# Helical Two-Revolutional Cyclical Surface

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*Abstract: Paper presents a family of helical two-revolutional cyclical surfaces, which are created by movement of the circle alongside the helical cycloidal curve, where circle is located in the curve normal plane and its centre is on this curve. Helical cycloidal curve can be created by simultaneous revolution of a point about two different axes 3o, 2o and by screwing about axis 1o in the space. Form of the helical cycloidal curve and also of the helical two-revolutional cyclical surface is dependent on the relative position of the three axes of revolutions, on multiples of angular velocities and orientations of separate revolutions. Analytic representation, classification of surfaces and some of their geometric properties are derived.*

*Keywords: revolution, angular velocity, cyclical surface*

## 1 Introduction

Helical two-revolutional cyclical surface can be created by movement of the circle alongside the helical cycloidal curve. Circle is located in the normal plane of the curve and its centre is on this curve.

Helical cycloidal curve can be created by simultaneous revolutions of a point about two different lines, axis ${}^3o$, ${}^2o$ and by screwing about axis ${}^1o$. Trajectories of the point P, which revolves about single axes of revolutions are circles ${}^2k$, ${}^3k$ located in the planes perpendicular to the axes of revolution ${}^2o$, ${}^3o$, trajectory of the point P, which screws about axis ${}^1o$ is helix ${}^1k$. With respect to the relative position of axes of revolutions these circles do not necessarily lie in one plane. Form of the helical cycloidal curve is dependent on the relative position of the axes ${}^1o$, ${}^2o$, ${}^3o$, on the orientations of the single revolutions and on their angular velocities, and also on the position of the revolving point P with respect to the axes of revolutions. In the next section there is described the creation of one type of the helical two-revolutional cyclical surface for particular relative position of the axes of revolutions (Figure 1).

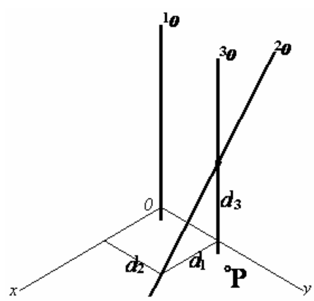



Figure 1
Position of the axes of revolutions

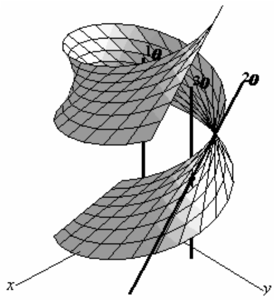



Figure 2
Linear oblique helical surface

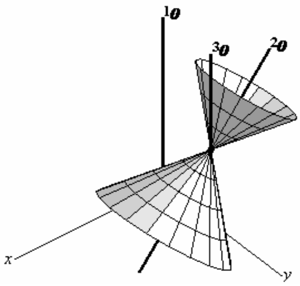



Figure 3
Conical surface of revolution

Let axis $^1o$ be fixed and $^1o = z$ in the Cartesian coordinate system $(O, x, y, z)$. Axis $^2o$ skew to $^1o$, $^2o \,/\, {}^1o$, creates a linear oblique helical surface by its screwing about axis $^1o$ with angular velocity $w_1 = v$, with orientation determined by parameter $q_1$ and screw height $h$ (Figure 2). Axis $^3o$ that is intersect to $^2o$, $^3o \times {}^2o$, creates a conical surface of revolution by revolution about axis $^2o$ with angular velocity $w_2 = m_1 w_1 = m_1 v$ and with orientation determined by parameter $q_2$ (Figure 3). Axis $^3o$ parallel to $^1o$, $^3o \parallel {}^1o$, creates a cylindrical helical surface of revolution by screwing about axis $^1o$ (Figure 4). In Figure 5 there are displayed all three surfaces together. Axis $^3o$, which revolves about axis $^2o$ and screws about axis $^1o$ simultaneously, creates a composed linear helical-revolutional suface (Figure 6). This surface has four identical branches, because axis $^3o$ revolves about axis $^2o$ with angular velocity, which is 4-multiple of angular velocity of revolution of the axis $^2o$ about axis $^1o$.

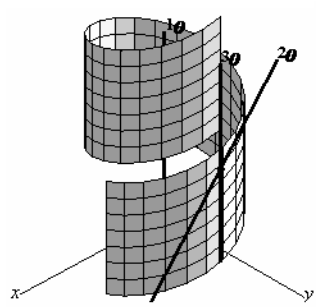



Figure 4
Cylindrical helical surface

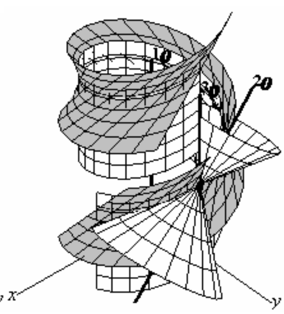



Figure 5
All three surfaces together

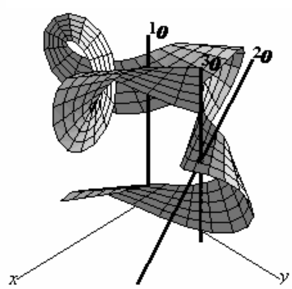



Figure 6
Composed linear helical-revolutional surface

Point P revolves about axis ${}^{3}o$ with angular velocity $w_3 = m_2 w_2 = m_2 m_1 v$ with orientation determined by parameter $q_3$, where perameters $q_1, q_2, q_3 = \pm 1$ (if $q_i = +1$, for i = 1,2,3, then revolution is right-handed, if $q_i = -1$, then revolution is left-handed). Trajectory of the point P movement created by its screwing about axis ${}^{1}o$ is helix ${}^{1}k$ (Figure 7), the circle ${}^{2}k$ is the trajectory of the point P movement about axis ${}^{2}o$ (Figure 8) and the circle ${}^{3}k$ is the trajectory of the point P movement about axis ${}^{3}o$ (Figure 9).

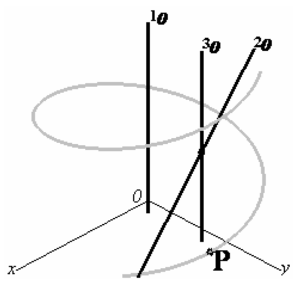



Figure 7
Helix ${}^{1}k$

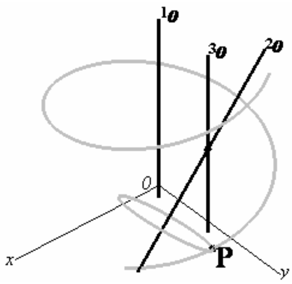



Figure 8
Helix ${}^{1}k$ and circle ${}^{2}k$

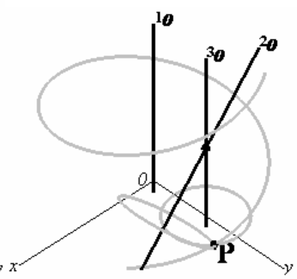



Figure 9
Helix ${}^{1}k$ and circles ${}^{2}k$, ${}^{3}k$

Curve $k$ as trajectory of the point **P** composite helical-two-revolutional movement is created by rolling of the circle ${}^{3}k$ on the circle ${}^{2}k$, which rolls on the helix ${}^{1}k$ simultaneously (Figure 10). Form of this helical cycloidal curve is dependent on the relative position of the axes ${}^{1}o, {}^{2}o, {}^{3}o$, on the orientations of the single revolutions and on their angular velocities, and also on the position of the revolving point P with respect to three axes of revolutions.

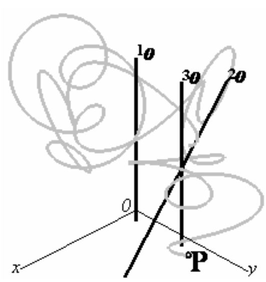



Figure 10
Trajectory of the point **P**

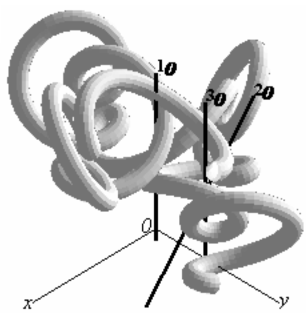



Figure 11
Helical two-revolutinal cyclical surface

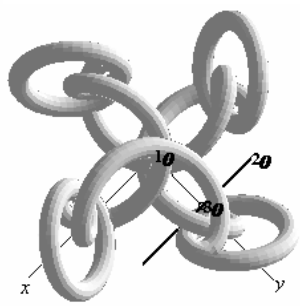



Figure 12
Wiev on it from above

Helical two-revolutinal cyclical surface can be created by moving a circle alongside the curve *k,* while the circle lies allways in the normal plane of the curve *k* and its centre is on the curve (Figure 11, in Figure 12 is view from above).

## 2 Classification of a Family of Helical Two-Revolutional Cyclical Surfaces

Classification of the family of helical two-revolutional cyclical surfaces can be done according to the relative position of axes of revolutions ${}^3o$, ${}^2o$ and ${}^1o$, which may be parallel, intersect or skew. Distribution of surfaces within the family is illustrated in the next Figure 13.

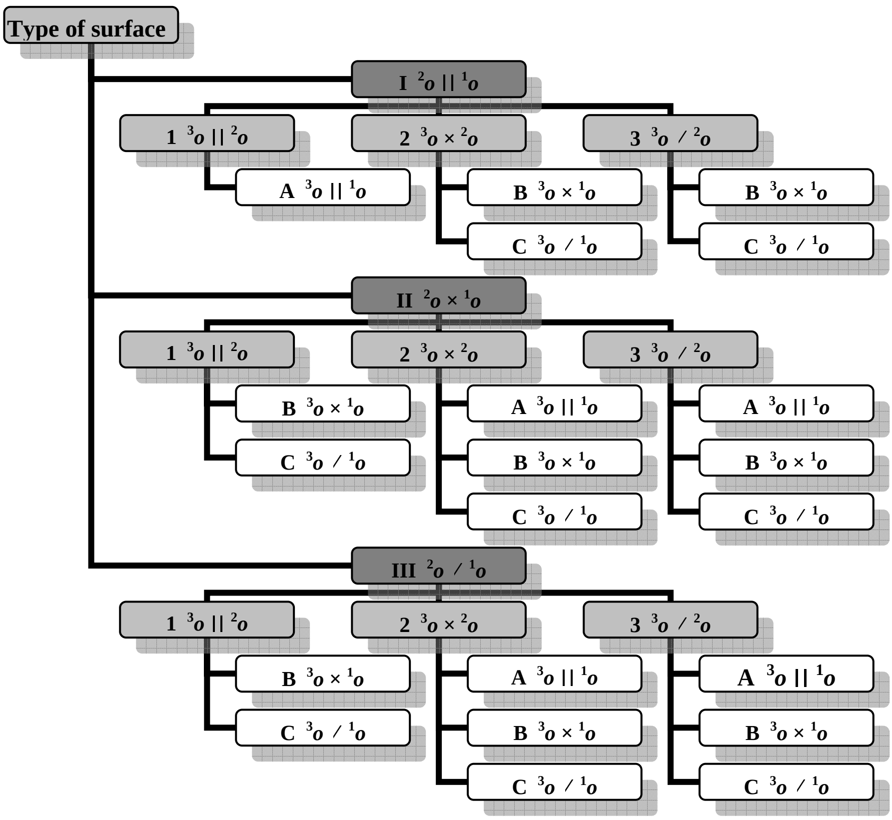



Figure 13
Classification of a family helical two-revolutional surfaces

Helical two-revolutional cyclical surfaces are distributed in the first level into the three types I, II, III with respect to the relative position of the axes ${}^2o$ and ${}^1o$.

Surfaces in all three subclasses I, II, III are distributed in the second level into the three types 1, 2, 3 with respect to the relative position of the axes ${}^3o$ and ${}^2o$.

Finaly, in the third level, each subgroup of types 1, 2, 3 can be further classified with respect to the relative position of the axes ${}^3o$ and ${}^1o$ into types A, B or C.

# 3 Analytical Representation of Helical Two-Revolutional Cyclical Surfaces

Let us derive the vector function of the helical two-revolutional cyclical surface for one particular position of the axes of revolutions and for one special position of the point P with respect of these axes, particularly for the surface of type III 2 A. Derivation of the vector function of all other types of surfaces is analogous.

Let the axes of revolution be in the following relative positions: ${}^1o = z$, ${}^2o \,/\, {}^1o$ (skew), ${}^3o \times {}^2o$ (intersect), ${}^3o \parallel {}^1o$ (parallel). The position of axis ${}^2o$ in the plane parallel to the coordinate plane (xz), ${}^2o \subset \nu'$, $\nu' \parallel \nu$, is determined by parameters $d_1$, $d_2$, $d_3$, which determine the position of the intersection points of axis ${}^2o$ with the coordinate planes ($xy$) and ($yz$) in the Cartesian coordinate system $(\mathrm{O}, x, y, z)$. Then $\alpha = \operatorname{arctg} d_3/d_1$ is the angle formed by axis ${}^2o$ with the coordinate plane ($xy$) and the position of axis ${}^3o$ is determined by parameter $d_2$, which is the distance between axes ${}^3o$ and ${}^1o$ (Figure 1).

Screwing about axis ${}^1o$ with angular velocity $w_1 = v$, in the direction determined by parameter $q_1 = \pm 1$, with screw height $h$ is represented by matrix

$$\mathbf{T}_1\left(w_1(v), q_1\right) = \mathbf{T}_z\left(w_1, q_1\right) . \mathbf{T}\left(0, 0, hv/2\pi\right), \tag{1}$$

where the matrix $\mathbf{T}_z\left(w_1, q_1\right)$ represents revolution about axis $z$ by angle $w_1$ in the direction determined by parameter $q_1$ and for $\mathrm{i} = 1$ it can be derived from (2)

$$\mathbf{T}_z\left(w_\mathrm{i}, q_\mathrm{i}\right) = \begin{pmatrix} \cos w_\mathrm{i} & q_\mathrm{i} \sin w_\mathrm{i} & 0 & 0 \\ -q_\mathrm{i} \sin w_\mathrm{i} & \cos w_\mathrm{i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

and matrix $\mathbf{T}(0, 0, hv/2\pi)$ is translation with vector $(0, 0, hv/2\pi)$ expressed in (6).

Revolution about axis ${}^{2}o$ with angular velocity $w_2 = m_1 w_1$, in the direction determined by parameter $q_2 = \pm 1$, is represented by matrix

$$\mathbf{T}_2(w_2(\mathrm{v}), q_2) = \mathbf{T}(-d_1, -d_2, 0).\mathbf{T}_{\mathrm{y}}(\alpha, +1).\mathbf{T}_{\mathrm{x}}(w_2, q_2).\mathbf{T}_{\mathrm{y}}(\alpha, \text{-}1).\mathbf{T}(d_1, d_2, 0), \qquad (3)$$

where the matrix $\mathbf{T}_{\mathrm{y}}(\alpha, \pm 1)$ expressed in (4) represents the revolution about axis $y$ by angle $\alpha$ in positive or negative direction, matrix $\mathbf{T}_{\mathrm{x}}(w_2, q_2)$ represents revolution about axis $x$ by angle $w_2 = m_1 v$ in the direction determined by parameter $q_2$ in (5), matrix $\mathbf{T}(\pm d_1, \pm d_2, 0)$ represents translation with translation vector $(\pm d_1, \pm d_2, 0)$ in (6).

$$\mathbf{T}_{\mathrm{y}}(\alpha, \pm 1) = \begin{pmatrix} \cos\alpha & 0 & \pm\sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \mp\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (4)$$

$$\mathbf{T}_x(w_2, q_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos w_2 & q_2 \sin w_2 & 0 \\ 0 & -q_2 \sin w_2 & \cos w_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (5)$$

$$\mathbf{T}(\pm d_{\mathrm{i}}, \pm d_{\mathrm{j}}, \pm d_{\mathrm{k}}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \pm d_{\mathrm{i}} & \pm d_{\mathrm{j}} & \pm d_{\mathrm{k}} & 1 \end{pmatrix}. \qquad (6)$$

Revolutionary movement of the point $\mathrm{P} = (x_0, y_0, z_0, 1)$ about axis ${}^{3}o$ with angular velocity $w_3 = m_2 w_2 = m_2 m_1 v$ and in the direction determined by parameter $q_3 = \pm 1$ is represented by matrix

$$\mathbf{T}_3(w_3(v), q_3) = \mathbf{T}(0, \text{-}d_2, 0).\ \mathbf{T}_z(w_3, q_3).\mathbf{T}(0, d_2, 0), \qquad (7)$$

where matrix $\mathbf{T}(0, \pm d_2, 0)$ in (6) represents translation with translation vector $(0, \pm d_2, 0)$, and matrix $\mathbf{T}_z(w_3, q_3)$ is for $\mathrm{i} = 3$ expressed by (2).

Vector function of the helical cycloidal curve $k$ created by simultaneous revolution of the point $\mathbf{P} = (x_0, y_0, z_0, 1)$ about axes ${}^{3}o$, ${}^{2}o$ and screwing about ${}^{1}o$ is

$$\mathbf{r}(v)=\mathbf{R}\,.\,\mathbf{T}_3\big(w_3(v),q_3\big).\,\mathbf{T}_2\big(w_2(v),q_2\big).\,\mathbf{T}_1\big(w_1(v),q_1\big),\ v\in\langle 0,2\pi\rangle, \tag{8}$$

where $\mathbf{T}_3\big(w_3(v),q_3\big)$, $\mathbf{T}_2\big(w_2(v),q_2\big)$, $\mathbf{T}_1\big(w_1(v),q_1\big)$ are matrices of particular revolutions and screwing expressed in (6), (3), (1) and $\mathbf{R}=(x_0, y_0, z_0, 1)$ is the positioning vector of the point P.

Let the new coordinate system be defined at the arbitrary regular point $\mathrm{P}\in k$, identical to the trihedron $(\mathrm{P}, t, n, b)$ determined by tangent $t$, basic normal $n$ and binormal $b$ to the curve $k$ with unit vectors expressed in (9)

$$\mathbf{t}(v)=(t_1,t_2,t_3)=\frac{\mathbf{r}'(v)}{|\mathbf{r}'(v)|}\ ,\mathbf{n}(v)=(n_1,n_2,n_3)=\frac{\mathbf{r}''(v)}{|\mathbf{r}''(v)|}\ ,\mathbf{b}(v)=(b_1,b_2,b_3)=\mathbf{t}(v)\times\mathbf{n}(v). \tag{9}$$

Helical two-revolutional cyclical surface can be created by movement of the circle $c=(\mathrm{P},r)$ with centre P and radius $r$ alongside the curve $k$ so that the circle is located in the normal plane of the curve in the point $\mathrm{P}\in k$, which is determined by basic normal $n$ and binormal $b$ to this curve. Vector function of this surface is

$$\mathbf{P}(u,v)=\mathbf{r}(v)+\big(n_1 r\cos u+b_1 r\sin u, n_2 r\cos u+b_2 r\sin u, n_3 r\cos u+b_3 r\sin u\big), \tag{10}$$

for $u\in\langle 0,2\pi\rangle$, $v\in\langle 0,2\pi\rangle$, where $\mathbf{r}(v)$ is vector function of the helical cycloidal curve $k$ expressed in (8).

Form of the helical cycloidal curve $k$ and created helical two-revolutional cyclical surface changes in dependence on the relative position of the axes of revolutions that are determined by parameters $d_i$, i = 1,2,3. Surface has $m_1$ identical external branches, where every branch has $m_2$ identical internal branches. Point **P** revolves about axis ${}^3o$ with angular velocity $w_3$, which is $m_2$-multiple of the angular velocity $w_2$ of the revolution about axis ${}^2o$ and $w_2$ is $m_1$-multiple of the angular velocity $w_1$ of the revolution about axis ${}^1o$. Many different forms of cycloidal cyclical surfaces can be created by change of their determining parameters.

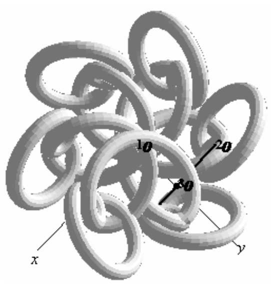



Figure 14
$m_1=6$ , $m_2=2$

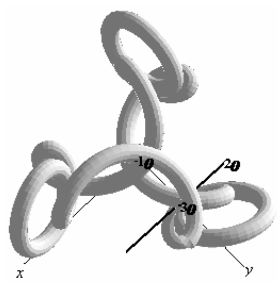



Figure 15
$m_1=3$ , $m_2=2$

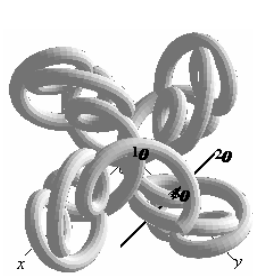



Figure 16
$m_1=4$ , $m_2=3$

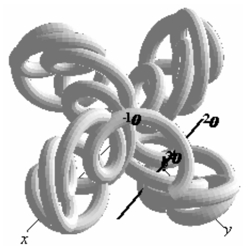



Figure 17
$m_1 = 4$ , $m_2 = 4$

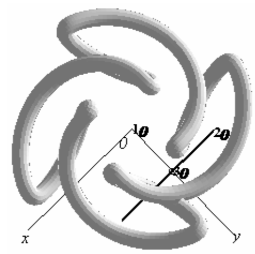



Figure 18
$m_1 = 4$ , $m_2 = 2$ , $q_2 = -1$ , $q_3 = -1$

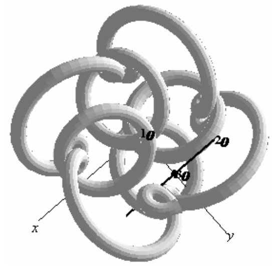



Figure 19
$m_1 = 4$ , $m_2 = 2$ , $q_2 = -1$ , $q_3 = +1$

Variations of the surface form are shown by change of some parameters of the surface of type III 2 A displayed in Figures 11 and 12. Presented surface is determined by parameters $m_1 = 4$ , $m_2 = 2$ , $q_1 = q_2 = +1$ , $q_3 = -1$ , then it has 4 external and 2 internal branches, and all three revolutions are not right-handed. Surface in Figure 14 is determined by parameter $m_1 = 6$ , $m_2 = 2$ , in Figure 15 by $m_1 = 3$ , $m_2 = 2$ , in Figure 16 by $m_1 = 4$ , $m_2 = 3$, in Figure 17 by $m_1 = 4$ , $m_2 = 4$ , then there are changes in the number of external and internal branches.

In Figure 18 depicted surface is determined by parameters $m_1 = 4$ , $m_2 = 2$ , $q_2 = -1$ , $q_3 = -1$ , in Figure 19 by $q_2 = -1$ and $q_3 = +1$ , then there are changes in the orientations of particular revolutions.

In Figure 20, there is presented surface with parameters identical to parameters of surface in Figures 11 and 12, but the position of the point P($x_0$, $y_0$, $z_0$, 1) was changed from ($d_1$/2, $d_2$/2, 0, 1) to ($d_1$, 0, 0, 1).

Surface with parameters $m_1 = 4$ , $m_2 = 6$ , $q_2 = -1$ , $q_3 = +1$ is illusrated in Figures 21 and 22, but relative position of the axes has been changed to position $^2o$ / $^1o$, $^2o \perp {}^1o$ and $^2o \perp {}^3o$. Surfaces in Figures 14-21 are displayed by view from above, because in these views the changes of parameters are more illustrative.

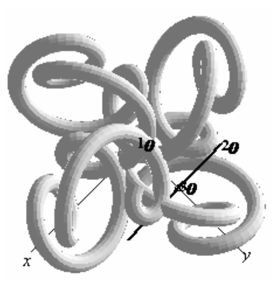



Figure 20
$m_1 = 4$ , $m_2 = 6$ , $q_2 = -1$ , $q_3 = +1$

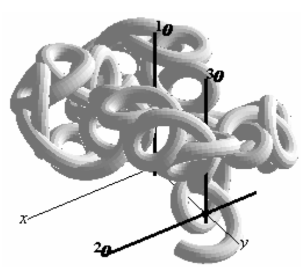



Figure 21
New position of the point **P**

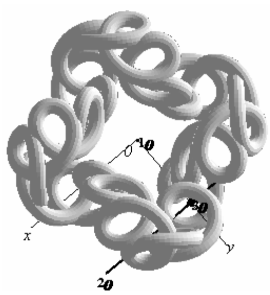
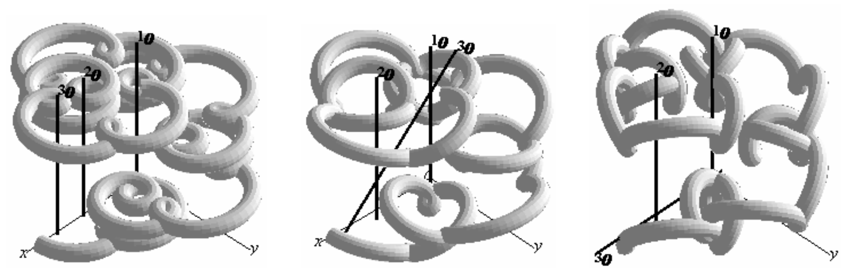


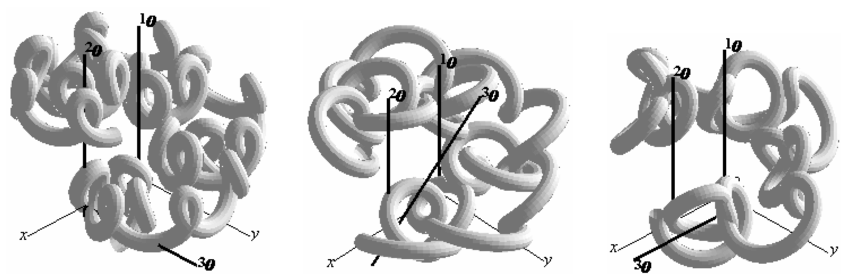

Figure 22
Wiev on it from above

## Conclusion

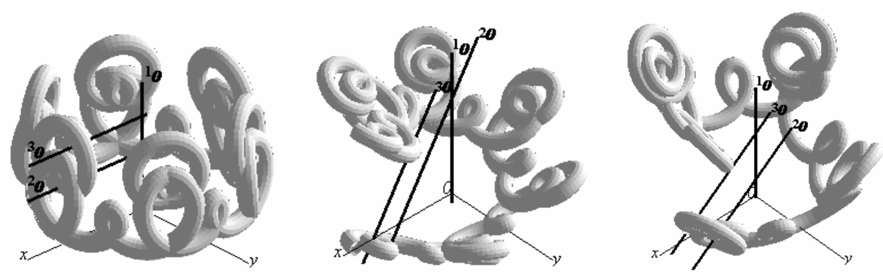
As the conclusion it can be summarised that the presented family of helical two-revolutional cyclical surfaces serves as an endlessly rich source of inspiration for artistic and design purposes. Their unusually complex forms obtained in a relatively simple way of composite spatial transformation. Special skew symmetry and harmonical periodicity reflect their simplistic generating priciple based on the naturally basic movement of our universe, revolution about an axis in the space. Several surface types from the presented classification frame are displayed in the Figures 23 a)-o) without commentary, as the most persuasive evidence.



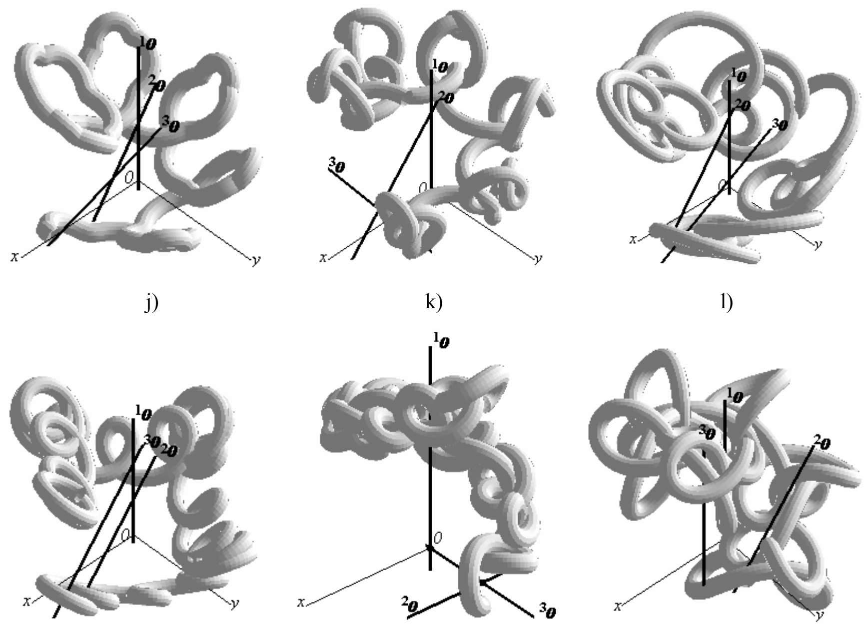
a) b) c)



d) e) f)



g) h) i)



j) k) l)

m) n) o)

Figure 23

Surface types from the presented classification frame

## Acknowledgement

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