New Approach to Fuzzy Decision Matrices

Pavla Rotterová, Ondřej Pavlačka

Department of Mathematical Analysis and Applications of Mathematics, Faculty of Science, Palacký University Olomouc, 17. listopadu 1192/12, 771 46 Olomouc, Czech Republic, pavla.rotterova01@upol.cz, ondrej.pavlacka@upol.cz

Abstract: Decision matrices represent a common tool for modeling decision-making problems under risk. They describe how the decision-maker's evaluations of the considered alternatives depend on the fact which of the possible and mutually disjoint states of the world will occur. The probabilities of the states of the world are assumed to be known. The alternatives are usually compared on the basis of the expected values and the variances of their evaluations. However, the states of the world as well as the alternatives evaluations are often described only vaguely. Therefore, we consider the following problem: the states of the world are modeled by fuzzy sets defined on the universal set on which the probability distribution is given, and the evaluations of the alternatives are expressed by fuzzy numbers. We show that the common approach to this problem, based on employing crisp probabilities of the fuzzy states of the world computed by the formula proposed by Zadeh, is not appropriate. Therefore, we introduce a new approach in which a fuzzy decision matrix does not describe discrete random variables but fuzzy rule bases. The problem is illustrated by an example.

Keywords: decision matrices; fuzzy decision matrices; decision making under risk; fuzzy states of the world; fuzzy rule bases system

1 Introduction

A decision matrix is often used as a tool of risk analysis in decision making under risk [3], [4], [7], [14]. It describes how the decision-maker's evaluations of the considered alternatives depend on the fact which of the possible and mutually disjoint states of the world will occur. The probabilities of occurrences of these states of the world are supposed to be known. Thus, the evaluations of the alternatives are discrete random variables. The alternatives are usually compared on the basis of the expected values and the variances of their evaluations. The decision-maker selects the alternative that maximizes his/her expected evaluation or maximizes the expected evaluation and simultaneously minimizes the variance.

In practical applications, the states of the world as well as the evaluations of the alternatives can be determined vaguely. The states of the world are mostly described verbally, like "the gross domestic product will increase moderately during next year". Sometimes, it can be problematic to express the evaluations of alternatives precisely because we may not have enough information. For instance, the evaluation under a certain state of the world can be described as “about 5%”. In some cases, it is more natural for a decision-maker to express the evaluations by selecting a term from a given linguistic scale.

The vaguely described pieces of information can be mathematically modeled by means of tools of fuzzy sets theory. Different views of uncertainty and fuzzy decisions in a decision matrix are discussed in [7]. Multiple attribute decision making problems, described by a decision matrix with crisp and fuzzy data, are analyzed in [1]. In [2], a fuzzy decision matrix is applied to a group decision making. An application of risk analysis with fuzzy sets employing the decision matrix is presented in [3]. In [4], the authors considered decision matrices with fuzzy targets. In [5], the hesitant fuzzy decision matrix, i.e. a decision matrix containing fuzzy sets with a different definition of membership function then the original one proposed by Zadeh [15], is considered.

A decision matrix with the fuzzy states of the world and the fuzzy evaluations of the alternatives under the particular fuzzy states of the world is called *a fuzzy decision matrix*. In [**Error! Reference source not found.**2], the authors considered a model where the fuzzy states of the world are expressed by fuzzy sets on the universal set on which the probability distribution is given. They proposed to proceed in the same way as in the case of the crisp (i.e. exactly described) states of the world; they set the probabilities of the fuzzy states of the world applying the formula proposed by Zadeh in [17]. Within this approach, the evaluations of the alternatives are understood as discrete random variables taking on fuzzy values with the probabilities of the fuzzy states of the world.

In [10], the authors showed that the Zadeh’s probabilities of fuzzy events lack the common interpretation of a probability measure. Another problem is a precise definition of "the occurrence of the particular fuzzy state of the world" (see the discussion in Section 3.3). Therefore, an alternative to how the information contained in a fuzzy decision matrix can be treated was proposed in [8]. The way is based on the idea that a fuzzy decision matrix does not determine discrete fuzzy random variables, but a system of fuzzy rule bases (a fuzzy rule base was introduced in [16]). However, only the crisp (i.e. not fuzzy) evaluations of alternatives were considered in [8] which makes the problem much simpler. The main aim of the paper is to extend this approach to the case where the evaluations of alternatives are expressed by fuzzy numbers, and to derive the formulas for correct computations of fuzzy expected values and fuzzy variances of evaluations of alternatives. The obtained fuzzy characterstics will be compared with those obtained by the approach considerd in [12].

The paper is organized as follows. A decision matrix tool is briefly recalled in Section 2. In Section 3, the common approach to the fuzzification of a decision matrix is analysed and the related problems are discussed. Our new approach to this problem is introduced and analysed in Section 4. In Section 5, both approaches are compared by an illustrative example.

2 Decision Matrices

In this section, let us describe a decision matrix as a tool for supporting a decision making under risk.

Let us consider *a probability space* $(\Omega, \mathscr{A}, P)$ where Ω denotes a non-empty universal set of all elementary events, $\mathscr{A}$ is a σ-algebra of subsets of Ω, i.e. $\mathscr{A}$ represents the set of all considered random events, and $P: \mathscr{A} \rightarrow [0,1]$ denotes a probability measure.

Now, let us describe a decision matrix under risk, considered e.g. in [3], [4], [7] and [14]. The decision matrix is shown in Table 1. In the matrix, $x_1, x_2, \ldots, x_n$ represent the alternatives of a decision-maker, $S_1, S_2, \ldots, S_m$, where $S_j \in \mathscr{A}$ for $j = 1, 2, \ldots, m$, denote the mutually disjoint states of the world, i.e. $S_j \cap S_k = \emptyset$ for any $j, k \in \{1, 2, \ldots, m\}$, $j \neq k$, and $\bigcup_{j=1}^{m} S_j = \Omega$, $p_1, p_2, \ldots, p_m$ stand for the probabilities of the states of the world $S_1, S_2, \ldots, S_m$, i.e. $p_j = P(S_j)$, and for any $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$, $h_{i,j}$ means the decision-maker's evaluation if he/she chooses the alternative x_i and the state of the world S_j occurs. The evaluation of the alternative x_i is commonly understood as a discrete random variable $H_i: \{S_1, S_2, \ldots, S_m\} \rightarrow \mathbb{R}$ which takes on the values $h_{i,j} = H_i(S_j)$ with the probabilities p_j, $j = 1, 2, \ldots, m$.

Table 1
Crisp decision matrix

	S_1 p_1	S_2 p_2	… …	S_m p_m		
x_1	$h_{1,1}$	$h_{1,2}$	…	$h_{1,m}$	EH_1	$var\ H_1$
x_2	$h_{2,1}$	$h_{2,2}$	…	$h_{2,m}$	EH_2	$var\ H_2$
…	…	…	…	…	…	…
x_n	$h_{n,1}$	$h_{n,2}$	…	$h_{n,m}$	EH_n	$var\ H_n$

The alternatives are usually compared on the basis of the expected values and the variances of their evaluations (an overview of decision making rules can be found e.g. in [**Error! Reference source not found.**]). The expected values of the decision-maker's evaluations, denoted by $EH_1, EH_2, \ldots, EH_n$, are given for any $i \in \{1, 2, \ldots, n\}$ by:

$$EH_i = \sum_{j=1}^{m} p_j \cdot h_{i,j}. \tag{1}$$

The variances of the decision-maker's evaluations, denoted by *var* H_1, *var* H_2, …, *var* H_n, are calculated for any $i \in \{1, 2, \dots, n\}$ as follows:

$$var\ H_i = \sum_{j=1}^{m} p_j \cdot (h_{i,j} - EH_i)^2. \tag{2}$$

The alternative that maximizes the expected evaluation and minimizes the variance of the evaluation is selected as the best one.

3 Fuzzy Decision Matrices

Now, let us describe the common approach to the generalization of a decision matrix to the case where the states of the world and the evaluations of the alternatives are expressed by fuzzy sets, considered e.g. in [**Error! Reference source not found.**2]. Within this approach, the probabilities of the fuzzy states of the world, computed by the formula proposed by Zadeh in [17], are used for computations of the characteristics of the evaluations of the alternatives.

3.1 Fuzzy States of the World

Vaguely defined states of the world can be mathematically expressed by fuzzy sets. *A fuzzy set A* on a non-empty set Ω is determined by its membership function μ_A: $\Omega \rightarrow [0, 1]$. Let us denote the family of all fuzzy sets on Ω by $\mathcal{F}(\Omega)$. *A support* of A and *a core* of A are given as $Supp\,A := \{\omega \in \Omega \mid \mu_A(\omega) > 0\}$ and $Core\,A := \{\omega \in \Omega \mid \mu_A(\omega) = 1\}$, respectively. A_α means *an α-cut* of A, i.e. $A_\alpha := \{\omega \in \Omega \mid \mu_A(\omega) \geq \alpha\}$ for any $\alpha \in (0,1]$.

Remark Any crisp set $A \subseteq \Omega$ can be seen as a fuzzy set $A \in \mathcal{F}(\Omega)$ of a special kind where its characteristic function χ_A coincides with the membership function μ_A of the fuzzy set. In fuzzy models, this convention allows us to consider also precisely described events given by crisp sets.

In fuzzy decision matrices, fuzzy states of the world are described by the fuzzy events. According to Zadeh [17], *a fuzzy event* $A \in \mathcal{F}(\Omega)$ is a fuzzy set whose α-cuts are random events, i.e. $A_\alpha \in \mathcal{A}$ for all $\alpha \in (0,1]$. As an analogy to a decomposition of the universal set Ω by crisp states of the world, the fuzzy states of the world, denoted by S_1, S_2, …, S_m, have to form *a fuzzy partition* of the universal set Ω, i.e. for any $\omega \in \Omega$, it has to hold that

$$\sum_{j=1}^{m} \mu_{S_j}(\omega) = 1. \tag{3}$$

Zadeh [17] extended the crisp probability measure P to the case of fuzzy events. Let us denote this extended measure by P_Z. A probability $P_Z(A)$ of a fuzzy event A is defined as follows:

$$P_Z(A) := E(\mu_A) = \int_{\omega\in\Omega} \mu_A(\omega) dP. \tag{4}$$

3.2 Fuzzy Evaluations of Alternatives under the Particular Fuzzy States of the World

As was mentioned in Introduction, it can be difficult for a decision-maker to evaluate each alternative under each state of the world by a real number. One reason can be a lack of information caused e.g. by inaccuracies of measurements or a lower quality of data transmissions. Another reason can be that it is more natural for the decision-maker to describe the evaluations linguistically rather than by numbers.

Linguistic terms or uncertain quantities can be mathematically modeled by fuzzy numbers. A *fuzzy number A* is a fuzzy set on the set of all real numbers $\mathbb{R}$ such that its core A is non-empty, its α-cuts A_α are closed intervals for any $\alpha \in (0, 1]$, and its support *Supp A* is bounded. The family of all fuzzy numbers on $\mathbb{R}$ will be denoted by $\mathcal{F}_N(\mathbb{R})$. In some models, fuzzy evaluations can be restricted only to a closed interval, mostly [0,1]. A *fuzzy number defined on the interval* $[a, b]$ is a fuzzy number whose α-cuts belong to the interval $[a, b]$ for all $\alpha \in (0,1]$. The family of all fuzzy numbers on the interval $[a, b]$ will be denoted by $\mathcal{F}_N([a, b])$.

Thus, there are two ways of expressing a fuzzy evaluation of an alternative. The first way is to specify the evaluation directly by a fuzzy number. For instance, some expert can evaluate the particular alternative directly by the fuzzy number "about five percent profit", whose membership function is shown in Figure 1.

The second possibility of expressing the fuzzy evaluation of the alternative consists in the fact that the evaluation is modeled by a linguistic variable (linguistic variables were introduced in [16]). A decision-maker evaluates the alternatives under the particular states of the world by appropriate linguistic terms whose mathematical meanings are described by fuzzy numbers. A set of linguistic terms $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_r$ forms *a linguistic scale on* $[a, b]$ if $T_1, T_2, \ldots, T_r \in \mathcal{F}_N([a, b])$ representing their mathematical meanings form a fuzzy partition of $[a, b]$.

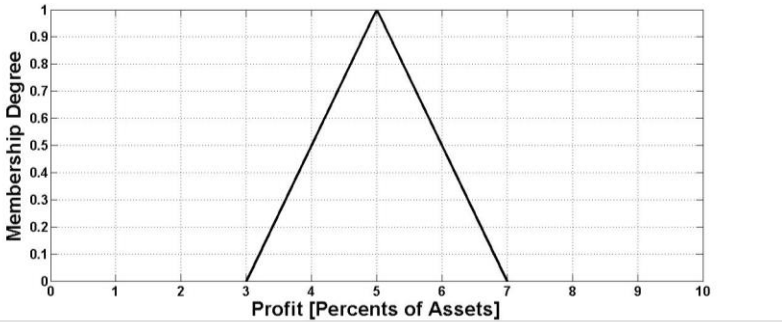

Figure 1
Example of an expertly specified evaluation

Example Let us consider a linguistic scale shown in Figure 2. This scale is formed by the linguistic terms "a big loss" (BL), "a small loss" (SL), "approximately without profit" (AWP), "a small profit" (SP), and "a big profit" (BP). In some cases, a selection of some linguistically described value like "a small profit" from the given linguistic scale can be more convenient for a decision-maker.

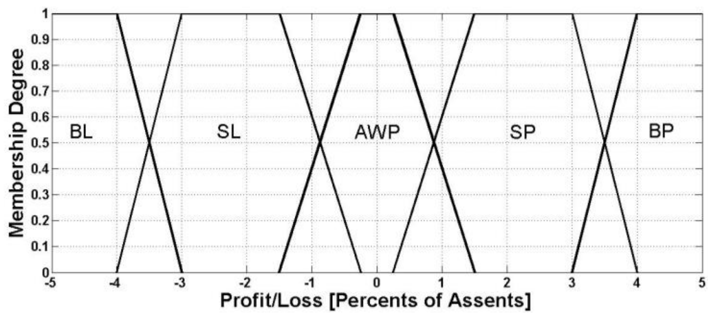

Figure 2
Example of a linguistic scale

3.3 Common Approach to a Fuzzy Decision Matrix

Let us describe a common approach to a fuzzy decision matrix that was considered e.g. in [**Error! Reference source not found.**2].

In the fuzzy decision matrix given in Table 2, x_1, x_2, …, x_n denote the alternatives of the decision-maker and S_1, S_2, …, S_m stand for the fuzzy states of the world. Probabilities of the fuzzy states of the world S_1, S_2, …, S_m, calculated according to (4), are denoted by p_{Z1}, p_{Z2}, …, p_{Zm}, i.e. $p_{Zj} = P_Z(S_j)$. For any $i \in \{1, 2, …, n\}$ and $j \in \{1, 2, …, m\}$, $H_{i,j}$ expresses the fuzzy evaluation of the alternative x_i under the fuzzy state of the world S_j.

Table 2
Fuzzy decision matrix

	S_1 p_{Z1}	S_2 p_{Z2}	… …	S_m p_{Zm}		
x_1	$H_{1,1}$	$H_{1,2}$	…	$H_{1,m}$	EH_1^Z	$var\ H_1^Z$
x_2	$H_{2,1}$	$H_{2,2}$	…	$H_{2,m}$	EH_2^Z	$var\ H_2^Z$
…	…	…	…	…	…	…
x_n	$H_{n,1}$	$H_{n,2}$	…	$H_{n,m}$	EH_n^Z	$var\ H_n^Z$

Thus, the evaluation of the alternative x_i is understood as a discrete fuzzy random variable H_i^Z: $\{S_1, S_2, \ldots, S_m\} \rightarrow \mathcal{F}_N(\mathbb{R})$ where $H_i^Z(S_j) = H_{i,j}$ for $j=1, 2, \ldots, m$. Its fuzzy expected value, denoted by EH_i^Z, is computed according to the generalized formula (1) where the probabilities p_j of the states of the world are replaced by the Zadeh's probabilities p_{Zj} of the fuzzy states of the world and the crisp evaluations $h_{i,j}$ are replaced by the fuzzy evaluations $H_{i,j}$, i.e.

$$EH_i^Z = \sum_{j=1}^{m} p_{Zj} \cdot H_{i,j}. \tag{5}$$

The α-cuts $EH_{i,\alpha}^Z = \left[Eh_{i,\alpha}^{ZL}, Eh_{i,\alpha}^{ZU}\right]$ are obtained for all $\alpha \in (0,1]$ as follows: Let $H_{i,j,\alpha} = \left[h_{i,j,\alpha}^L, h_{i,j,\alpha}^U\right]$, $j = 1, 2, \ldots, m$. The boundary values of $EH_{i,\alpha}^Z$ are obtained by

$$Eh_{i,\alpha}^{ZL} = \sum_{j=1}^{m} p_{Zj} \cdot h_{i,j,\alpha}^L \tag{6}$$

and

$$Eh_{i,\alpha}^{ZU} = \sum_{j=1}^{m} p_{Zj} \cdot h_{i,j,\alpha}^U. \tag{7}$$

Computation of the fuzzy variance $var\ H_i^Z$ is more complex. It was shown in [9] that the formulas proposed in [**Error! Reference source not found.**2] were not correct because the relationships between the fuzzy evaluations $H_{i,1}, H_{i,2}, \ldots, H_{i,m}$, and the fuzzy expected evaluation EH_i^Z were not involved in the calculation. This fact causes that the uncertainty of the resulting fuzzy variance is falsely increased. The proper formulas for the computation of the fuzzy variance were proposed in [9]. For any $i \in \{1, 2, \ldots, n\}$ and any $\alpha \in (0,1]$, the α-cut of the fuzzy variance $var\ H_{i,\alpha}^Z = \left[var\ h_{i,\alpha}^{ZL}, var\ h_{i,\alpha}^{ZU}\right]$ has to be calculated as follows: Let us denote

$$z_i\left(h_{i,1}, h_{i,2}, \ldots, h_{i,m}\right) = \sum_{j=1}^{m} p_{Zj} \cdot \left(h_{i,j} - \sum_{k=1}^{m} p_{Zk} \cdot h_{i,k}\right)^2. \tag{8}$$

Then,

$$var\ h_{i,\alpha}^{ZL} = \min \left\{ z_i \left(h_{i,1}, h_{i,2}, \ldots, h_{i,m} \right) \mid h_{i,j} \in H_{\mathrm{i,j},\alpha}, j = 1,2,..,m \right\} \tag{9}$$

and

$$var\ h_{i,\alpha}^{ZU} = \max \left\{ z_i \left(h_{i,1}, h_{i,2}, \ldots, h_{i,m} \right) \mid h_{i,j} \in H_{\mathrm{i,j},\alpha}, j = 1,2,..,m \right\}. \tag{10}$$

As it is written in section 2, the element $h_{i,j}$ of the matrix given in table 1 describes the decision-maker's evaluation of the alternative x_i if the state of the world S_j occurs. If we consider the fuzzy states of the world instead of the crisp ones, a natural question arises: What does it mean to say "if the fuzzy state of the world S_j occurs"? Let us suppose that some $\omega \in \Omega$ has occurred. If $\mu_{S_j}(\omega)=1$, then it is clear that the evaluation of the alternative x_i is exactly $h_{i,j}$. However, what is the evaluation of x_i if $0 < \mu_{S_j}(\omega) < 1$ (which also means that $0 < \mu_{S_k}(\omega) < 1$ for some $k \neq j$)? Thus, perhaps it is not appropriate in the case of a decision matrix with the fuzzy states of the world to treat the evaluation of x_i as a discrete random variable H_i^Z that takes on the fuzzy values $H_{i,1}, H_{i,2}, \ldots, H_{i,m}$.

Moreover, it was pointed out by Rotterová and Pavlačka [10] that the Zadeh's probabilities $p_{Z1}, p_{Z2}, \ldots, p_{Zm}$ of the fuzzy states of the world express the expected membership degrees in which the particular states of the world will occur. Thus, they do not have in general the common probabilistic interpretation - a measure of a chance that a given event will occur in the future, which is desirable in the case of a decision matrix.

Therefore, we cannot say that the values EH_1^Z, EH_2^Z, ..., EH_n^Z, given by (6) and (7), and $var\ H_1^Z$, $var\ H_2^Z$, ..., $var\ H_n^Z$, given by (9) and (10), express the expected values and variances of evaluations of the alternatives, respectively. Ordering of the alternatives based on these characteristics is questionable.

4 Fuzzy Rule Bases System Determined by the Fuzzy Decision Matrix

In this section, let us introduce a different approach to the model of decision making under risk described by the decision matrix with fuzzy states of the world presented in Table 2. Taking the problems discussed in the previous section into account, we suggest not to treat the evaluation of the i^{th} alternative x_i, $i \in \{1, 2, \ldots, n\}$, as a discrete random variable H_i^Z taking on the fuzzy values $H_{i,1}$, $H_{i,2}$, ..., $H_{i,m}$ with the probabilities $p_{Z1}, p_{Z2}, \ldots, p_{Zm}$. Instead of this, we propose to understand the information about the evaluation of the alternative x_i as the following fuzzy rule base:

If the state of the world is S_1, then the evaluation of x_i is $H_{i,1}$.
If the state of the world is S_2, then the evaluation of x_i is $H_{i,2}$.
$\vdots$ (11)
If the state of the world is S_m, then the evaluation of x_i is $H_{i,m}$.

In [8], it was shown that in the case of the fuzzy decision matrix with crisp evaluations under the particular fuzzy states of the world, it is appropriate to use the *Sugeno's method of fuzzy inference*, introduced in [11]. The obtained output from the fuzzy rule base was expressed by a real number.

In the paper, we deal with the fuzzy evaluations of the alternatives under the fuzzy states of the world. Thus, the so-called *generalised Sugeno's method of fuzzy inference,* introduced in [13], should be applied for obtaining an output from the fuzzy rule base (11). According to this method, the evaluation of an alternative x_i for a given $\omega \in \Omega$ is computed in the following way:

$$H_i^S(\omega) = \frac{\sum_{j=1}^{m} \mu_{S_j}(\omega) \cdot H_{i,j}}{\sum_{j=1}^{m} \mu_{S_j}(\omega)} = \sum_{j=1}^{m} \mu_{S_j}(\omega) \cdot H_{i,j}. \tag{12}$$

For any $\alpha \in (0,1]$, let us denote $H_{i,j,\alpha} = \left[h_{i,j,\alpha}^L, h_{i,j,\alpha}^U\right]$, $j = 1, 2, \ldots, m$, and $H_{i,\alpha}^S(\omega) = \left[h_{i,\alpha}^{SL}(\omega), h_{i,\alpha}^{SU}(\omega)\right]$. Then, the boundary values of $H_{i,\alpha}^S(\omega)$ are computed as follows:

$$h_{i,\alpha}^{SL}(\omega) = \sum_{j=1}^{m} \mu_{S_j}(\omega) \cdot h_{i,j,\alpha}^L$$

and

$$h_{i,\alpha}^{SU}(\omega) = \sum_{j=1}^{m} \mu_{S_j}(\omega) \cdot h_{i,j,\alpha}^U.$$

Remark In the formula (12), the denominator equals to one due to the assumption that the fuzzy states of the world $S_1, S_2, \ldots, S_m$ form a fuzzy partition of Ω. It is worth to note that in our approach, this assumption can be omitted.

Since we operate within the given probability space $(\Omega, \mathcal{A}, P)$, H_i^S is a fuzzy random variable such that $H_i^S\colon \Omega \to \mathcal{F}_N(\mathbb{R})$.

Remark It can be easily seen from (12) that in the case of the crisp states of the world S_j, $j = 1, 2, \ldots, m$, and the crisp evaluations $h_{i,j}$, $i = 1, 2, \ldots, n$, under the particular fuzzy states of the world, the fuzzy random variables H_i^S coincide with discrete random variables H_i taking on the values $h_{i,j}$ with the probabilities p_j,

$j = 1, 2, \ldots, m$. Thus, this new approach can be seen as an extension of a decision matrix to the case of the fuzzy states of the world and the fuzzy evaluations of alternatives where appropriate.

Analogously, as in the common approach to the fuzzy decision matrix, the ordering of the alternatives $x_1, x_2, \ldots, x_n$ can be based on the fuzzy expected values and the fuzzy variances of the random variables H_i^S, $i = 1, 2, \ldots, n$. Let us introduce the formulas for computations of the α-cuts of EH_i^S and $var\ H_i^S$.

For any $\alpha \in (0,1]$, the α-cut of the fuzzy expected output from the fuzzy rule base given by (11), denoted by $EH_{i,\alpha}^S = \left[Eh_{i,\alpha}^{SL}, Eh_{i,\alpha}^{SU}\right]$, is obtained as follows:

$$Eh_{i,\alpha}^{SL} = \min\left\{\int_{\omega\in\Omega}\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}\,dP \,\middle|\, h_{i,j}\in H_{\mathrm{i,j},\alpha}, j = 1,2,..,m\right\}$$
$$= \int_{\omega\in\Omega}\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j,\alpha}^{L}\,dP \qquad (13)$$

and

$$Eh_{i,\alpha}^{SU} = \max\left\{\int_{\omega\in\Omega}\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}\,dP \,\middle|\, h_{i,j}\in H_{\mathrm{i,j},\alpha}, j = 1,2,..,m\right\}$$
$$= \int_{\omega\in\Omega}\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j,\alpha}^{U}\,dP. \qquad (14)$$

The α-cut $var\ H_{i,\alpha}^S = \left[var\ h_{i,\alpha}^{SL}, var\ h_{i,\alpha}^{SU}\right]$ of the fuzzy variance of the output from the fuzzy rule base is obtained as follows: Let us denote

$$\mathrm{s}_i\left(h_{i,1}, h_{i,2}, \ldots, h_{i,m}\right)$$
$$= \int_{\omega\in\Omega}\left(\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j} - \int_{t\in\Omega}\sum_{k=1}^{m}\mu_{S_k}(t)\cdot h_{i,k}\,dP\right)^2 dP. \qquad (15)$$

Then,

$$var\ h_{i,\alpha}^{SL} = \min\left\{\mathrm{s}_i\left(h_{i,1}, h_{i,2}, \ldots, h_{i,m}\right) \mid h_{i,j}\in H_{\mathrm{i,j},\alpha}, j = 1,2,..,m\right\} \qquad (16)$$

and

$$var\ h_{i,\alpha}^{SU} = \max\left\{\mathrm{s}_i\left(h_{i,1}, h_{i,2}, \ldots, h_{i,m}\right) \mid h_{i,j}\in H_{\mathrm{i,j},\alpha}, j = 1,2,..,m\right\}. \qquad (17)$$

Now, let us compare the fuzzy expected values EH_i^Z and EH_i^S, and the fuzzy variances $var\ H_i^Z$ and $var\ H_i^S$.

Theorem 1 For $i = 1, 2, \ldots, n$, the expected fuzzy evaluation EH_i^Z and the expected output from the fuzzy rule base EH_i^S coincide.

Proof For any $\alpha \in (0,1]$, let $EH_{i,\alpha}^S = \left[Eh_{i,\alpha}^{SL}, Eh_{i,\alpha}^{SU}\right]$ be the α-cut of the expected output from the fuzzy rule base and $EH_{i,\alpha}^Z = \left[Eh_{i,\alpha}^{ZL}, Eh_{i,\alpha}^{ZU}\right]$be the α-cut of the fuzzy expected evaluation. For the boundary values of $EH_{i,\alpha}^S$, it holds:

$$Eh_{i,\alpha}^{SL} = \int_{\omega\in\Omega} \sum_{j=1}^{m} \mu_{S_j}(\omega)\cdot h_{i,j,\alpha}^L dP = \sum_{j=1}^{m} \int_{\omega\in\Omega} \mu_{S_j}(\omega) dP \cdot h_{i,j,\alpha}^L$$

$$= \sum_{j=1}^{m} p_{Zj} \cdot h_{i,j,\alpha}^L = Eh_{i,\alpha}^{ZL}$$

and

$$Eh_{i,\alpha}^{SU} = \int_{\omega\in\Omega} \sum_{j=1}^{m} \mu_{S_j}(\omega)\cdot h_{i,j,\alpha}^U dP = \sum_{j=1}^{m} \int_{\omega\in\Omega} \mu_{S_j}(\omega) dP \cdot h_{i,j,\alpha}^U$$

$$= \sum_{j=1}^{m} p_{Zj} \cdot h_{i,j,\alpha}^U = Eh_{i,\alpha}^{ZU}.$$

Thus, all the α-cuts are the same. Therefore, $EH_i^S = EH_i^Z$. □

In [8], the authors showed that in the case of a fuzzy decision matrix where the evaluations under the particular fuzzy states of the world are expressed by real numbers, the variances $var\, H_i^Z$ and $var\, H_i^S$ are real numbers as well, and $var\, H_i^Z \geq var\, H_i^S$. Now, let us compare the fuzzy variances $var\, H_i^Z$ and $var\, H_i^S$.

Theorem 2 For $i = 1, 2, \ldots, n$, the fuzzy variance $var\, H_i^Z$ of the fuzzy evaluation is greater or equals to the fuzzy variance $var\, H_i^S$ of the output from the fuzzy rule base (11).

Proof Let $z_i(h_{i,1}, h_{i,2}, \ldots, h_{i,m})$ and $\mathrm{s}_i(h_{i,1}, h_{i,2}, \ldots, h_{i,m})$ be the auxiliary functions defined by (8) and (15), respectively. For the sake of simplicity, let us denote for a given $h_{i,j} \in H_{\mathrm{i,j},\alpha}$, $j = 1, 2, \ldots, m$,

$$Eh_i = \sum_{j=1}^{m} p_{Zj} \cdot h_{i,j} = \int_{\omega\in\Omega} \sum_{j=1}^{m} \mu_{S_j}(\omega)\cdot h_{i,j} dP.$$

We can express the difference of $z_i(h_{i,1}, h_{i,2}, \ldots, h_{i,m})$ and $\mathrm{s}_i(h_{i,1}, h_{i,2}, \ldots, h_{i,m})$ as follows:

$$\mathrm{d}_i\left(h_{i,1},h_{i,2},\ldots,h_{i,m}\right)=z_i\left(h_{i,1},h_{i,2},\ldots,h_{i,m}\right)-s_i\left(h_{i,1},h_{i,2},\ldots,h_{i,m}\right)$$

$$=\sum_{j=1}^{m} p_{Zj}\cdot\left(h_{i,j}-Eh_i\right)^2-\int_{\omega\in\Omega}\left(\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}-Eh_i\right)^2 dP$$

$$=\sum_{j=1}^{m}\int_{\omega\in\Omega}\mu_{S_j}(\omega)dP\cdot\left(h_{i,j}^2-2\cdot h_{i,j}\cdot Eh_i+\left(Eh_i\right)^2\right)$$

$$-\int_{\omega\in\Omega}\left(\left(\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}\right)^2-2\cdot\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}\cdot Eh_i+\left(Eh_i\right)^2\right)dP$$

$$=\int_{\omega\in\Omega}\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}^2 dP-2\cdot Eh_i\cdot\int_{\omega\in\Omega}\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}dP$$

$$+\left(Eh_i\right)^2\cdot\int_{\omega\in\Omega}\sum_{j=1}^{m}\mu_{S_j}(\omega)dP-\int_{\omega\in\Omega}\left(\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}\right)^2 dP$$

$$+2\cdot Eh_i\cdot\int_{\omega\in\Omega}\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}dP-\left(Eh_i\right)^2\cdot\int_{\omega\in\Omega}dP$$

$$=\int_{\omega\in\Omega}\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}^2 dP-\left(Eh_i\right)^2-\int_{\omega\in\Omega}\left(\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}\right)^2 dP$$

$$+\left(Eh_i\right)^2=\int_{\omega\in\Omega}\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}^2 dP-\int_{\omega\in\Omega}\left(\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}\right)^2 dP$$

$$=\int_{\omega\in\Omega}\left(\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}^2-\left(\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}\right)^2\right)dP$$

where relations (3), (13), (14) and the following relation from measure theory:

$$\int_{\omega\in\Omega}dP=P(\Omega)=1,$$

were applied.

The integrand $\left(\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}^2-\left(\sum_{j=1}^{m}\mu_{S_j}(\omega)\cdot h_{i,j}\right)^2\right)$ is clearly non-negative (it represents the variance of a discrete random variable that takes on the values $h_{i,j}$, $j=1, 2, \ldots, m$, with the "probabilities" $\mu_{S_j}(\omega)$, $j=1, 2, \ldots, m$). It is equal to zero if and only if $h_{i,j} = h_{i,k}$ for any $j \neq k$ such that both p_{Zj} and p_{Zk} are positive. Thus, the function $d_i(h_{i,1}, h_{i,2}, \ldots, h_{i,m})$ is always non-negative.

However, $d_i(h_{i,1}, h_{i,2}, \ldots, h_{i,m})$ is the auxiliary function for computation of the fuzzy difference D_i between $var\ H_i^Z$ and $var\ H_i^S$. For any $\alpha \in (0,1]$, the α-cut of the fuzzy difference $D_{i,\alpha} = \left[d_{i,\alpha}^L, d_{i,\alpha}^U\right]$ is given as follows:

$$d_{i,\alpha}^L = \min\left\{\mathrm{d}_i\left(h_{i,1}, h_{i,2}, \ldots, h_{i,m}\right) \mid h_{i,j} \in H_{\mathrm{i,j},\alpha}, j = 1,2,..,m\right\}$$

and

$$d_{i,\alpha}^U = \max\left\{d_i\left(h_{i,1}, h_{i,2}, \ldots, h_{i,m}\right) \mid h_{i,j} \in H_{\mathrm{i,j},\alpha}, j = 1,2,..,m\right\}.$$

Due to the non-negativity of the auxiliary function d_i, the α-cut of the fuzzy difference $D_{i,\alpha}$ contains only non-negative values, i.e. $var\ H_{i,\alpha}^Z \geq var\ H_{i,\alpha}^S$. Hence, $var\ H_i^Z \geq var\ H_i^S$. □

Thus, although the fuzzy expected values EH_i^Z and EH_i^S coincide, the fuzzy variances $var\ H_i^Z$ and $var\ H_i^S$ differ in general. This can affect the ranking of the considered alternatives, which is illustrated by the example in Section 5.

Now, let us focus on the interpretation of EH_i^S and $var\ H_i^S$. Both characteristics describe a random variable that explains outputs from the fuzzy rule base (11). There are no such interpretational problems as those discussed in the previous section. So this approach seems to be more appropriate for the practical use.

5 Illustrative Example

Let us illustrate the difference between both described approaches on the similar problem as was considered in [9]. Let us compare two stocks, A and B, with respect to their future yields. We consider the following states of the economy: "great economic drop" (GD), "economic drop" (D), "economic stagnation" (S), "economic growth" (G), and "great economic growth" (GG). Let us assume that the considered states of the economy are given only by the development of the gross domestic product, abbreviated as GDP. Further, we assume that the next year prediction of GDP development [%] shows a normally distributed growth of GDP with parameters $\mu = 1.5$ and $\sigma = 2$.

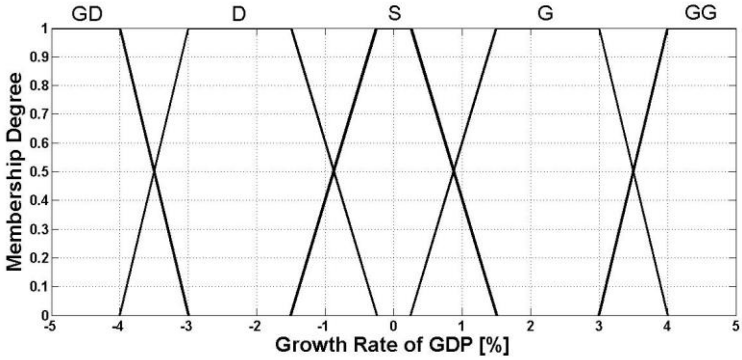

Figure 3
Linguistic scale of the states of the economy

A considered state of the economy can be expressed by a *trapezoidal fuzzy number* which is determined by its *significant values* a_1, a_2, a_3, and a_4 such that $a_1 \leq a_2 \leq a_3 \leq a_4$. The membership function of any trapezoidal fuzzy number $A \in \mathcal{F}_N(\mathbb{R})$ is for any $x \in \mathbb{R}$ in the form as follows:

$$\mu_A(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & \text{if } x \in [a_1, a_2), \\ 1 & \text{if } x \in [a_2, a_3], \\ \dfrac{a_4 - x}{a_4 - a_3} & \text{if } x \in (a_3, a_4], \\ 0 & \text{otherwise.} \end{cases}$$

The trapezoidal fuzzy number A determined by its significant values is denoted further by (a_1, a_2, a_3, a_4).

Let us assume that the states of the economy are mathematically expressed by trapezoidal fuzzy numbers that form a linguistic scale shown in Figure 3. Moreover, let us consider that the predictions of future stock yields (in %) are set expertly.

Significant values of the fuzzy states of the economy and of the fuzzy stock yields are shown in Table 3. The probabilities of the fuzzy states of the economy were calculated according to the formula (4) and are used only in the calculation of the characteristics of the output with respect to the common approach described in Section 3.

Table 3
Considered fuzzy decision matrix

Economy states	GD = (-∞, -∞, -4, -3)				D = (-4, -3, -1.5, -0.25)			
Probabilities	0.0067				0.1146			
A yield (%)	-36	-34	-31	-16	-20	-17	-10	0
B yield (%)	-45	-40	-32	-25	-22	-17	-11	0
Economy states	S = (-1.5, -0.25, 0.25, 1.5)				G = (0.25, 1.5, 3, 4)			
Probabilities	0.2579				0.4596			
A yield (%)	-5	-3	3	10	6	12	17	24
B yield (%)	-5	-3	3	5	8	12	16	18
Economy states	GG = (3, 4, ∞, ∞)							
Probabilities	0.1612							
A yield (%)	22	27	34	36				
B yield (%)	20	26	33	40				

The resultant fuzzy expected values and the fuzzy variances can be compared, for instance, according to their centers of gravity. The center of gravity of a fuzzy number $A \in \mathcal{F}_N(\mathbb{R})$ is a real number cog_A given as follows:

$$cog_A = \frac{\int_{-\infty}^{\infty} x \cdot \mu_A(x)dx}{\int_{-\infty}^{\infty} \mu_A(x)dx}.$$

The fuzzy expected values *EA* and *EB*, computed by the formulas (6) and (7) (or (13) and (14)) are trapezoidal fuzzy numbers. Their significant values are given in Table 4. The fuzzy variances $var\ A^Z$ and $var\ B^Z$, obtained by the formulas (9) and (10), as well as $var\ A^S$ and $var\ B^S$, computed by (16) and (17), are not trapezoidal fuzzy numbers. Their membership functions are shown in Figures 4 and 5. The significant values of the fuzzy variances are also given in Table 4 (by these significant values we understand end points of the core and of the closure of the support). We can see that the fuzzy variances of the outputs from the fuzzy rule bases reach lower values than the variances obtained by the common approach.

From the results given in Table 4, it is obvious that the center of gravity of the fuzzy expected value *EA* is greater than the center of gravity of the fuzzy expected value *EB*. Therefore, without considering the variances the decision-maker should prefer the stock A.

Table 4
Resultant stocks characteristics

Stock Characteristic	Significant Values (%)				Centre of Gravity
EA	2.48	6.92	12.71	19.30	10.44
EB	2.79	6.72	11.97	15.84	9.33
$var\ A^Z$	38.48	117.61	255.30	365.28	195.21
$var\ B^Z$	40.12	117.06	245.53	369.68	194.83
$var\ A^S$	33.44	105.75	229.30	324.36	173.98
$var\ B^S$	35.41	105.23	220.70	332.41	174.98

In this example, we can also see that the change in the fuzzy variance computation can cause a change in the decision-maker's preferences. Based on $var\ A^Z$ and $var\ B^Z$, the decision-maker is not able to make a decision on the basis of the rule of the expected value and the variance described in Section 2, while based on $var\ A^S$ and $var\ B^S$, the decision-maker should prefer the stock A (the higher expected value and the lower variance than the stock B compared on the basis of centers of gravity of variances approximated by trapezoidal fuzzy numbers).

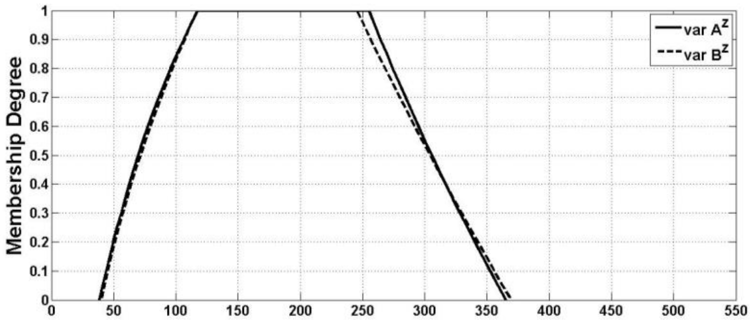

Figure 4
Membership functions of $var\ A^Z$ and $var\ B^Z$

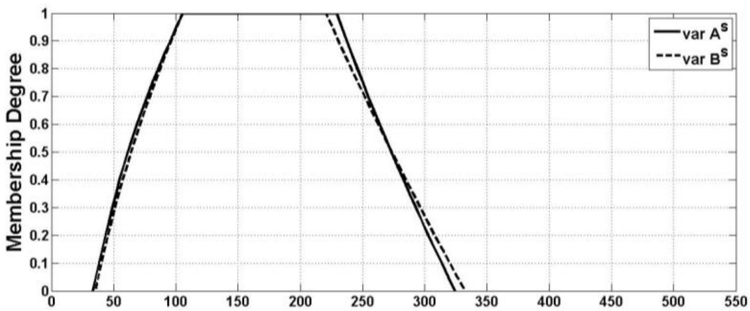

Figure 5
Membership functions of $var\ A^S$ and $var\ B^S$

Conclusions

We have dealt with the problem of the extension of a decision matrix for the case of the fuzzy states of the world and the fuzzy evaluations of the alternatives. We have analyzed the common approach to this problem proposed in [**Error! Reference source not found.**2] that is based on applying the Zadeh's probabilities of the fuzzy states of the world. We have found out that the meaning of the obtained characteristics of the evaluations of the alternatives, namely the fuzzy expected values and the fuzzy variances, is questionable. Therefore, we have introduced a new approach that is based on the idea that a fuzzy decision matrix does not determine discrete fuzzy random variables, but fuzzy rule bases. In such a case, the obtained characteristics of the evaluations, based on which the alternatives are compared, are clearly interpretable. We have proved that the resulting expected values of the evaluations are for both approaches the same, whereas the variances generally differ. In the numerical example, we have shown that the final ordering of the alternatives, according to both approaches, can be different.

Future work in this field will be focused on the case, where the underlying probability measure is fuzzy. For instance, the parameters of the underlying probability distribution, like μ and σ in the case of the normal distribution considered in the numerical example in Section 5, could be expertly set with fuzzy numbers.

Acknowledgement

This work was supported by the project No. GA 14-02424S of the Grant Agency of the Czech Republic *Methods of Operations Research for Decision Support under Uncertainty* and by the grant IGA_PrF_2016_025 *Mathematical Models* of the *Internal Grant Agency of Palacký University Olomouc*.

References

[1] Chen, S., Hwang, Ch.: *Fuzzy Multiple Attribute Decision Making Methods: Methods and Applications*. Springer Berlin Heidelberg, 1992

[2] Cheng, Ch.: A simple fuzzy group decision making method. In: *Fuzzy Systems Conference Proceedings 1999*. IEEE International, Seoul, South Korea, 1999, 910-915

[3] Ganoulis, J.: *Engineering Risk Analysis of Water Pollution: Probabilities and Fuzzy Sets*. VCH, Weinheim, 2008

[4] Huynh, V. et. al.: A Fuzzy Target Based Model for Decision Making Under Uncertainty. *Fuzzy Optimization and Decision Making* **6**, 3 (2007), 255-278

[5] Liao, H., Xu, Z.: *Fuzzy Decision Making Methodologies and Applications*. Springer Singapore, 2017

[6] Matos, M. A.: Decision under risk as a multicriteria problem. *European Journal of Operational Research* **181** (2007) 1516-1529

[7] Özkan, I., Türken, I. B.: Uncertainty and fuzzy decisions. In: *Chaos Theory in Politics. Understanding Complex Systems*. (S. Banerjeeet al. eds.), Springer Netherlands, Dordrecht, 2014, 17-27

[8] Pavlačka, O., Rotterová, P.: Fuzzy Decision Matrices Viewed as Fuzzy Rule-Based Systems. In: *Proceedings of the 34th International Conference Mathematical Methods in Economics*. (A. Kocourek, M. Vavroušek, eds.), Technical University of Liberec, Liberec, 2016, 641-646

[9] Rotterová, P., Pavlačka, O.: Computing Fuzzy Variances of Evaluations of Alternatives in Fuzzy Decision Matrices. In: *Proceedings of the 34th International Conference Mathematical Methods in Economics*. (A. Kocourek, M. Vavroušek, eds.), Technical University of Liberec, Liberec, 2016, 735-740

[10] Rotterová, P., Pavlačka, O.: Probabilities of Fuzzy Events and Their Use in Decision Matrices. *International Journal of Mathematics in Operational Research* **9**, 4 (2016) 423-435

[11] Sugeno, M.: *Industrial applications of fuzzy control*. Elsevier Science Pub. Co., New York, 1985

[12] Talašová, J., Pavlačka, O.: Fuzzy Probability Spaces and Their Applications in Decision Making. *Austrian Journal of Statistics* **35**, 2&3 (2006) 347-356

[13] Talašová, J.: *Fuzzy methods of multiple criteria evaluation and decision making* (in Czech) Publishing House of Palacký University, Olomouc, 2003

[14] Yoon, K. P., Hwang, Ch.: *Multiple Attribute Decision Making: An Introduction*. SAGE Publications, California, 1995

[15] Zadeh, L. A.: Fuzzy sets, *Information and Control* **8**, 3 (1965) 338-353

[16] Zadeh, L. A.: The Concept of a Linguistic Variable and its Application to Approximate Reasoning - I. *Information Sciences* **8** (1975) 199-245

[17] Zadeh, L. A.: Probability Measures of Fuzzy Events. *Journal of Mathematical Analysis and Applications* **23**, 2 (1968) 421-427