y - R)(T - J_\pi \pi r_Y) -$$
$$- \alpha^2 J_\pi^2 \pi^2 (R - J_Y)^2 - 2\alpha\beta I_K J_\pi \pi T + 2\alpha I_K J_\pi^2 \pi^2 (J_Y - R) -$$
$$- \beta I_K^2 J_\pi \pi - I_K^2 J_\pi^2 \pi^2$$

$$f_3(\alpha,\beta) = \alpha^2\beta^2 I_K T U + \alpha^2 \beta I_K J_\pi \pi [T(P - J_Y) + U(R - J_Y)] +$$
$$+ \alpha^2 I_K J_\pi^2 \pi^2 (R - J_Y)(P - J_Y) + \alpha\beta^2 I_K^2 U +$$
$$+ \alpha\beta I_K^2 J_\pi \pi (P - J_Y + U) + \alpha I_K^2 J_\pi^2 \pi^2 (P - J_Y)$$

where

$$P = c(1-\tau) + J_Y - 1 < 0,$$

$$R = P + I_Y + I_r r_Y < 0,$$

$$T = R - J_\pi \pi r_Y < 0,$$

$$U = P - J_\pi \pi r_Y < 0.$$

We see that for an arbitrary γ exists $\hat{\alpha}$ such that $f_1(\hat{\alpha},0) = 0,\ \hat{\alpha} > 0.$

Put $F(\alpha,\beta,\gamma) = f_1(\alpha,\beta) + f_2(\alpha,\beta)\frac{1}{\gamma} + f_3(\alpha,\beta)\frac{1}{\gamma^2} = 0.$

Instead of γ introduce $\vartheta = \frac{1}{\gamma}$ and consider an equation $\Phi(\alpha,\beta,\vartheta) = 0$ when

$$\Phi(\alpha,\beta,\vartheta) = \begin{cases} f_1(\alpha,\beta), & \vartheta = 0 \\ f_1(\alpha,\beta) + f_2(\alpha,\beta)\vartheta + f_3(\alpha,\beta)\vartheta^2, & \vartheta \neq 0. \end{cases}$$

We see that $\Phi(\alpha,\beta,\vartheta) = 0$ is equivalent to $F(\alpha,\beta,\gamma) = 0.$ Analyze $\Phi(\alpha,\beta,\vartheta) = 0.$

It holds:

1. $\Phi(\hat{\alpha},\beta = 0,\vartheta = 0) = 0$
2. $\dfrac{\partial\Phi(\hat{\alpha},\beta = 0,\vartheta = 0)}{\partial\alpha} = -J_\pi^2\pi^2(J_Y - R) \neq 0.$

By the implicit function theorem there exists a function $\alpha = f(\beta,\vartheta)$ in a small neighbourhood of $(\beta = 0,\vartheta = 0)$ such that $\hat{\alpha} = f(0,0)$ and $\Phi(f(\beta,\vartheta),\beta,\vartheta) = 0.$

We see that for a sufficiently large γ_0 of parameter γ and sufficiently small β_0 of parameter β there exists value α_0 of parameter α such that the triple is $(\alpha_0,\beta_0,\gamma_0)$ the critical triple of the model (3). The following theorem gives sufficient conditions for the existence of a critical triple of the model (3).

Theorem 1. *Let the condition* $\pi_{\pi^e} - 1 < 0$ *is satisfied. If parameter* $\beta < \dfrac{-J_Y}{r_Y}$ *is sufficiently small and parameter* γ *is sufficiently large, then there exists a critical triple* $(\alpha_0, \beta_0, \gamma_0)$ *of model* (5).

3 Existence of Limit Cycles and their Stability

According to the assumption (iv) the model (5) can be itemized in the form

$$\dot{Y}_1 = \alpha\Big\{\big[I_Y + I_r r_Y - \big(1 - c(1-\tau) - J_Y\big) + J_\pi \pi_Y\big]Y_1 + I_K K_1 + J_\pi \pi_{\pi^e}\pi_1^e\Big\} +$$
$$+\frac{1}{2}\alpha\left(I_Y^{(2)} + J_Y^{(2)}\right)Y_1^2 + \frac{1}{2}\alpha J_{\pi^e}^{(2)}\left(\pi_1^e\right)^2 + \frac{1}{6}\alpha\left(I_Y^{(3)} + J_Y^{3}\right)Y_1^3 +$$
$$+\frac{1}{6}\alpha J_{\pi^e}^{(3)}\left(\pi_1^e\right)^3 + \frac{1}{24}\alpha\left(I_Y^{(4)} + J_Y^{(4)}\right)Y_1^4 + \frac{1}{24}\alpha J_{\pi^e}^{(4)}\left(\pi_1^e\right)^4 + O_5\left(Y_1, \pi_1^e\right)$$
$$\dot{K}_1 = \left(I_Y + I_r r_Y\right)Y_1 + I_K K_1 + \frac{1}{2}I_Y^{(2)}Y_1^2 + \frac{1}{6}I_Y^{(3)}Y_1^3 + \frac{1}{24}I_Y^{(4)}Y_1^4 +$$
$$+O_5\left(Y_1\right) \tag{9}$$
$$\dot{\pi}_1^e = \gamma\pi_Y Y_1 + \gamma\left(\pi_{\pi^e} - 1\right)\pi_1^e + \frac{1}{2}\gamma\pi_Y^{(2)}Y_1^2 + \frac{1}{6}\gamma\pi_Y^{(3)}Y_1^3 + \frac{1}{24}\gamma\pi_Y^{(4)}Y_1^4 +$$
$$+O_5\left(Y_1\right),$$

where

$$I_Y^{(2)} = \frac{\partial^2 I(E^*)}{\partial Y^2},\ I_Y^{(3)} = \frac{\partial^3 I(E^*)}{\partial Y^3},\ I_Y^{(4)} = \frac{\partial^4 I(E^*)}{\partial Y^4},\ J_Y^{(2)} = \frac{\partial^2 J(E^*)}{\partial Y^2},$$
$$J_Y^{(3)} = \frac{\partial^3 J(E^*)}{\partial Y^3},\ J_Y^{(4)} = \frac{\partial^4 J(E^*)}{\partial Y^4},\ J_{\pi^e}^{(2)} = \frac{\partial^2 J(E^*)}{\partial(\pi^e)^2},\ J_{\pi^e}^{(3)} = \frac{\partial^3 J(E^*)}{\partial(\pi^e)^3},$$
$$J_{\pi^e}^{(4)} = \frac{\partial^4 J(E^*)}{\partial(\pi^e)^4},\ \pi_Y^{(2)} = \frac{\partial^2 \pi(E^*)}{\partial Y^2},\ \pi_Y^{(3)} = \frac{\partial^3 \pi(E^*)}{\partial Y^3},\ \pi_Y^{(4)} = \frac{\partial^4 \pi(E^*)}{\partial Y^4}.$$

Consider a critical triple $(\alpha_0, \beta_0, \gamma_0)$ of the model (3). Let us investigate the behavior of Y_1, K_1 and π_1^e around the equilibrium $E_1^* = (0,0,0)$ with respect to the parameter $\alpha, \alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon), \varepsilon > 0,$ and the fixed parameters $\beta = \beta_0,\ \gamma = \gamma_0$.

After the shifting α_0 into the origin by relation $\alpha_1 = \alpha - \alpha_0$, the model (9) becomes

$$\begin{aligned}
\dot{Y}_1 = {} & \alpha_0\left[I_Y + I_r r_Y - (1 - c(1-\tau) - J_Y) + J_\pi \pi_Y\right]Y_1 + \alpha_0 I_K K_1 + \\
& + \alpha_0 J_\pi \pi_{\pi^e} \pi_1^e + \left[I_Y + I_r r_Y - (1 - c(1-\tau) - J_Y) + J_\pi \pi_Y\right]Y_1 \alpha_1 + \\
& + I_K K_1 \alpha_1 + J_\pi \pi_{\pi^e} \pi_1^e \alpha_1 + \frac{1}{2}\alpha_0\left(I_Y^{(2)} + J_Y^{(2)}\right)Y_1^2 + \frac{1}{2}\alpha_0 J_{\pi^e}^{(2)}\left(\pi_1^e\right)^2 + \\
& + \frac{1}{2}\left(I_Y^{(2)} + J_Y^{(2)}\right)Y_1^2\alpha_1 + \frac{1}{2}J_{\pi^e}^{(2)}\left(\pi_1^e\right)^2\alpha_1 + \frac{1}{6}\alpha_0\left(I_Y^{(3)} + J_Y^{(3)}\right)Y_1^3 + \\
& + \frac{1}{6}\alpha_0 J_{\pi^e}^{(3)}\left(\pi_1^e\right)^3 + \frac{1}{6}\left(I_Y^{(3)} + J_Y^{(3)}\right)Y_1^3\alpha_1 + \frac{1}{6}J_{\pi^e}^{(3)}\left(\pi_1^e\right)^3\alpha_1 + \\
& + \frac{1}{24}\alpha_0\left(I_Y^{(4)} + J_Y^{(4)}\right)Y_1^4 + \frac{1}{24}\alpha_0 J_{\pi^e}^{(4)}\left(\pi_1^e\right)^4 + O_5\left(Y_1, \pi_1^e, \alpha_1\right) \\
\dot{K}_1 = {} & \left(I_Y + I_r r_Y\right)Y_1 + I_K K_1 + \frac{1}{2}I_Y^{(2)}Y_1^2 + \frac{1}{6}I_Y^{(3)}Y_1^3 + \frac{1}{24}I_Y^{(4)}Y_1^4 + O_5(Y_1) \\
\dot{\pi}_1^e = {} & \gamma_0 \pi_Y Y_1 + \gamma_0\left(\pi_{\pi^e} - 1\right)\pi_1^e + \frac{1}{2}\gamma_0 \pi_Y^{(2)} Y_1^2 + \frac{1}{6}\gamma_0 \pi_Y^{(3)} Y_1^3 + \frac{1}{24}\gamma_0 \pi_Y^{(4)} Y_1^4 + \\
& + O_5(Y_1).
\end{aligned} \tag{10}$$

Denote the eigenvalues of (6) as

$$\lambda_1 = \xi(\alpha,\beta,\gamma) + i\omega(\alpha,\beta,\gamma), \lambda_2 = \xi(\alpha,\beta,\gamma) - i\omega(\alpha,\beta,\gamma),$$
$$\lambda_3 = \lambda_3(\alpha,\beta,\gamma)$$

and let

$$\lambda_1 = i\omega_0, \lambda_2 = -i\omega_0, \omega_0 = \omega(\alpha_0,\beta_0,\gamma_0), \lambda_{30} = \lambda_3(\alpha_0,\beta_0,\gamma_0).$$

Express the model (10) in the form

$$\dot{\mathbf{x}} = \mathbf{A}(\alpha_0,\beta_0,\gamma_0)\mathbf{x} + \widetilde{\mathbf{Y}}(\mathbf{x},\alpha_1),$$

where

$$\mathbf{x} = \begin{pmatrix} Y_1 \\ K_1 \\ \pi_1^e \end{pmatrix}, \quad \widetilde{\mathbf{Y}} = \begin{pmatrix} \widetilde{Y}_1 \\ \widetilde{Y}_2 \\ \widetilde{Y}_3 \end{pmatrix}.$$

Consider a matrix $\mathbf{M} = (m_{ij})$, $i, j = 1,2,3,$ which transfers the matrix $\mathbf{A}(\alpha_0, \beta_0, \gamma_0)$ into its Jordan form $\mathbf{J}$, and its inverse matrix $\mathbf{M}^{-1} = (m_{ij}^{-1})$.

By the transformation $\mathbf{x} = \mathbf{My}$, $\mathbf{y} = \begin{pmatrix} Y_2 \\ K_2 \\ \pi_2^e \end{pmatrix}$ we obtain

$$\dot{\mathbf{y}} = \mathbf{Jy} + \mathbf{F}(\mathbf{y}, \alpha_1), \tag{11}$$

Where

$$\mathbf{J} = \begin{pmatrix} i\omega_0 & 0 & 0 \\ 0 & -i\omega_0 & 0 \\ 0 & 0 & \lambda_{30} \end{pmatrix},$$

$$\mathbf{F}(\mathbf{y}, \alpha_1) = \begin{pmatrix} F_1(\mathbf{y}, \alpha_1) \\ F_2(\mathbf{y}, \alpha_1) \\ F_3(\mathbf{y}, \alpha_1) \end{pmatrix} = \begin{pmatrix} m_{11}^{-1}\widetilde{Y}_1 + m_{12}^{-1}\widetilde{Y}_2 + m_{13}^{-1}\widetilde{Y}_3 \\ m_{21}^{-1}\widetilde{Y}_1 + m_{22}^{-1}\widetilde{Y}_2 + m_{23}^{-1}\widetilde{Y}_3 \\ m_{31}^{-1}\widetilde{Y}_1 + m_{32}^{-1}\widetilde{Y}_2 + m_{33}^{-1}\widetilde{Y}_3 \end{pmatrix},$$

$K_2 = \overline{Y}_2$, $F_2 = \overline{F}_1$, and F_3 is real function (the symbol "$-$" means complex conjugate expression).

Theorem 2. *There exists a polynomial transformation*

$$Y_2 = Y_3 + h_1(Y_3, K_3, \alpha_1)$$

$$K_2 = K_3 + h_2(Y_3, K_3, \alpha_1) \tag{12}$$

$$\pi_2^e = \pi_3^e + h_3(Y_3, K_3, \alpha_1),$$

where $h_j(Y_3, K_3, \alpha_1)$, $j = 1,2,3,$ *are nonlinear polynomials with constant coefficients of the kind*

$$h_j(Y_3, K_3, \alpha_1) = \sum_{m_1+m_2+m_3 \geq 2, m_3 \in \{0,1\}}^{4-2m_3} h_j^{(m_1,m_2,m_3)} Y_3^{m_1} K_3^{m_2} \alpha_1^{m_3}, \quad j = 1,2,3, h_2 = \overline{h}_1,$$

which transforms the model

$$\begin{aligned} \dot{Y}_2 &= i\omega_0 Y_2 + F_1(Y_2, K_2, \pi_2^e, \alpha_1) \\ \dot{K}_2 &= -i\omega_0 K_2 + F_2(Y_2, K_2, \pi_2^e, \alpha_1) \\ \dot{\pi}_2^e &= \lambda_{30}\pi_2^e + F_3(Y_2, K_2, \pi_2^e, \alpha_1) \end{aligned} \tag{11}$$

into its partial normal form on a center manifold

$$\begin{aligned} \dot{Y}_3 &= i\omega_0 Y_3 + \delta_1 Y_3 \alpha_1 + \delta_2 Y_3^2 K_3 + U^{\circ}(Y_3, K_3, \pi_3^e, \alpha_1) + U^{*}(Y_3, K_3, \pi_3^e, \alpha_1) \\ \dot{K}_3 &= -i\omega_0 K_3 + \overline{\delta}_1 K_3 \alpha_1 + \overline{\delta}_2 Y_3 K_3^2 + \overline{U}^{\circ}(Y_3, K_3, \pi_3^e, \alpha_1) + \\ &\quad + \overline{U}^{*}(Y_3, K_3, \pi_3^e, \alpha_1) \\ \dot{\pi}_3^e &= \lambda_{30}\pi_3^e + V^{\circ}(Y_3, K_3, \pi_3^e, \alpha_1) + V^{*}(Y_3, K_3, \pi_3^e, \alpha_1), \end{aligned} \tag{13}$$

where $U^{\circ}(Y_3, K_3, 0, \alpha_1) = V^{\circ}(Y_3, K_3, 0, \alpha_1) = 0,$

$$U^{*}\left(\sqrt{\alpha_1} Y_3, \sqrt{\alpha_1} K_3, \sqrt{\alpha_1}\pi_3^e, \alpha_1\right) = \left(\sqrt{\alpha_1}\right)^5 \widetilde{U}(Y_3, K_3, \pi_3^e, \alpha_1),$$

$$V^{*}\left(\sqrt{\alpha_1} Y_3, \sqrt{\alpha_1} K_3, \sqrt{\alpha_1}\pi_3^e, \alpha_1\right) = \left(\sqrt{\alpha_1}\right)^5 \widetilde{V}(Y_3, K_3, \pi_3^e, \alpha_1),$$

and $\widetilde{U}, \widetilde{V}$ *are continuous functions.*

The resonant coefficients δ_1 *and* δ_2 *in the model (13) are determined by the formulae*

$$\delta_1 = m_{11}^{-1}\left\{\left[I_Y + I_r r_Y + c(1-\tau) - 1 + J_Y + J_\pi \pi_Y\right] m_{11} + I_k m_{21} + J_\pi \pi_{\pi^e} m_{31}\right\}$$

$$\begin{aligned} \delta_2 = m_{11}^{-1}&\left[\frac{1}{2}\alpha_0\left(I_Y^{(3)} + J_Y^{(3)}\right) m_{11}^2 m_{12} + \frac{1}{2}\alpha_0 J_{\pi^e}^{(3)} m_{31}^2 m_{32} + \right. \\ &\left. + \alpha_0\left(I_Y^{(2)} + J_Y^{(2)}\right) A + \alpha_0 J_{\pi^e}^{(2)} B\right] + \\ &+ m_{12}^{-1}\left[\frac{1}{2} I_Y^{(3)} m_{11}^2 m_{12} + I_Y^{(2)} A\right] + m_{13}^{-1}\left[\frac{1}{2}\gamma_0 \pi_Y^{(3)} m_{11}^2 m_{12} + \gamma_0 \pi_Y^{(2)} A\right], \end{aligned}$$

where

$$h_1^{(1,1,0)} = -\frac{1}{i\omega_0}\left(Cm_{11}^{-1} + Dm_{12}^{-1} + Em_{13}^{-1}\right)$$

$$h_1^{(2,0,0)} = \frac{1}{2i\omega_0}\left(Fm_{11}^{-1} + Gm_{12}^{-1} + Hm_{13}^{-1}\right)$$

$$h_2^{(1,1,0)} = \frac{1}{i\omega_0}\left(Cm_{21}^{-1} + Dm_{22}^{-1} + Em_{23}^{-1}\right)$$

$$h_2^{(2,0,0)} = \frac{1}{6i\omega_0}\left(Fm_{21}^{-1} + Gm_{22}^{-1} + Hm_{23}^{-1}\right)$$

$$h_3^{(1,1,0)} = -\frac{1}{\lambda_{30}}\left(Cm_{31}^{-1} + Dm_{32}^{-1} + Em_{33}^{-1}\right)$$

$$h_3^{(2,0,0)} = \frac{1}{2\left(2i\omega_0 - \lambda_{30}\right)}\left(Fm_{31}^{-1} + Gm_{32}^{-1} + Hm_{33}^{-1}\right),$$

and

$$A = m_{11}^2 h_1^{(1,1,0)} + m_{12}^2 h_2^{(2,0,0)} + m_{11}m_{12}\left(h_2^{(1,1,0)} + h_1^{(2,0,0)}\right) +$$

$$+ m_{11}m_{13}h_3^{(1,1,0)} + m_{12}m_{13}h_3^{(2,0,0)}$$

$$B = m_{31}^2 h_1^{(1,1,0)} + m_{32}^2 h_2^{(2,0,0)} + m_{31}m_{32}\left(h_2^{(1,1,0)} + h_1^{(2,0,0)}\right) +$$

$$+ m_{31}m_{33}h_3^{(1,1,0)} + m_{32}m_{33}h_3^{(2,0,0)}$$

$$C = m_{11}m_{12}\alpha_0\left(I_Y^{(2)} + J_Y^{(2)}\right) + m_{31}m_{32}\alpha_0 J_{\pi^e}^{(2)}$$

$$D = m_{11}m_{12}I_Y^{(2)}$$

$$E = m_{11}m_{12}\gamma_0\pi_Y^{(2)}$$

$$F = m_{11}^2\alpha_0\left(I_Y^{(2)} + J_Y^{(2)}\right) + m_{31}^2\alpha_0 J_{\pi^e}^{(2)}$$

$$G = m_{11}^2 I_Y^{(2)}$$

$$H = m_{11}^2\gamma_0\pi_Y^{(2)},$$

while all partial derivatives are calculated at the values $Y_1 = K_1 = \pi_1^e = 0, \alpha_1 = 0.$

Proof. The unknown terms $h_j^{(m_1,m_2,m_3)}, j=1,2,3$ and the resonant terms δ_1,δ_2 can be found by the standard procedure which is described for the example in [2]. As the whole process of finding them is rather elaborate, we do not present it here.

In polar coordinates $Y_3 = re^{i\varphi}, K_3 = re^{-i\varphi}$ the model (13) can be written as

$$\begin{aligned}
\dot{r} &= r\left(ar^2 + b\alpha_1\right) + R^{\circ}\left(r,\varphi,\pi_3^e,\alpha_1\right) + R^{*}\left(r,\varphi,\pi_3^e,\alpha_1\right) \\
\dot{\varphi} &= \omega_0 + c\alpha_1 + dr^2 + \frac{1}{r}\left[\Phi^{\circ}\left(r,\varphi,\pi_3^e,\alpha_1\right) + \Phi^{*}\left(r,\varphi,\pi_3^e,\alpha_1\right)\right] \\
\dot{\pi}_3^e &= \lambda_{30}\pi_3^e + W^{\circ}\left(r,\varphi,\pi_3^e,\alpha_1\right) + W^{*}\left(r,\varphi,\pi_3^e,\alpha_1\right),
\end{aligned} \tag{14}$$

where $a = \operatorname{Re}\delta_2, b = \operatorname{Re}\delta_1$. The equation

$$ar^2 + b\alpha_1 = 0 \tag{15}$$

is the bifurcation equation of the model (14). It determines the behaviour of solutions in a neigbourhood of the equilibrium point of the model (5). Utilizing the results from the bifurcation theory [3], [11] we can formulate the following theorem.

Theorem 3. *Let the coefficients a,b in the bifurcation equation (15) exist.*

1) *If $a<0$ then there exists a stable limit cycle for every small enough $\alpha_1>0$, if b is positive and for every small enough $\alpha_1<0$, if b is negative.*

2) *If $a>0$ then there exists an unstable limit cycle for every small enough $\alpha_1<0$, if b is positive and for every small enough $\alpha_1>0$, if b is negative.*

Conclusions

The main contributions of this paper are the results in Theorem 1and in Theorem 2. Theorem 1 gives sufficient conditions for the existence of a critical triple of the model (3). Theorem 2 gives the formulae for the calculation of the first two resonant coefficients of the model. These theorems are important for the investigation of the existence of limit cycles which are interpreted as business cycles in economics. We intend to show an application of both the model of flexible exchange rates and the model of fixed exchange rates on selected countries.

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References

[1] Asada, T.: Kaldorian Dynamics in an Open Economy, Journal of Economics No. 3, Springer-Verlag, Printed in Austria, pp. 239-269, 1995

[2] Bibikov, Yu. N.: Local Theory of Nonlinear Analytic Ordinary Differential Equations, Springer-Verlag, Berlin, 1979

[3] Bibikov, Yu. N.: Multi-frequency non-linear oscillations and their bifurcations, The Publishing House of the Saint Petersburg University, Saint Petersburg, 1991

[4] Gandolfo, G.: Economic Dynamics, Springer-Verlag, Berlin, 1997

[5] Liu, W. M.: Criterion of Hopf Bifurcations without using Eigenvalues, Journal of Mathematical Analysis and Applications, 182, pp. 250-256, 1994

[6] Makovínyiová, K.: Existence of an Equilibrium and Business Cycles in the Economy Model, in Proceedings of 10th International Scientific Conference Applications of Mathematics and Statistics in Economy, Poprad, Slovakia, August 30-31, 2007, pp. 145-153

[7] Makovínyiová, K., Zimka, R.: On the existence of Hopf bifurcation in an open economy model, Minisymposia and contributed talks, Comenius University Press, Bratislava, Slovakia, pp. 229-236, 2007

[8] Makovínyiová, K., Zimka, R.: On the necessary condition for the bifurcation of a torus in a macroeconomic model, in Proceedings of 12th International Scientific Conference Applications of Mathematics and Statistics in Economy, Uherské Hradiště, Czech Republic, August 26-28, 2009, pp. 269-278

[9] Maličký, P., Zimka, R.: On the existence of business cycles in Asada's two-regional model, Nonlinear Analysis: Real World Applications, Volume 11, Issue 4, pp. 2787-2795, 2010

[10] Medveďová, P.: Dynamic Model of a Small Open Economy Under Fixed Exchange Rates, in Proceedings of 12th International Scientific Conference Applications of Mathematics and Statistics in Economy, Uherské Hradiště, Czech Republic, August 26-28, 2009, pp. 305-313

[11] Wiggins, S.: Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer-Verlag, Berlin, 1990

[12] Maličký, P., Zimka, R.: Can Business Cycles Arise in a Two-Regional Model with Fixed Exchange Rates?, Tatra Mountains Mathematical Publications 43 (2009), ISSN 1210-3195, pp. 123-135

[13] L'ubica Lesáková: Innovations in Small and Medium Enterprises in Slovakia, in Acta Polytechnica Hungarica, Vol. 6, No. 3, 2009, pp. 23-34