Directional Monotonicity of Fuzzy Implications

Katarzyna Miś

Institute of Mathematics, University of Silesia in Katowice
Bankowa 14, 40-007 Katowice, Poland, kmis@us.edu.pl

Abstract: In this paper we consider special fuzzy implications as directional increasing functions and we introduce the notion of inversely special fuzzy implications as directional decreasing functions. We recall some results connected with special R-implications shown by Sainio et al. [A characterization of fuzzy implications generated by generalized quantifiers, Fuzzy Sets and Systems 159, 2008, pp. 491-499] and we present several new results connected with inversely special R-implications. Also, we discuss this new property for other families of fuzzy implications like (S,N)-implications, f-implications and g-implications.

Keywords: fuzzy implications; special implications; inversely special implications; directional monotonicity

1 Introduction

Standard monotonicity is one of the key properties of any function. Some functions used in fuzzy logic like t-norms, t-conorms, copulas are increasing in each variable. However, a very important fuzzy connective, a fuzzy implication, is hybrid monotonic – it is decreasing in the first variable and increasing in the second one. For such functions, among others, a notion of directional monotonicity was introduced. Our motivation is the article *Directional monotonicity of fusion functions* (Bustince et al. [3]) in which the authors investigated it deeper for different families of functions. In our paper we refer it to fuzzy implications. It turns out that there are some directional increasing and decreasing implications among which we support with examples.

In this paper we consider special fuzzy implications as directional increasing functions and inversely special fuzzy implications as directional decreasing functions (see Section 3). The first notion was introduced in 1996 by Hájek and Kohout [5]. Later, Sainio et al. [11] and Jayaram and Mesiar [6] showed some results concerning R-implications, which we cite here in Section 4. In Section 5 we formulate main new results for inversely special R-implications, while in Section 6 we consider other classes of inversely special fuzzy implications: (S,N)-implications, f-implications and g-implications.

2. Basic Definitions

This section contains definitions, properties and characterizations of directional monotonicity, fuzzy connectives and convex functions that will be used in the main part of this paper.

2.1 A Directional Monotonicity

The notion of the directional monotonicty was introduced in 2015 for functions which are not monotonic in each variable. Such functions are for example weighted arithmetic means, OWA operators, the Choquet and Sugeno integrals. As we mentioned before, fuzzy implications are monotonic in each variable separately, but not together. However, all these types of functions can be monotonic in a way described below.

Definition 1 *(Bustince et al. [3, Definition 2]).* Let $n \in \mathbb{N}, n \geq 2, \mathbb{I}$ be the unit interval $[0,1]$ and $r \in \mathbb{R}^n$ such that $r = (r_1, \dots, r_n) \neq (0, \dots, 0)$. A function $F: \mathbb{I}^n \longrightarrow \mathbb{I}$ is:

i. r-increasing, if for all $x \in \mathbb{I}^n$ and $c > 0$ such that $x + cr \in \mathbb{I}^n$, it holds that
$$F(x + cr) \geq F(x)$$
ii. r-decreasing, if for all $x \in \mathbb{I}^n$ and $c > 0$ such that $x + cr \in \mathbb{I}^n$, it holds that
$F(x + cr) \leq F(x)$.

Lemma 2 Let $r = (r_1, \dots, r_1), r_1 > 0$ and $n \in \mathbb{N}, n \geq 2$. A function $F: \mathbb{I}^n \longrightarrow \mathbb{I}$ is:

1) r-increasing if and only if it is $\mathbb{1}$-increasing,
2) r-decreasing if and only if it is $\mathbb{1}$-decreasing,

where $\mathbb{1} = \underbrace{(1, \dots, 1)}_{n}$

Proof. We show it only for r-increasing functions. The proof for r-decreasing functions is parallel. Let $x = (x_1, \dots, x_n)$ for $x_1, \dots, x_n \in \mathbb{I}$. If a function F is $(r_1, \dots, r_1)$-increasing, then for $c > 0$ such that $(x_1 + cr_1, \dots, x_n + cr_1) \in \mathbb{I}^n$ we have

$$F(x_1 + cr_1, \dots, x_n + cr_1) \geq F(x_1, \dots, x_n)$$
$$F(x_1 + d \cdot 1, \dots, x_n + d \cdot 1) \geq F(x_1, \dots, x_n)$$

Hence, F is $\mathbb{1}$-increasing, for $d > 0$ and $d = cr_1$.

If F is $\mathbb{1}$-increasing, then for applicable $c > 0$ we have

$$F(x_1 + c, \dots, x_n + c) \geq F(x_1, \dots, x_n)$$
$$F(x_1 + d \cdot r_1, \dots, x_n + d \cdot r_1) \geq F(x_1, \dots, x_n)$$

where $d = \frac{c}{r_1}$ and $d > 0$. Therefore, F is r-increasing.

Let us consider a notion of directional monotonicity for two different types of functions.

Example 3

1. The Fodor implication is given by the formula
$$I_{FD}(x,y) = \begin{cases} 1, & x \le y \\ \max\{1-x, y\}, & x > y \end{cases} \quad \text{for } x, y \in [0,1]$$
For $x = 0.2,\ y = 0.1,\ c = 0.4$ we have
$$I_{FD}(x,y) = 0.8 > I_{FD}(x+c, y+c) = 0.5$$
For the same x, y and $c = 0.71$ we have
$$I_{FD}(x,y) = 0.8 < I_{FD}(x+c, y+c) = 0.81$$
Therefore, I_{FD} is not $(1,1)$-increasing neither $(1,1)$-decreasing.
2. The Goguen implication, given by the formula
$I_{GG}(x,y) = \begin{cases} 1, x \le y \\ \frac{y}{x}, x > y \end{cases}$ for $x, y \in [0,1]$, is (r_1, r_2)-increasing for $r_1, r_2 \ge 0$ such that $r_2 \ge r_1$. Indeed, for $x \le y$ and $c > 0$ such that $x + cr_1$, $y + cr_2 \in [0,1]$ we have $x + cr_1 \le y + cr_2$ when $r_1 \le r_2$ and then $I(x,y) = 1 \le 1 = I(x + cr_1, y + cr_2)$. For $x > y$ and applicable $c > 0$ we have $\frac{y}{x} \le \frac{y+cr_2}{x+cr_1} \Leftrightarrow r_1 y - r_2 x \le 0$, which is true if $r_1 \le r_2$.
3. Let $F: [0,1]^2 \to [0,1]$ be a function given by the formula
$F(x,y) = (1-\lambda) \cdot \max\{x,y\} + \lambda \cdot \min\{x,y\}, \lambda \in [0,1]$ (see [2]).
Then it is r-decreasing for all $r \in [0,1]^2$ such that $r = (r_1, r_2)$ and $r_1 + \frac{\lambda}{1-\lambda} r_2 \le 0,\ r_1 + \frac{1-\lambda}{\lambda} r_2 \le 0$. Indeed, for all $r_1, r_2, x, y \in [0,1]$ and $c > 0$ such that $x + cr_1, y + cr_2 \in [0,1]$ we have
$$(1-\lambda) \cdot \max\{x,y\} + \lambda \cdot \min\{x,y\} \ge$$
$$(1-\lambda) \cdot \max\{x + cr_1, y + cr_2\} + \lambda \cdot \min\{x + cr_1, y + cr_2\}$$
this leads us to the following inequalities:
$r_1 \le -\frac{\lambda}{1-\lambda} r_2$, when $x \ge y$ and $r_1 \le -\frac{1-\lambda}{\lambda} r_2$ for $x < y$

The notion of the directional monotonicity is a generalization of another one, i.e., weak monotonicity (see [12]). Thanks to Lemma 2 we can say that weak monotonic function is a directional one in the direction of the vector $\mathbb{1}$.

More general facts and properties of directional monotonic functions can be found in Bustince et al. [3].

2.2 Fuzzy Connectives

We assume that the reader is familiar with the classical results concerning basic fuzzy logic connectives, but to make this work more self-contained, we place some of them here.

Definition 4 *(Fodor and Roubens [4]).* A function $N: [0,1] \longrightarrow [0,1]$ is called a fuzzy negation if

- $N(0) = 1$ and $N(1) = 0$
- N is decreasing

The basic example of a fuzzy negation is the classical strong negation N_C, i.e.,

$N_C(x) = 1 - x, x \in [0,1]$.

2.2.1 T- norms, t-conorms and Copulas

This part contains basic definitions and theorems, which are necessary to define some families of fuzzy implications.

Definition 5 *(Fodor and Roubens [4]).* A function $T: [0,1]^2 \longrightarrow [0,1]$ is called a triangular norm (t-norm) if it satisfies the following conditions:

- $T(1, x) = x$ for all $x \in [0,1]$
- $T(x, y) = T(y, x)$ for all $x, y \in [0,1]$
- $T(x, y) \leq T(u, v)$ for all $0 \leq x \leq u \leq 1, 0 \leq y \leq v \leq 1$
- $T\big(x, T(y, z)\big) = T(T(x, y), z)$ for all $x, y, z \in [0,1]$

Definition 6 *(Fodor and Roubens [4]).* A function $S: [0,1]^2 \longrightarrow [0,1]$ is called a triangular conorm (t-conorm) if it satisfies the following conditions:

- $S(0, x) = x$ for all $x \in [0,1]$
- $S(x, y) = S(y, x)$ for all $x, y \in [0,1]$
- $S(x, y) \leq S(u, v)$ for all $0 \leq x \leq u \leq 1, 0 \leq y \leq v \leq 1$
- $S\big(x, S(y, z)\big) = S(S(x, y), z)$ for all $x, y, z \in [0,1]$

Definition 7 *(Klement et al. [7, Definitions 2.9, 2.13]).* A t-norm T is said to be:

- Archimedean, if for each $(x, y) \in (0,1)^2$ there is an $n \in \mathbb{N}$ such that $x_T^{[n]} < y$, where by the notation $x_T^{[n]}$ we understand $x_T^{[n]} = \begin{cases} 1, & \text{if } n = 0 \\ x, & \text{if } n = 1 \\ T\Big(x, x_T^{[n-1]}\Big), & \text{if } n > 1 \end{cases}$
- Nilpotent, if it is continuous and for each $x \in (0,1)$ there is an $n \in \mathbb{N}$ such that $x_T^{[n]} = 0$.
- Strict, if it is continuous and strictly monotonic, i.e., $T(x, y) < T(x, z)$ whenever $x > 0$ and $y < z$.

The following theorem is usually used to characterize continuous Archimedean t-norms, its first proof can be found in the article written by Ling [9].

Theorem 8 *(Klement et al. [7, Theorem 5.1]).* For a function $T: [0,1]^2 \longrightarrow [0,1]$ the following statements are equivalent:

i. T is a continuous Archimedean t-norm

ii. T has a continuous additive generator, i.e., there exists a continuous, strictly decreasing function $f: [0,1] \longrightarrow [0, \infty]$ with $f(1) = 0$ such that $T(x, y) = f^{-1}(\min\{f(x) + f(y), f(0)\})$, for $x, y \in [0,1]$. Moreover, such representation is unique up to a positive multiplicative constant.

The following theorem tells about a method of constructing new t-norms from some family of given t-norms.

Theorem 9 *(Klement et al. [7, Theorem 3.43])*. Let $(T_\alpha)_{\alpha \in A}$ be a family of t-norms and $((a_\alpha, e_\alpha))_{\alpha \in A}$ be a family of non-empty, pairwise disjoint open subintervals of [0,1]. Then the following function $T: [0,1]^2 \longrightarrow [0,1]$ is a t-norm:

$$T(x, y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot T_\alpha\left(\frac{x - a_\alpha}{e_\alpha - a_\alpha}, \frac{y - a_\alpha}{e_\alpha - a_\alpha}\right), & \text{if } (x, y) \in [a_\alpha, e_\alpha]^2 \\ \min\{x, y\}, & \text{otherwise.} \end{cases} \tag{1}$$

This theorem allows us to formulate the following definition.

Definition 10 *(Klement et al. [7, Definition 3.44])*. Let $(T_\alpha)_{\alpha \in A}$ be a family of t-norms and $((a_\alpha, e_\alpha))_{\alpha \in A}$ be a family of non-empty, pairwise disjoint open subintervals of [0,1]. The t-norm T defined by (1) is called the ordinal sum of the summands $< a_\alpha, e_\alpha, T_\alpha >, \alpha \in A$, and we shall write $T = (< a_\alpha, e_\alpha, T_\alpha >)_{\alpha \in A}$.

In the following theorem, we recall a very important characterization of continuous t-norms.

Theorem 11 *(Klement et al. [7, Theorem 5.11])*. For a function $T: [0,1]^2 \to [0,1]$ the following statements are equivalent:

i. T is a continuous t-norm.
ii. T is uniquely representable as an ordinal sum of continuous Archimedean t-norms, i.e., T is defined by a formula (1).

We present a definition of a copula below. This notion is necessary to show its relationship with t-norms.

Definition 12 *(Klement et al. [7, Definition 9.4])*. A function $C: [0,1]^2 \longrightarrow [0,1]$ is a copula if, for all $x, y, u, v \in [0,1]$ with $x \leq u$ and $y \leq v$, it satisfies the following conditions:

- $C(x, y) + C(u, v) \geq C(x, v) + C(u, y)$
- $C(x, 0) = C(0, x) = 0$
- $C(x, 1) = C(1, x) = x$

Definition 13. A function $f: [0,1]^2 \longrightarrow [0,1]$ is said to be 1-Lipschitz if it satisfies the Lipschitz property with constant 1 i.e.,

$|f(x_1, y_1) - f(x_2, y_2)| \leq |x_1 - x_2| + |y_1 - y_2|$ for all $x_1, x_2, y_1, y_2 \in [0,1]$.

The next theorem is the full characterization of t-norms which are copulas.

Theorem 14 *(Moynihan [10, Theorem 3.1], Klement et al [7, Theorem 9.10])*. For a t-norm T the following statements are equivalent:

i. T is a copula.
ii. T is 1-Lipschitz.

2.2.2 Convex Functions

This section contains known theorems, which describe continuous and convex functions. Properties presented here are needed in the next part of the work for additive generators of t-norms.

Definition 15 *(Kuczma [8, p.130])*. Let $D \subset \mathbb{R}^n, n \in \mathbb{N}$ be a convex and open set. A function $f: D \longrightarrow \mathbb{R}$ is called convex if it satisfies the Jensen's functional inequality $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$ for all $x, y \in D$.

Definition 16 *(Kuczma [8, p.130])*. Let $D \subset \mathbb{R}^n, n \in \mathbb{N}$ be a convex and open set. A function $f: D \longrightarrow \mathbb{R}$ is called concave if it satisfies the following functional inequality $f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}$ for all $x, y \in D$.

Theorem 17 *(Kuczma [8, Theorem 7.1.1])*. For a function $f: D \longrightarrow \mathbb{R}$ the following statements are equivalent:

i. f is convex and continuous.
ii. For all $\lambda \in [0,1]$ and all $x, y \in D$ it holds
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \tag{2}$$

The following characterization is true for continuous functions.

Theorem 18 *(Kuczma [8, Theorems 7.3.2 and 7.3.3])*. For a continuous function $f: [0,1] \longrightarrow \mathbb{R}$ the following statements are equivalent:

i. f is convex.
ii. f satisfies the inequality
$$f(y+\varepsilon) - f(y) \leq f(x+\varepsilon) - f(x), \tag{3}$$
for all $x, y \in [0,1]$ such that $y \leq x$ and all $\varepsilon > 0$ such that $x+\varepsilon, y+\varepsilon \in [0,1]$.

It is well-known that a function f is convex, if and only if, $-f$ is concave. Therefore, the analogous theorem can be formulated for concave functions.

Theorem 19. For a continuous function $f: [0,1] \longrightarrow \mathbb{R}$ the following statements are equivalent:

i. f is concave.
ii. f satisfies the inequality
$$f(y+\varepsilon) - f(y) \geq f(x+\varepsilon) - f(x), \tag{4}$$
for all $x, y \in [0,1]$ such that $y \leq x$ and all $\varepsilon > 0$ such that $x+\varepsilon, y+\varepsilon \in [0,1]$.

2.2.3 Fuzzy Implications

In this part we present main definitions connected with fuzzy implications.

Definition 20 *(Fodor and Roubens [4], Baczyński and Jayaram [1])*. A function $I: [0,1]^2 \longrightarrow [0,1]$ is called a fuzzy implication if it satisfies, for all $x,\ x_1,\ x_2, y, y_1, y_2 \in [0,1]$, the following conditions:

- if $x_1 \leq x_2$, then $I(x_1, y) \geq I(x_2, y)$
- if $y_1 \leq y_2$, then $I(x, y_1) \leq I(x, y_2)$
- $I(0,0) = 1$
- $I(1,1) = 1$
- $I(1,0) = 0$

Below, we cite one result that will be useful in the last part of our paper.

Theorem 21 *(Baczyński and Jayaram [1])*. Let $\phi: [0,1] \longrightarrow [0,1]$ be an increasing bijection. If I is a fuzzy implication, then the ϕ-conjugate of I given by formula $I_\phi(x, y) = \phi^{-1}(I(\phi(x), \phi(y)))$ for $x, y \in [0,1]$ is also a fuzzy implication.

Now, we present definitions of some families of fuzzy implications that will appear later.

Definition 22 *(Baczyński and Jayaram [1])*. A function $I: [0,1]^2 \longrightarrow [0,1]$ is called an R-implication if there exists a t-norm T such that

$$I(x, y) = \sup\{t \in [0,1]: T(x, t) \leq y\}, \quad \text{for } x, y \in [0,1] \tag{5}$$

If I is generated from a t-norm T, then it will be denoted by I_T.

Definition 23 *(Baczyński and Jayaram [1])*. A function $I: [0,1]^2 \longrightarrow [0,1]$ is called an (S, N)-implication if there exists a t-conorm S and a fuzzy negation N such that

$$I(x, y) = S(N(x), y), \quad \text{for } x, y \in [0,1]. \tag{6}$$

3 Special and Inversely Special Implications

As we mentioned before, the notion of directional monotonicity was introduced in 2015 (Bustince et al. [3]). However, earlier, in 1996, it appeared for fuzzy implications in the article by Hájek and Kohout [5], investigated in 2007 by Sainio et al. [11] and also in 2009 by Jayaram and Mesiar [6]. The authors suggested the following notion.

Definition 24 *(Sainio et al. [11])*. A fuzzy implication I is called special if

$$\forall_{\varepsilon>0}\ \forall_{x,y\in[0,1]}\ (x + \varepsilon, y + \varepsilon \in [0,1] \Rightarrow I(x, y) \leq I(x + \varepsilon, y + \varepsilon)). \tag{SP}$$

According to the Definition 1 we can say that special implications are (1,1)-increasing functions.

Below, we give some examples of special implications.

Example 25

1. The Łukasiewicz implication given by the formula
   $I_Ł(x,y) = \min\{1, 1-x+y\}$, for $x, y \in [0,1]$ (7)
   is a special implication (see [6]). Note that $I_Ł(x,y) = I_Ł(x+\varepsilon, y+\varepsilon)$ for $\varepsilon > 0$ and $x+\varepsilon, y+\varepsilon \in [0,1]$.
2. The Gödel implication given by the formula
   $I_G(x,y) = \begin{cases} 1, & x \le y \\ y, & x > y \end{cases}$, for $x, y \in [0,1]$,
   is special. Indeed, $I(x,y) = I(x+\varepsilon, y+\varepsilon)$ for $x \le y$ and suitable $\varepsilon > 0$. We also have $I(x,y) = y \le \mathrm{y} + \varepsilon = \mathrm{I}(\mathrm{x}+\varepsilon, \mathrm{y}+\varepsilon)$ for $x > y$ and proper $\varepsilon > 0$.

Analogously, we formulate the notion for fuzzy implications which are (1,1)-decreasing functions.

Definition 26 A fuzzy implication $I: [0,1]^2 \longrightarrow [0,1]$ is called inversely special if

$$\forall_{\varepsilon>0}\ \forall_{x,y\in[0,1]}\ (x+\varepsilon, y+\varepsilon \in [0,1] \Rightarrow I(x,y) \ge I(x+\varepsilon, y+\varepsilon)). \quad \text{(ISP)}$$

Below we show several examples of inversely special implications, which belong to different families of fuzzy implications.

Example 27

1. The Łukasiewicz implication $I_Ł$ is inveresly special (see Example 25).
2. Let S be a t-conorm, N the fuzzy negation given by
   $N(x) = \begin{cases} 0, & x = 1 \\ 1, & x < 1 \end{cases}$.
   Then the (S,N)-implication given by $I(x,y) = S(N(x),y) = \begin{cases} 1, x < 1 \\ y, x = 1 \end{cases}$ for $x, y \in [0,1]$ is inversely special. Indeed, for $x, y < 1$ and $\varepsilon > 0$ such that $x+\varepsilon < 1$ we have $1 = I(x,y) \ge I(x+\varepsilon, y+\varepsilon) = 1$. The condition (ISP) holds also for $x, y < 1$ such that $x+\varepsilon = 1$, since in this case $I(x,y) = 1 \ge y+\varepsilon = I(x+\varepsilon, y+\varepsilon)$. Note that this implication is also the R-implication generated from the drastic product t-norm T_D given by the formula $T_D(x,y) = \begin{cases} 0, & (x,y) \in [0,1)^2 \\ \min\{x,y\}, & \text{otherwise} \end{cases}$ for $x, y \in [0,1]$.
3. It is easy to check that the Rescher implication given by the formula
   $I_{RS}(x,y) = \begin{cases} 1, & x \le y \\ 0, & x > y \end{cases}$, for $x, y \in [0,1]$ is inversely special.
4. Note that the Gödel implication (see Example 25) is not inversely special. Let us take $x = 0.5$, $y = 0.3$ and $\varepsilon = 0.2$, then $I(x,y) = 0.3 < I(x+\varepsilon, y+\varepsilon) = 0.5$.

Lemma 28 Let I be a fuzzy implication. If I is special or inversely special, then it satisfies

- The identity principle i.e., $I(x,x) = 1$ for all $x \in [0,1]$ (IP)
- The left ordering property i.e., $\forall_{x,y\in[0,1]}\ (x \le y \Rightarrow I(x,y) = 1)$

Proof. We show it for inversely special implications (as for special ones it is similar). Let I be an inversely special implication and take $x \in [0,1)$ and $\varepsilon > 0$. Let us fix $\varepsilon = 1 - x > 0$, we have

$$1 \ge I(x,x) \ge I(x+\varepsilon, x+\varepsilon) = I(1,1) = 1$$

Of course $I(1,1) = 1$, hence $I(x,x) = 1$ for $x \in [0,1]$, so I satisfies (IP).

To show the second condition, let us take $x, y \in [0,1]$ such that $x \le y$, then

$$1 \ge I(x,y) \ge I(y,y) = 1$$

because of the monotonicity of I. Therefore, $I(x,y) = 1$

Note that the fuzzy implication I from Example 27 point 2 satisfies the left neutrality property, i.e.,

$$I(1,y) = y, \text{ for } y \in [0,1] \tag{NP}$$

However, it does not satisfy the ordering property i.e., the following equality

$$\forall_{x,y\in[0,1]}\ (x \le y \Leftrightarrow I(x,y) = 1) \tag{OP}$$

Indeed, $I(x,y) = 1$ for $x = 0.9$ and $y = 0.5$. This makes a difference between fuzzy implications satisfying (ISP) and (SP). If a special implication satisfies (NP) then it satisfies (OP) as well (see [*6, Proposition 2.7*]). Here, as we have seen, it can be opposite.

For all fuzzy implications, the following result is true.

Theorem 29 *(cf. Jayaram and Mesiar [6, Theorem 9.6])*. For an increasing bijection $\phi: [0,1] \longrightarrow [0,1]$ the following statements are equivalent:

i. For each inversely special implication I, the implication I_ϕ is an inversely special fuzzy implication.
ii. ϕ is convex.

Proof. $(i. \Rightarrow ii.)$ We can take any fuzzy implication which satisfies (ISP), so let us consider the Łukasiewicz implication $I_Ł$. Assume that $(I_Ł)_\phi$ is inversely special for some increasing bijection ϕ. Let us fix arbitrarily $x, y \in [0,1]$ such that $x \ge y$ and take any $\varepsilon > 0$ such that $x + \varepsilon, y + \varepsilon \in [0,1]$. From (ISP) for $(I_Ł)_\phi$ we obtain

$$\begin{aligned}\phi^{-1}\big(1 - \phi(x) + \phi(y)\big) = (I_Ł)_\phi(x,y) &\ge (I_Ł)_\phi(x+\varepsilon, y+\varepsilon)\\ &= \phi^{-1}\big(1 - \phi(x+\varepsilon) + \phi(y+\varepsilon)\big)\end{aligned}$$

thus, by the monotonicity of ϕ^{-1}, we have

$$1 - \phi(x) + \phi(y) \geq 1 - \phi(x + \varepsilon) + \phi(y + \varepsilon)$$

hence

$$\phi(x + \varepsilon) - \phi(x) \geq \phi(y + \varepsilon) - \phi(y)$$

and ϕ is convex in virtue of Theorem 18.

$(ii. \Rightarrow i.)$ Since I is inversely special, then it satisfies the left ordering property. We show that I_ϕ satisfies it too. Let us take $x \in [0,1)$ and define $\varepsilon = 1 - \phi(x) > 0$. From (ISP) for I we obtain

$$1 \geq I\big(\phi(x), \phi(x)\big) \geq I(\phi(x) + \varepsilon, \phi(x) + \varepsilon) = I(1,1) = 1$$

Of course

$$I_\phi(x, x) = \phi^{-1}\Big(I\big(\phi(x), \phi(x)\big)\Big) = \phi^{-1}(1) = 1$$

Thus, $1 = I_\phi(x, x) \leq I_\phi(x, y) \leq 1$ for any $x, y \in [0,1]$ such that $x \leq y$, because of the monotonicity of the fuzzy implication I_ϕ, hence $I_\phi(x, y) = 1$ for $x \leq y$.

Therefore, it remains to show that I_ϕ is inversely special for $x, y \in [0,1]$ such that $x > y$. To do this let us fix arbitrarily $x, y \in [0,1)$ such that $x > y$ (the case when $x = 1$ is not applicable in the definition of (ISP)). We know that

$$I\big(\phi(x), \phi(y)\big) \geq I(\phi(x) + \delta, \phi(y) + \delta)$$

for any $\delta > 0$ such that $\phi(x) + \delta, \phi(y) + \delta \in [0,1]$. Let us take any $\varepsilon > 0$ such that $x + \varepsilon \leq 1$. Bijection ϕ is in particular continuous, so from our assumption on convexity and by Theorem 18 we have $\phi(y + \varepsilon) \geq \phi(y) + \phi(x + \varepsilon) - \phi(x)$. Now, for $\delta = \phi(x + \varepsilon) - \phi(x) > 0$ we have

$$\begin{aligned} I\big(\phi(x), \phi(y)\big) &\geq I(\phi(x + \varepsilon), \phi(y) + \phi(x + \varepsilon) - \phi(x)) \\ &\geq I\big(\phi(x + \varepsilon), \phi(y + \varepsilon)\big) \end{aligned}$$

Therefore $\phi^{-1}\Big(I\big(\phi(x), \phi(y)\big)\Big) \geq \phi^{-1}\Big(I\big(\phi(x + \varepsilon), \phi(y + \varepsilon)\big)\Big)$ and thus I_ϕ is inversely special.

4 Characterizations of Special R-implications

First, we cite characterizations of special implications that are R-implications generated from specific t-norms.

Theorem 30 *(Sainio et al. [11, Proposition 2])*. For a continuous Archimedean t-norm *T* the following statements are equivalent:

i. The R-implication I_T satisfies (SP).

ii. The continuous additive generator of T is a convex function.

Theorem 31 *(Sainio et al. [11, Theorem 2])*. For a continuous t-norm T the following statements are equivalent:

i. The R-implication I_T satisfies (SP).
ii. T is the ordinal sum of the summands $< a_\alpha, e_\alpha, T_\alpha >, \alpha \in A$, where each T_α is generated by a convex additive generator f_α.

In particular, when *A* is the empty set, then *T* is the minimum t-norm and I_T is the Gödel implication which is special. As a corollary, they received the following result.

Theorem 32 *(Sainio et al. [11, Corollary 2])*. For a left-continuous t-norm T the following statements are equivalent:

i. The R-implication I_T satisfies (SP).
ii. T is 1-Lipschitz.

As an easy corollary we receive the following fact.

Corollary 33 For a left-continuous t-norm T the following statements are equivalent:

i. The R-implication I_T satisfies (SP).
ii. T is a copula.

5 Characterizations of Inversely Special R-implications

In this section, we present new results for inversely special fuzzy implications which are in some sense equivalents of results from previous section. The following remark says about such R-implications generated from 1-Lipschitz t-norms.

Theorem 34 The Łukasiewicz implication given by (7) is the only one R-implication generated from a 1-Lipschitz t-norm that is inversely special.

Proof. Let us take a 1-Lipschitz t-norm T and consider the R-implication generated from it. For the simplificity let us denote it by *I*. First notice that for every R-implication we have

$$I(1, y) = \sup\{t \in [0,1] = T(1, t) \leq y\} = y$$

for y∈ [0,1]. From Theorem 32 we know that *I* is special. Let us take $x, y \in [0,1)$ such that $x > y$ and $\varepsilon = 1 - x > 0$. Since *I* satisfies (SP) we can write

$$I(x, y) \leq I(x + \varepsilon, y + \varepsilon) = I(1,1 - x + y) = 1 - x + y = I_{Ł}(x, y)$$

Also, I satisfies (ISP). Therefore

$$I(x,y) \geq I(x+\varepsilon, y+\varepsilon) = I(1,1-x+y) = 1-x+y = I_{Ł}(x,y)$$

for $x > y$. Therefore $I(x,y) = I_{Ł}(x,y)$ for all $x,y \in [0,1)$ such that $x > y$. From Lemma 28 we know that $I(x,y) = 1$ for $x \leq y$. Hence $I(x,y) = I_{Ł}(x,y)$ for all $x,y \in [0,1]$.

For some R-implications generated from continuous t-norms, we can formulate a characterization of inversely special implications in the analogous way to special ones (compare the following result with Theorem 30).

Theorem 35 For a continuous Archimedean t-norm T the following statements are equivalent:

i. The R-implication I_T satisfies (ISP).
ii. The continuous additive generator of T is a concave function.

Proof. $(i. \Rightarrow ii.)$ Let T be a continuous Archimedean t-norm and I_T be the R-implication generated from T. Also, let $f\colon [0,1] \longrightarrow [0,1]$ be the additive generator of T, i.e., $T(x,y) = f^{-1}(\min\{f(x)+f(y), f(0)\})$, for $x,y \in [0,1]$. Hence, by Theorem 2.5.21 in [1] we obtain

$$I_T(x,y) = f^{-1}(\max\{f(y)-f(x), 0\}), \text{ for all } x,y \in [0,1]$$

From Theorem 19 it is enough to show the condition (4). Let us fix arbitrarily $x,y \in [0,1]$ such that $x \geq y$. Then $f(x) \leq f(y)$, so $f(y) - f(x) \geq 0$ and hence

$$I_T(x,y) = f^{-1}\big(f(y) - f(x)\big)$$

for such x,y. Since I_T is inversely special, for any $\varepsilon > 0$ such that $x+\varepsilon, y+\varepsilon \in [0,1]$, we receive

$$I_T(x,y) \geq I_T(x+\varepsilon, y+\varepsilon)$$

so

$$f^{-1}\big(f(y) - f(x)\big) \geq f^{-1}\big(f(y+\varepsilon) - f(x+\varepsilon)\big)$$

f^{-1} is also a decreasing function, therefore

$$f(y) - f(x) \leq f(y+\varepsilon) - f(x+\varepsilon)$$

hence

$$f(y+\varepsilon) - f(y) \geq f(x+\varepsilon) - f(x)$$

thus, by Theorem 19, f is a concave function.

$(ii. \Rightarrow i.)$ Let us assume that f is a concave function and by Theorem 19 we have

$$f(y+\varepsilon) - f(y) \geq f(x+\varepsilon) - f(x)$$

for $x,y \in [0,1]$, $x \geq y$ and applicable $\varepsilon > 0$. Hence

$$f^{-1}\big(f(y)-f(x)\big) \geq f^{-1}\big(f(y+\varepsilon)-f(x+\varepsilon)\big)$$

thus

$$I_T(x,y) \geq I_T(x+\varepsilon, y+\varepsilon)$$

We know that $I_T(x,y) = 1$ for $x < y$ and therefore I_T is inversely special.

Now we consider continuous t-norms (compare the following result with Theorem 31)

Theorem 36 For a continuous t-norm T the following statements are equivalent:

i. The R-implication I_T satisfies (ISP).
ii. T is continuous Archimedean with a concave generator.

Proof. $(i. \Rightarrow ii.)$ Let us take a continuous t-norm T and consider the R-implication I_T generated from this T. From Theorem 11 we know that T can be represented as an ordinal sum of continuous Archimedean t-norms. Then I_T is given by the following formula (see [1, Theorem 2.5.24]):

$$I_T(x,y) = \begin{cases} 1, & x \leq y \\ a_\alpha + (e_\alpha - a_\alpha) \cdot I_{T_\alpha}\left(\dfrac{x-a_\alpha}{e_\alpha - a_\alpha}, \dfrac{y-a_\alpha}{e_\alpha - a_\alpha}\right), & (x,y) \in [a_\alpha, e_\alpha]^2 \\ y, & \text{otherwise.} \end{cases}$$

Let us consider three cases with respect to the index set A.

1. If $A = \emptyset$, then $I_T = I_G$ (see Example 25 point 2). However, we have shown that I_G is special but not inversely special.
2. If $\bar{\bar{A}} = 1$ and $a_\alpha = 0, e_\alpha = 1$, then $T = < 0,1,T >$ and T is a continuous Archimedean t-norm. In this case, in virtue of Theorem 35, I_T is inversely special if and only if T has a concave generator.
3. In all other situations we consider two possibilities.
   a. There exists $\alpha 0 \in A$ such that $a_{\alpha 0} = 0$ and $e_{\alpha 0} < 1$. Let us take $x \in (e_{\alpha 0}, 1)$, $y \in (a_{\alpha 0}, e_{\alpha 0})$, then there exists $\varepsilon > 0$ such that $x + \varepsilon \in (e_{\alpha 0}, 1)$, $y + \varepsilon \in (a_{\alpha 0}, e_{\alpha 0})$. Then $I_T(x,y) = y < y + \varepsilon = I_T(x+\varepsilon, y+\varepsilon)$, so I_T does not satisfy (ISP) in this case.
   b. $a_\alpha > 0$, for all $\alpha \in A$. Let $a_{\alpha 0} = \min\{a_\alpha : \alpha \in A\}$ and $e_{\alpha 0} = \min\{e_\alpha : \alpha \in A\}$. Consider $x \in (a_{\alpha 0}, e_{\alpha 0})$ such that $\frac{x}{2} < a_{\alpha 0}$. Then there exists $\varepsilon > 0$ such that $x + \varepsilon \in (a_{\alpha 0}, e_{\alpha 0})$ and $\frac{x+\varepsilon}{2} < a_{\alpha 0}$. Then $I_T\left(x, \frac{x}{2}\right) = \frac{x}{2} < \frac{x}{2} + \frac{\varepsilon}{2} = I_T(x+\varepsilon, \frac{x+\varepsilon}{2})$.

As we have shown, if I_T is represented by a proper ordinal sum (case 3) it is not inversely special. Therefore I_T is inversely special if and only if T is continuous Archimedean with a concave generator.

$(ii. \Rightarrow i.)$ This follows from Theorem 35.

Based on the above results, we can formulate the following fact.

Theorem 37 For a continuous t-norm T the following statements are equivalent:

i. The R-implication I_T satisfies (ISP).
ii. The R-implication I_T is ϕ-conjugate with the Łukasiewicz implication, where ϕ is convex.

Proof. $(i. \Rightarrow ii.)$ Let T be a continuous t-norm. From Theorem 36 we know that if the R-implication generated from a continuous t-norm T satisfies (ISP), then T is also Archimedean. Among all such t-norms there are only two classes – nilpotent and strict t-norms (see [7, Theorem 2.18]). Let f be the additive generator of T. If T is nilpotent, then $f(0) < \infty$ and if T is strict, then $f(0) = \infty$ (see [7, Proposition 3.29]). We also know from Theorem 36 that f is concave, so by Theorem 18 the condition (4), i.e., the inequality

$$f(y+\varepsilon) - f(y) \geq f(x+\varepsilon) - f(x)$$

is true for $x, y \in [0,1]$ such that $y \leq x$ and $\varepsilon > 0$ such that $x+\varepsilon, y+\varepsilon \in [0,1]$. Let $y = 0$ and take any $x \in (0,1)$ and $\varepsilon \in (0,1)$ such that $x+\varepsilon \in (0,1)$. From the above inequality we obtain that

$$f(0) \leq f(x) - f(x+\varepsilon) - f(\varepsilon)$$

Therefore, $f(0) < \infty$ (because $f(x), f(x+\varepsilon), f(\varepsilon) < \infty$) and T must be nilpotent. Hence I_T is ϕ-conjugate with the Łukasiewicz implication (see [1, Lemma 2.5.23]). Moreover, we can define ϕ in the following way $\phi(x) = 1 - \frac{f(x)}{f(0)}$ for $x \in [0,1]$. Also if f is concave, then of course ϕ is convex.

$(ii. \Rightarrow i.)$ For any increasing bijection ϕ, by Proposition 2.5.10 in [1], the function $(I_{Ł})_\phi$ is a continuous R-implication generated from the continuous t-norm ϕ-conjugate with the Łukasiewicz t-norm, i.e., $(T_{Ł})_\phi(x,y) = \phi^{-1}(\max\{\phi(x) + \phi(y) - 1, 0\})$, $x, y \in [0,1]$. From Theorem 29 we know that $(I_{Ł})_\phi$ satisfies (ISP), if ϕ is convex.

Now, we present a proposition which contains a little more general characterization of directional decreasing R-implications generated from continuous t-norms.

Proposition 38 Let $\varepsilon, \varepsilon_1, \varepsilon_2 > 0, \varepsilon_2 \leq \varepsilon \leq \varepsilon_1$, T be a continuous t-norm and I_T be the R-implication generated from T. Then the following statements are equivalent:

i. I_T is an inversely special implication.
ii. I_T is $(\varepsilon_1, \varepsilon)$-decreasing.
iii. I_T is $(\varepsilon, \varepsilon_2)$-decreasing.

Proof. $(i. \Rightarrow ii.)$ If an R-implication I_T is inversely special, then t-norm T is continuous Archimedean with an additive generator f. Therefore for $x, y \in [0,1]$, $x \geq y$ and applicable $\varepsilon > 0$ we can write the following inequality

$$f^{-1}\big(f(y)-f(x)\big) \geq f^{-1}\big(f(y+\varepsilon)-f(x+\varepsilon)\big)$$

thus for proper $\varepsilon_1 > 0$ we have

$$f(y)-f(x) \leq f(y+\varepsilon)-f(x+\varepsilon) \leq f(y+\varepsilon)-f(x+\varepsilon_1)$$

because f is strictly decreasing. Further, $f^{-1}\big(f(y)-f(x)\big) \geq f^{-1}\big(f(y+\varepsilon)-f(x+\varepsilon_1)\big)$, what means $I_T(x,y) \geq I_T(x+\varepsilon_1, y+\varepsilon)$ in this case. Moreover, we have

$$1 = I_T(x,y) \geq I_T(x+\varepsilon_1, y+\varepsilon)$$

for any $x < y$ and applicable $\varepsilon > 0$. Proofs $(ii. \Rightarrow i.), (iii. \Rightarrow i.)$ are obvious and $(i. \Rightarrow iii.)$ is analogous to the above one.

The similar result can be formulated for special implications and the proof is similar to the above one.

Proposition 39 Let $\varepsilon, \varepsilon_1, \varepsilon_2 > 0, \varepsilon_1 \leq \varepsilon \leq \varepsilon_2$, T be a continuous t-norm and I_T be the R-implication generated from T. Then the following statements are equivalent:

i. I_T is a special implication.
ii. I_T is $(\varepsilon_1, \varepsilon)$-increasing.
iii. I_T is $(\varepsilon, \varepsilon_2)$-increasing.

6 Other Classes of Inversely Special Implications

In this part we consider different families of fuzzy implications, i.e., (S,N)-implications, f-implications and g-implications. We will use the following theorems to characterize inversely special (S,N)-implications.

Theorem 40 *(Baczyński and Jayaram [1, Theorem 2.4.17])*. For a t-conorm S and a fuzzy negation N the following statements are equivalent:

i. $I_{S,N}$ is a continuous (S,N)-implication that satisfies (IP).
ii. S is a nilpotent t-conorm and $N \geq N_S$, where N_S is the natural negation of S (see Definition 2.3.1 in [1]).

Theorem 41 *(Baczyński and Jayaram [1, Theorem 2.4.20])*. For a function $I: [0,1]^2 \to [0,1]$ the following statements are equivalent.

i. I is an (S,N)-implication obtained from a nilpotent t-conorm S and its natural negation N_S.
ii. I is ϕ-conjugate with the Łukasiewicz implication.

Thanks to these results, we can formulate the following corollary.

Corollary 42 For a function $I\colon[0,1]^2 \to [0,1]$ the following statements are equivalent.

i. I is an inversely special (S,N)-implication obtained from a continuous t-conorm S and its natural negation N_S.
ii. I is ϕ-conjugate with the Łukasiewicz implication, where ϕ is convex.

Proof. $(i. \Rightarrow ii.)$ Let I be an inversely special (S,N)-implication obtained from a continuous t-conorm S and its natural negation N_S. If I satisfies (ISP), then by Lemma 28 it satisfies (IP). I is in particular continuous, so in virtue of Theorem 40 the t-conorm S is nilpotent. Now, from Theorem 41 and Theorem 29 we obtain the thesis.

$(ii. \Rightarrow i.)$ By Theorem 2.4.5 in [1] the function $(I_Ł)_\phi$ is the (S,N)-implication obtained from the ϕ-conjugate Łukasiewicz t-conorm $S_Ł$ (which is in particular continuous) and its natural negation. From Theorem 29 we know that $(I_Ł)_\phi$ satisfies (ISP), if ϕ is convex.

A fuzzy implication ϕ-conjugate with an R-implication generated from any t-norm is also an R-implication (see [1, Proposition 2.5.10]). Since all implications ϕ-conjugate with the Łukasiewicz implication are R-implications, (S,N)-implications satisfying (ISP) and generated from a continuous t-conorm and the natural negation of this t-conorm are a subclass of inversely special R-implications.

Now let us consider f-implications and g-implications. We will see there are no inversely special implications among them.

First, let us recall some definitions and their properties.

Definition 43 *(Baczyński and Jayaram [1, Definition 3.1.1])*. Let $f\colon[0,1] \to [0,1]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I\colon[0,1]^2 \to [0,1]$ defined by

$$I(x,y) = f^{-1}\left(x \cdot f(y)\right)$$

for $x, y \in [0,1]$ with understanding $0 \cdot \infty = 0$, is called an f-generated implication. The function f itself is called an f-generator of the I. In such case, to emphasize the apparent relation, we will write I_f.

Theorem 44 *(Baczyński and Jayaram [1, Theorem 3.1.7])*. If f is an f-generator, then I_f does not satisfy (IP).

From the above theorem and Lemma 28 it is clear that all f-implications are not inversely special.

Corollary 45 There is no f-implication satisfying (ISP).

Definition 46 *(Baczyński and Jayaram [1, Definition 3.2.1]).* Let $g\colon [0,1] \longrightarrow [0,1]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I\colon [0,1]^2 \longrightarrow [0,1]$ defined by

$$I(x,y) = g^{-1}\left(\min\left\{\frac{1}{x} \cdot g(y), g(1)\right\}\right)$$

for $x, y \in [0,1]$, with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is called a g-generated implication. The function g itself is called a g-generator of the I and we will write I_g instead of I.

Theorem 47 *(Baczyński and Jayaram [1, Theorem 3.2.9])*. If g is a g-generator, then the following statements are equivalent:

i. I_g satisfies (OP).
ii. I_g is a Goguen implication.

Now we can prove the following fact.

Proposition 48 There is no g-implication satisfying (ISP).

Proof. Let us suppose that there is a g-implication I_g which is inversely special. Then it satisfies the left ordering property. From Theorem 47 we know that if g-implication satisfies (OP), then it is the Goguen implication, which is not inversely special. That means that $I_g$does not satisfy the following condition:

$I_g(x,y) = 1 \Rightarrow x \leq y$ for $x, y \in [0,1]$.

Therefore, there exist $x, y \in [0,1]$ such that $I_g(x,y) = 1$ and $x > y$. Observe that $x < 1$ and $y > 0$, since $I_g(1,y) = y$ and $I_g(x,0)$=0 if $x > 0$. Thus there exists $\varepsilon > 0$ such that $x' = x - \varepsilon > 0$, $y' = y - \varepsilon \geq 0$ and $x' > y'$. We assumed that I_g satisfies (ISP) and hence $I_g(x',y') \geq I_g(x' + \varepsilon, y' + \varepsilon) = I_g(x,y) = 1$. Furthermore, we can take $\varepsilon = y$ and then $x' = x - y$, $y' = 0$ and we get

$$0 = I_g(x - y, 0) = I_g(x', y') \geq I_g(x,y) = 1,$$

a contradiction. Therefore I_g cannot satisfy (ISP).

Conclusions

In this paper we have investigated special and inversely special implications, as directional monotonics, and we have provided some examples of them. We have characterized all inversely special R-implications generated from continuous t-norms. Also, we have considered other families of fuzzy implications. Our conclusion is that there are no inversely special implications other than R-implications in the set. Finally, we have shown some generalizations of inversely special implications as directional monotonic functions.

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