Robust Fixed Point Transformations in Adaptive Control Using Local Basin of Attraction

József K. Tar, János F. Bitó

Institute of Intelligent Engineering Systems, John von Neumann Faculty of Informatics, Budapest Tech, Bécsi út 96/B, H-1034 Budapest, Hungary
tar.jozsef@nik.bmf.hu, bito@bmf.hu

László Nádai

Transportation Informatics and Telematics Knowledge Center, Budapest Tech, Bécsi út 96/B, H-1034 Budapest, Hungary, nadai@bmf.hu

José A. Tenreiro Machado

Department of Electrotechnical Engineering, Institute of Engineering of Porto, Rua Dr. Antonio Bernardino de Almeida, 4200-072 Porto, Portugal
jtm@isep.ipp.pt

Abstract: A further step towards a novel approach to adaptive nonlinear control developed at Budapest Tech in the past few years is reported. This approach obviates the use of the complicated Lyapunov function technique that normally provides global stability of convergence at the costs of both formal and essential restrictions, by applying Cauchy sequences of local, bounded basin of attraction in an iterative control that is free of such restrictions. Its main point is the creation of a robust iterative sequence that only slightly depends on the features of the controlled system and mainly is determined be the control parameters applied. It is shown that as far as its operation is considered the proposed method can be located between the robust Variable Structure / Sliding Mode and the adaptive Slotine-Li control in the case of robots or other Classical Mechanical Systems. The operation of these method is comparatively analyzed for a wheel + connected mass system in which this latter component is "stabilized" along one of the spokes of the wheel in the radial direction by an elastic spring. The robustness of these methods is also investigated againts unknown external disturbances of quite significant amplitudes. The numerical simulations substantiate the superiority of the robust fixed point transformations in the terms of accuracy, simplicity, and smoothness of the control signals applied.

Keywords: Adaptive Slotine-Li Robot Control, Variable Structure / Sliding Mode Controller, Fixed Point Transformations, Cauchy Sequences, Banach Spaces

1 Introduction: The Common Features of the Robust Variable Structure / Sliding Mode and the Slotine-Li Adaptive Robot Controllers

In the analitical modeling based "classical" control approaches two typical "limit cases" can be distinguished: the simple robust *Variable Structure / Sliding Mode (VS/SM)* controllers and the sophisticated adaptive approaches as e.g. Slotine's and Li's adaptive robot control.

The VS/SM controllers originate from the past Soviet Union of the sixties and obtained wide-spread applications even in these days (e.g. [1, 2, 3]). Their main idea is the application of a rough bang-bang type control in which the controlled system receives drastic flaps from the controller whenever its state crosses a switching surface that finely slides in the phase space according to some kinematical prescriptions. The really important factor is the proper timing of the "flaps", while their absolute value or extent is of minor significance. Mathematically speaking these controllers define some "*Error Metrics*" of the trajectory tracking error of the systam as $\mathbf{S} := (d/dt + \mathbf{\Lambda})^{m-1}\mathbf{e}$, $\mathbf{e}$:=$\mathbf{q}$-$\mathbf{q}^N$, in which $\mathbf{q}$ and $\mathbf{q}^N$ denote the *actual* and the *nominal* coordinates in the given time instant, respectively, $\mathbf{\Lambda}$ is a positive definite matrix or a positive scalar, while *m* denotes the *order of the system*, that is the order of the time-derivative of the generalized coordinates that can directly be manipulated by some physical agents as e.g. torques or forces in the case of Classical Mechanical sytems. On the basis of the available rough model normally strong torque/force overestimation is applied to drive $\mathbf{S}$ to the vicinity of $\mathbf{0}$ during finite time by approximating some simple differential equation prescribed for $d\mathbf{S}/dt$ necessarily containing $d^m\mathbf{q}/dt^m$ that normally can physically be manipulated in the case of an m^{th} order system. The precise realization of this differential equation has no practical significance: as soon as $\mathbf{S}$ approximates $\mathbf{0}$, due to its definition various, decreasing order derivatives of the form $(d/dt + \mathbf{\Lambda})^k\mathbf{e}$ have to converge to zero, too, so finally the tracking error $\mathbf{e}$ itself must converge to $\mathbf{0}$ with characteristic exponents determined by $\mathbf{\Lambda}$. The torque/force overestimations normally cause some chattering that can be reduced or obviated at the price of deteriorated tracking accuracy by smoothing the control signal. This controller does not invest any energy into observing the behavior and learning the fine structure of the controlled system. It is content with the available rough system model that usually works well on various reasons. For instance, in Classical Mechanics the Euler - Lagrange equations of motion take the following form

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{Q} \qquad (1)$$

In which $\mathbf{H}$ is a positive definite symmetric matrix that codes the *inertia* properties of the controlled system, the term containing the matrix $\mathbf{C}$ describes the Coriolis forces, while $\mathbf{g}$ stands for the gravitational effects and any other effects that can be deduced from some potential energy. Since for the modification of $\mathbf{q}$ and $d\mathbf{q}/dt$ considerable time is nedeed due to the inertia of the system, by instantaneously modifying the generalized force $\mathbf{Q}$ by $\Delta\mathbf{Q}$, the 2nd time - derivatives $d^2\mathbf{q}/dt^2$ can instantaneously be modified by $\Delta d^2\mathbf{q}/dt^2$. Since $\mathbf{H}$ is positive definite, the angle included by the vectors $\Delta\mathbf{Q}$ and $\Delta d^2\mathbf{q}/dt^2$ is *acute*, i.e. these vectors approximately have the same direction, therefore without precisely knowing the system's dynamic model the desired $\Delta d^2\mathbf{q}/dt^2$ modifications can be approximated if the appropriate $\Delta\mathbf{Q}$ forces are calculated from the available rough model. Normally, the desired relaxation of $\mathbf{S}$ can be prescribed as

$$\frac{dS_i}{dt} \approx -K\,\mathrm{sgn}(S_i)\tanh\left(\frac{S_i}{w}\right) \tag{2}$$

in which the parameter $K>0$ determines the speed of relaxation, and w is the smoothing parameter.

Essentially the same properties of (1) are utilized by the very sophisticated adaptive control developed by Slotine and Li [4]. By the use of a similar positive definite $\mathbf{\Lambda}$ matrix to the time-derivative of the nominal trajectory a linear feedback correction is kinematically given as $\mathbf{v} := \dot{\mathbf{q}}^N - \mathbf{\Lambda e}$, and the available model of the inertia matrix, marked be the "hat symbol" (^) is used for computing the appropriate contribution to the driving forces from its time derivative according to the equation

$$\mathbf{Q} = \hat{\mathbf{H}}(\mathbf{q})\dot{\mathbf{v}} + \hat{\mathbf{C}}(\mathbf{q},\dot{\mathbf{q}})\mathbf{v} + \hat{\mathbf{g}}(\mathbf{q}) - \mathbf{K}_D[\dot{\mathbf{e}} + \mathbf{\Lambda e}] \equiv \mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{v},\dot{\mathbf{v}})\hat{\mathbf{p}} - \mathbf{K}_D[\dot{\mathbf{e}} + \mathbf{\Lambda e}]. \tag{3}$$

Furthermore, a kind of attempt for linearization can be observed in the 2nd term of the right hand side of (3): the available model for $\hat{\mathbf{C}}$ contains the measurable component of the physical state $\dot{\mathbf{q}}$, and it yields a correction that is "proportional" to $\mathbf{v}$, too. The gravitational term is estimated according to the available model. Since the estimation of the model inertia may be very inaccurate, with a positive definite $\mathbf{K}_D$ matrix further additive derivative and proportional error feedback is applied for keeping the errors at bay when the model is very imprecise. It can be observed that this term exactly corresponds to the error metrics of the robust controller [case m=2 in the definition of $\mathbf{S}$], and it plays similar role in the control though its contribution is not maximized as in the case of the robust controller. As well as the *Adapative Inverse Dynamics Control*, this approach also utilizes the fact that *at least the analytical form of the dynamic model is exactly known*, only the available inertia and other dynamical data are imprecise, so *in the possession of the precise kinematic model* the $\mathbf{Y}$ array can exactly be calculated, while the array $\hat{\mathbf{p}}$ contains the estimation for the relevant dynamical data. Since it is assumed that the so calculated $\mathbf{Q}$ is the only contribution to the generalized forces and no additional external perturbations are present, (1) and (3) are also related to the actual state propagation of the system, from which the parameter estimation errors can be expressed as

$$\mathbf{H\dot{S}}+\mathbf{CS}+\mathbf{K}_D\mathbf{S}=\tilde{\mathbf{H}}\dot{\mathbf{v}}+\tilde{\mathbf{C}}\mathbf{v}+\tilde{\mathbf{g}}=\mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{v},\dot{\mathbf{v}})\tilde{\mathbf{p}}\,, \tag{4}$$

in which the "tilde" symbol (~) denotes the estimation error of the appropriate quantities. An excellent "trick" applied in the Slotine-Li approach is that instead of introducing a more or less "arbitrary" positive definite matrix for constructing the Lyapunov function, the exact symmetric positive definite matrix **H** is used for this purpose. It is very important to realize that the unknown **H** itself is not used in the calculation the control signal. Only the fact of its existence and known properties are used in the Lyapunov function [5, 6]:

$$U:=\frac{1}{2}\mathbf{S}^T\mathbf{HS}+\frac{1}{2}\tilde{\mathbf{p}}^T\mathbf{\Gamma}\tilde{\mathbf{p}}\,, \tag{5}$$

with some symmetric positive definite matrix $\mathbf{\Gamma}$. This immediately yields the derivative

$$\dot{U}:=\mathbf{S}^T\left[\frac{1}{2}\dot{\mathbf{H}}\mathbf{S}+\mathbf{H}\dot{\mathbf{S}}\right]+\tilde{\mathbf{p}}^T\mathbf{\Gamma}\dot{\tilde{\mathbf{p}}}\,, \tag{6}$$

into which from (4) $\mathbf{H}\dot{\mathbf{S}}$ can be substituted, that leads to

$$\dot{U}=\mathbf{S}^T\left[\frac{1}{2}\dot{\mathbf{H}}-\mathbf{C}\right]\mathbf{S}-\mathbf{S}^T\mathbf{K}_D\mathbf{S}+\mathbf{S}^T\mathbf{Y}\tilde{\mathbf{p}}+\tilde{\mathbf{p}}^T\mathbf{\Gamma}\dot{\tilde{\mathbf{p}}}\,. \tag{7}$$

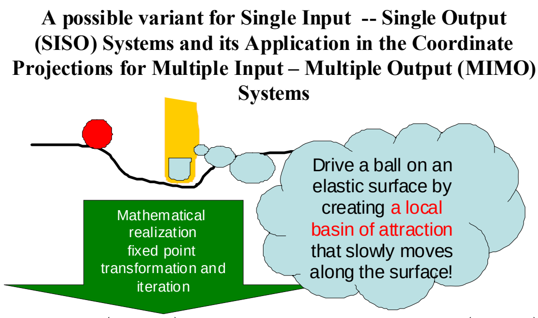
The next great idea is the realization of the fact that the matrix $\left(\frac{1}{2}\dot{\mathbf{H}}-\mathbf{C}\right)$ is skew symmetric therefore yields zero contribution in (7). Since the next term is negative definite, the parameter tuning rule is obtained by making the sum of the third and fourth terms equal to zero:

$$\tilde{\mathbf{p}}^T\left(\mathbf{Y}^T\mathbf{S}+\mathbf{\Gamma}\dot{\tilde{\mathbf{p}}}\right)=0\Rightarrow\dot{\hat{\mathbf{p}}}=-\mathbf{\Gamma}^{-1}\mathbf{Y}^T\mathbf{S}\,. \tag{8}$$

To sum up, this method makes it possible that following a relatively long tuning phase the role of the correcting feedback terms becomes more and more insignificant as the estimation errors converge to zero. In comparison with the robust SM/VS controllers this approach seems to be very ambitious: it intends to learn the exact analytical model, furthermore it utilizes its subtle details that are not really taken into account in the case of the robust controller. However, this approach can work in the learning pase because **H** is symmetric positive definite matrix.

The above "limit cases" inspired the idea of finding a stage of medium complexity for control purposes: by utilizing the fact that **H** is symmetric positive definite, a simple controller can be developed that does not guarantee global asymptotic stability, does not want to learn the exact dynamic model of the system, but in the same time applies less drastic feedback than the robust controller, and works with local and bounded basins of attraction for a convergent sequence of control actions. In the sequel this idea is detailed.

2 The Excitation - Response Scheme and Fixed Point Transformations



$$x_{n+1} = G\left(x_n; x^d\right); \text{if } x^d = f\left(x_*\right) \text{then } x_* = G\left(x_*; x^d\right)$$

$$x_n \to x_*$$

Figure 1

The idea of local deformations resulting in local basins of attraction: pressing the valance with one's finger local deformation can be brought about; by varying the location of the deformation the small ball can be kept moving along a desired trajectory on the valance

Each control task can be formulated by using the concepts of the appropriate "excitation" $\mathbf{Q}$ of the controlled system to which it is expected to respond by some prescribed or "desired response" $\mathbf{r}^d$. The appropriate excitation can be computed by the use of some dynamic model $\mathbf{Q} = \varphi\left(\mathbf{r}^d\right)$. Since normally this model is neither complete nor exact, the actual response determined by the system's exact dynamics, ψ, results in a *realized response* $\mathbf{r}^r$ that differs from the desired one: $\mathbf{r}^r = \psi\left(\varphi\left(\mathbf{r}^d\right)\right) := \mathbf{f}\left(\mathbf{r}^d\right)$. It is worth noting that these functions may contain various hidden parameters that partly correspond to the dynamic model of the system, and partly pertain to unknown external dynamic forces acting on it. Due to phenomenological reasons the controller can manipulate or "deform" the input value from $\mathbf{r}^d$ so that $\mathbf{r}^d = \mathbf{f}\left(\mathbf{r}_*^d\right)$. Other possibility is the manipulation of the output of the rough model as $\mathbf{r}^d = \psi\left(\varphi_*\left(\mathbf{r}^d\right)\right)$. For *Single Input – Single Output (SISO)* systems in [7] particular local deformations were proposed for bringing about local basins of attraction for iterative series the main idea of which is sketched in Fig. 1: driving a small ball on an elastic surface by local basin of attraction created by local deformation that slowly is meandering along some trajectory. The local basins of attrcation in [7] were created by the mathematical transformations

$$
\begin{aligned}
x_{n+1} &= h_\pm\left(x_n;x^d\right) := \frac{x^d-\Delta_\pm}{f\left(x_n\right)-\Delta_\pm}\left(x_n-D_-\right)+D_-,\\
x_{n+1} &= g_\pm\left(x_n;x^d\right) := \frac{f\left(x_n\right)-\Delta_\pm}{x^d-\Delta_\pm}\left(x_n-D_-\right)+D_-
\end{aligned}
\quad . \tag{9}
$$

of which the solution of the equation $f\left(x_*\right)=x^d$ evidently is a fixed point. For achieving convergent sequences in the vicinity of x_* the contractivity of these transformations is required in a local region around x_*. If $\left|G'\left(x;x^d\right)\right| \le K < 1$, then

$$
\left|G(b)-G(a)\right| = \left|\int_a^b G'(\tau)d\tau\right| \le \int_a^b \left|G'(\tau)\right|d\tau \le K\left|b-a\right| \tag{10}
$$

and we obtain a self-convergent (Cauchy) sequence that in a *complete linear metric space* (Banach space) converges to one of the fixed points u ($x_n \to u$) since

$$
\begin{aligned}
&\left|G(u)-u\right| = \left|G(u)-x_n+x_n-u\right| \le \left|G(u)-x_n\right|+\left|x_n-u\right| \le\\
&\le \left|G(u)-G\left(x_{n-1}\right)\right|+\left|x_n-u\right| \le K\left|u-x_{n-1}\right|+\left|x_n-u\right| \to 0 \text{ as } x_n \to u
\end{aligned}
\quad . \tag{11}
$$

For guaranteeing such a convergence the derivatives of (9) have to be considered around x_* in (12). It is evident that if $\left|f'(x)\right|$ is small enough around x_* (that can be guaranteed by properly choosing the parameters of the model φ), and its sign is known, too, the control parameters $\Delta_\pm$ and D_- can be chosen accordingly. Certain successful applications of these transformations were repeorted e.g. in [8, 9]. It the function $f(x)$ has "nice behavior", it is not difficult to find proper control paramteters by numerical simulations starting with big $|\Delta_\pm|$ and small $|D_-|$ values.

$$
\begin{aligned}
h'_\pm\left(x;x^d\right) &= \frac{x^d-\Delta_\pm}{f(x)-\Delta_\pm} - \frac{\left(x^d-\Delta_\pm\right)f'(x)}{\left[f(x)-\Delta_\pm\right]^2}\left(x-D_-\right),\\
h'_\pm\left(x_*;x^d\right) &= 1-\frac{\left(x-D_-\right)f'\left(x_*\right)}{x^d-\Delta_\pm},\\
g'_\pm\left(x;x^d\right) &= \frac{f(x)-\Delta_\pm}{x^d-\Delta_\pm} + \frac{f'(x)}{x^d-\Delta_\pm}\left(x-D_-\right),\\
g'_\pm\left(x_*;x^d\right) &= 1+\frac{f'\left(x_*\right)}{x^d-\Delta_\pm}\left(x_*-D_-\right)
\end{aligned}
\quad . \tag{12}
$$

As it can well be seen, this approach is akin to the VS/SM controller in the sense that none of these controllers wish to identify or learn the parameters of the analytical model of the controlled system. However, there is a considerable difference between them: this novel one does not wish to apply drastic bang-bang type control signals for precise trajectory tracking: it finely adjusts the control signal by observing the controlled system's behavior only in the given, temporal situation. In its philosophy this approach is similar to the idea of "Madarász's situational control" (e.g. [10]) that is applicable in various fields, with the difference that in our case the "control situation" can continuously vary and there is no need for the elaboration of some hierarchical rules for "typical situations".

In spite of these initial successes, it can be stated that the transformations defined in (9) have some deficiencies. As it can well be seen from (12) the derivatives $h'_\pm$ and $g'_\pm$ in the vicinity of x_* behave mainly according to the properties of the not very well known function $f(x)$. If the control leaves the local basin of attraction it becomes divergent, but it is very difficult to recognize the details of this divergence. With other words, the solution proposed in (9) are not very robust against the properties of $f(x)$, therefore it would be expedient to apply more drastic fixed point transformations with more robust behavior. For this purpose the replacement of the simple affine or fractional functions of (9) with more saturated ones, possibly obtained from the exponential function were considered.

The first idea in this direction was rougly sketched in [11] in the form of

$$\begin{gathered}
G\left(x;x^d\right)=\left(x+K\right)\exp\left(-A\left|f\left(x\right)-x^d\right|\right)-K\\
G\left(x_*;x^d\right)=x_*,\quad G\left(-K;x^d\right)=-K\\
G'\left(x;x^d\right)=\exp\left(-A\left|f\left(x\right)-x^d\right|\right)\left\{1-\operatorname{sgn}\left(f\left(x\right)-x^d\right)A\left(x+K\right)f'\left(x\right)\right\}.\\
G'\left(x_*;x^d\right)=1,\quad G'\left(x_*\pm\varepsilon;x^d\right)\approx 1\mp A\left(x_*+K\right)f'\left(x\right)
\end{gathered}\tag{13}$$

With an affine approximation of $f(x)$ in the vicinity of x_* it was shown that this transformation normally can have a "proper" (x_*) and an "false" fixed point ($-K$); the false fixed point has some global basin of attraction that is the whole real axis with the exception of the basin of attraction of x_* that is a small, bounded region only. This structure promised identifiable loss of proper convergence (in this case convergence approaches the false fixed point $-K$) instead of an indefinite divergence. However, it also had the disadvantage that the proper fixed point (x_*) is just located at the boundary of its basin of attracttion, and since G' in x_* is 1, the speed of convergence of the iteration slows down as x_* is approached. To obviate such disadvantages better transformations were proposed in [12] in the form

$$
\begin{aligned}
G\left(x;x^d\right) &= (x+K)\left[1+B\tanh\left(A\left[f(x)-x^d\right]\right)\right]-K \\
G\left(x_*;x^d\right) &= x_*, \quad G\left(-K;x^d\right)=-K \\
G' &= (x+K)\frac{ABf'(x)}{\cosh^2\left(A\left[f(x)-x^d\right]\right)}+\left[1+B\tanh\left(A\left[f(x)-x^d\right]\right)\right] \\
G'\left(x_*;x^d\right) &= (x_*+K)ABf'(x_*)+1
\end{aligned} \quad . \tag{14}
$$

It is evident that the transformation defined in (14) also has a proper and a false fixed point, but by properly manipulating the *A*, *B* and *K* control parameters the good fixed point can be located within its basin of attraction, and the requirement of $|G'(x_*)|<1$ can be guaranteed, too. This means that the iteration can have considerable speed of convergence even nearby x_*, and the strongly saturated *tanh* function can make it more robust in its vicinity, that is the properties of $f(x)$ have less influence on the behavior of *G*.

In [12] the transformation (14) first was tested by simulations for the control of the Φ^6-Type Van der Pol oscillator that is a nonlinear electric circuit the simplest version of which first was modeled by Van der Pol in 1927 [13]. This equipment is a popular paradigm in nonlinear control since it is able to generate chaiotic oscillations for pure sinusoidal excitation (e.g. [14]). The equations of motion of this system to some extent is similar to that of the Classical Mechanical systems. For instance, it is possible to develop the *Adaptive Inverse Dynamic Controller* for it, but is not similar enough to Classical Mechanics for developing a more sophisticated Slotine - Li controller for its study. On this reason, for the sake of more definite comparisons in the present paper we apply our method for a wheel plus a point-like mass affixed to one of its rays by some stiff spring. In the next section the model of this sytem is described.

3 The Wheel – Mass – Spring System's Model

The Euler – Lagrange equations of motion of the system prepared for the application of the Slotine – Li controller with Θ=50 [$kg\times m^2$] momentum of the wheel while rotating around axis q_1 [*rad*], mass m=5 [*kg*] kept at zero spring force at one of the spokes at r_0=1 [*m*] by a spring of stiffness k=50 [*N/m*] and gravitational acceleration g=9.81 [m/s^2] with radial displacement along the ray q_2 [*m*] are as follows:

$$
\underbrace{\begin{bmatrix}\Theta+mq_2^2 & 0\\ 0 & m\end{bmatrix}}_{\mathbf{H}(\mathbf{q})}\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix}+\underbrace{\begin{bmatrix}mq_2\dot{q}_2 & mq_2\dot{q}_1\\ mq_2\dot{q}_1 & 0\end{bmatrix}}_{\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}+\underbrace{\begin{bmatrix}mgq_2\cos q_1\\ mg\sin q_1+k(q_2-r_0)\end{bmatrix}}_{\mathbf{g}(\mathbf{q})}=\begin{bmatrix}Q_1\\ Q_2\end{bmatrix} \tag{15}
$$

in which the generalized forces are Q_1 torque [$N\times m$] and Q_2 force [*N*]. The approximate model parameters were $\hat{\Theta}=5$ [$kg\times m^2$], $\hat{m}=0.5$ [*kg*], $\hat{r}_0=0.8$ [*m*],

$\hat{k} = 5$ [*N/m*], $\hat{g} = 1$ [*m/s*^{2}], i.e. they very roughly differred to the "exact ones." The operation of the various control approaches will be comparatively analyzed according to this setting in the sequel.

4 Computational Results

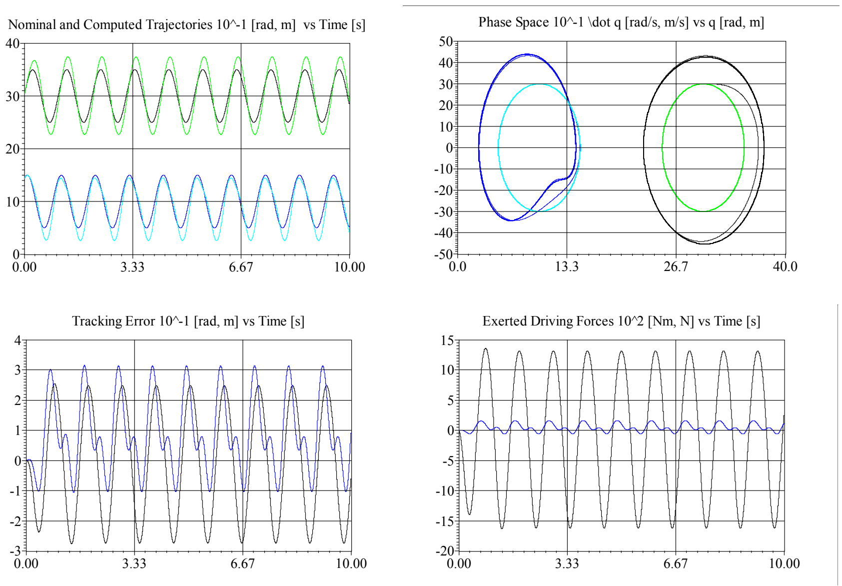


Figure 2

The operation of the simple PD – type controller without disturbance forces

Since the three various methods to be compared have different control parameters, no exact comparisons can be done. However we tried to experimentally find some settings that provided the best results for each mothod considered. For the non-adaptive, non-robust control the desired trajectory tracking property was prescribed in purely kinematic terms according to a simple PD – type controller as $\ddot{\mathbf{q}}^d = \ddot{\mathbf{q}}^N + 2\Lambda\left(\dot{\mathbf{q}}^N - \dot{\mathbf{q}}\right) + \Lambda^2\left(\mathbf{q}^N - \mathbf{q}\right)$ with Λ=30 [s^{-1}]. The first set of investigations happened without any disturbance forces. As it can be seen in Fig. 2 the very inaccurate estimation of the model values yielded great error in tracking the trajectories, the phase trajectories. Improvement with respect to the data of Fig. 2 can characterize the operation of the other methods considered. In the simulatons simple Euler integration was aplied with the discrete time resolution of 1 *ms*.

4.1 The VS/SM Controller's Operation

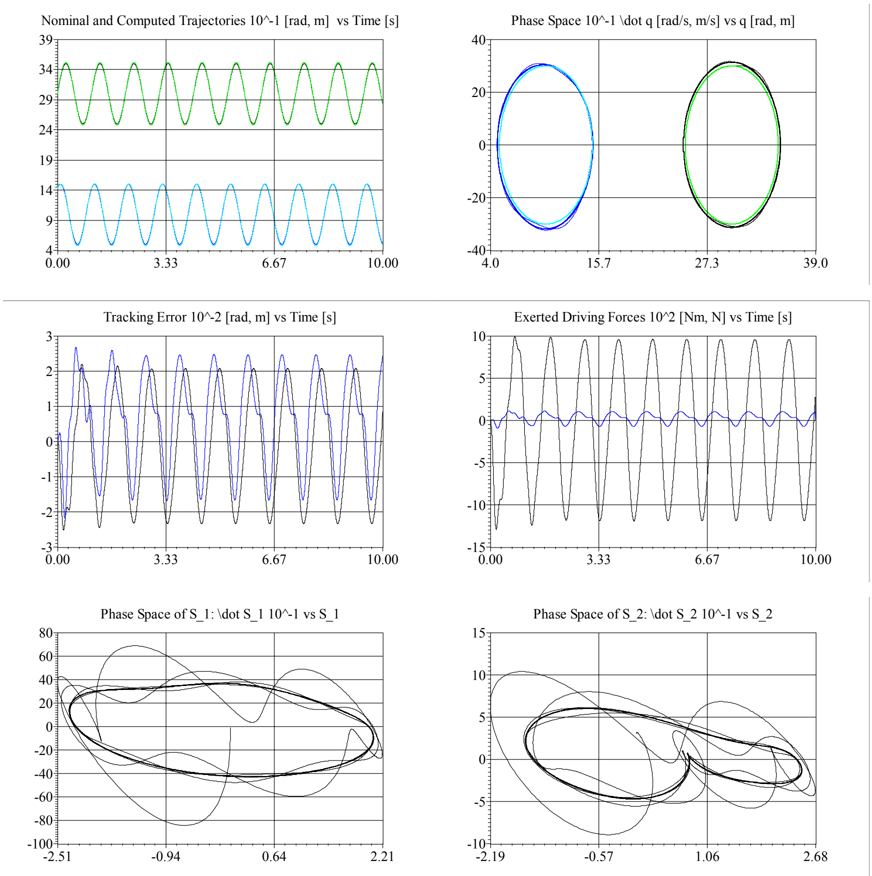


Figure 3
The operation of the robust VS/SM – type controller without external disturbance forces

In the case of the VS/SM controller the following control parameters were used: Λ=30 [s^{-1}], w=0.1, K=800 according to the notation used in (2). The results are given in Fig. 3 that is the counterpart of Fig. 2 with the exception that it also displays the phase spaces of the two components of the error metrics. It is evident that though these quantities are very far from the prescription expressed by (2), they are kept at bay near zero and the curves are "concentrated" along a curve with some negative slope. Figure 3 well testifies that the simple smoothed VS/SM controller works well and considerably increases the tracking accuracy without causing drastic chattering.

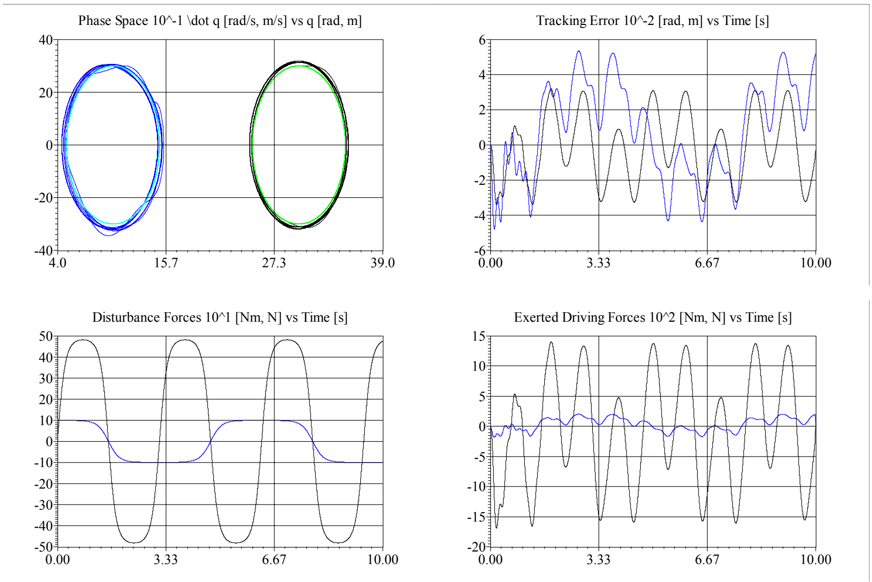


Figure 4

The operation of the robust VS/SM – type controller with external disturbance forces

The results displayed in Fig. 4 belong to consideable disturbance forces that are well compensated by this simple controller, too. The "negative" distortion of the disturbance forces can well be recognized in the exerted control forces, too.

4.2 The Adaptive Slotine - Li Controller's Operation

This very sophisticated controller has a plenty of control parameters as far as the tuning of the estimation of the dynamic data is concerned. Their number strictly depends on the analytical model of the system. In our case a matrix **Y** of size 2×5 and five directly tuned parameters were used as given in (16). For the sake of simplicity ve used the following settings: Λ=30 [s^{-1}], K_D=200 as denoted in (3). For the parameter tuning the positive definite Γ matrix has the following number of independent parameters if the possible diagonalization with orthogonal matrices is considered ($\boldsymbol{\Gamma} = \mathbf{ODO}^T$): 5+(5^2-5)/2=15. For the simplicity we used only a $\Gamma=\mathit{\Gamma}\mathbf{I}$ structure with $\mathit{\Gamma}$=0.05 that allows relatively fast learning/tuning.

$$\mathbf{Y} = \begin{bmatrix} \dot{v}_1 & q_2^2\dot{v}_1 + q_2\dot{q}_2 v_1 + q_2\dot{q}_1 v_2 & q_2\cos q_1 & 0 & 0 \\ 0 & \dot{v}_2 + q_2\dot{q}_1 v_1 & \sin q_1 & q_2 & -1 \end{bmatrix} \begin{bmatrix} \Theta \\ m \\ mg \\ k \\ kr_0 \end{bmatrix} \tag{16}$$

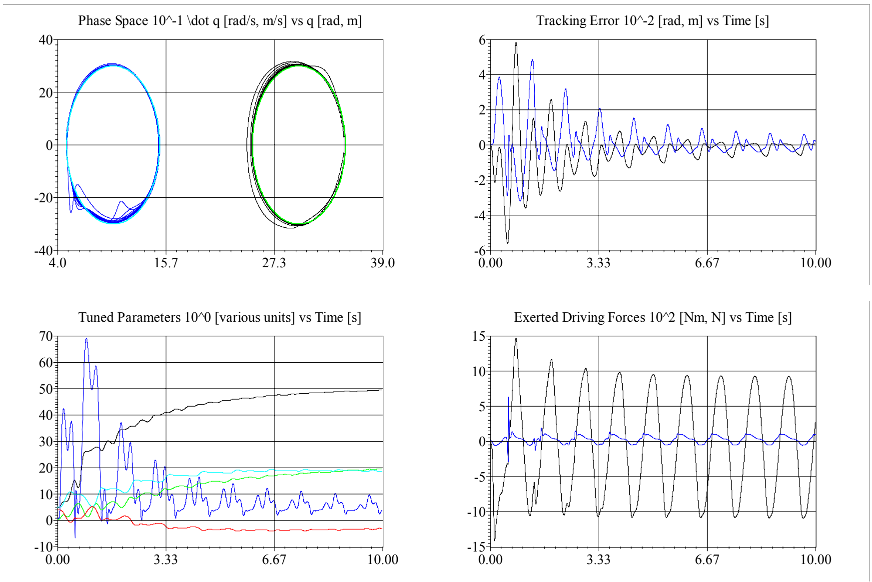


Figure 5

The operation of the adaptive Slotine – Li controller without external disturbance forces

The operation of the Slotine – Li controller without disturbance forces is depicted in Fig. 5. It can well be seen that it slowly "learns" the dynamics of the controlled system and following a transient phase it yields precise trajectory and phase trajectory tracking with proper generalized forces. However, due to the suppositon that the generalized forces are exactly known and are equal with that exerted by the controller the external disturbances can fob this sophisticated controller. This can well be seen in Fig. 6 that describes the time-dependence of the Lyapunov function that otherwise is not used in the calculation of the control signal. The disturbance – free case corresponds to decreasing Lyaunov function in accordance with the theoretical expectations. In the case of external disturbances some fluctuations appear in it. Figure 7 is the counterpart of Fig. 5 when the same external disturbance forces were applied to the system as that in Fig. 4. Both the accuracies of trajectory and phase trajectory tracking became degraded, furthermore, the parameter tuning mechanuism tried to compensate the effects of the external disturbances in quite improper manner.

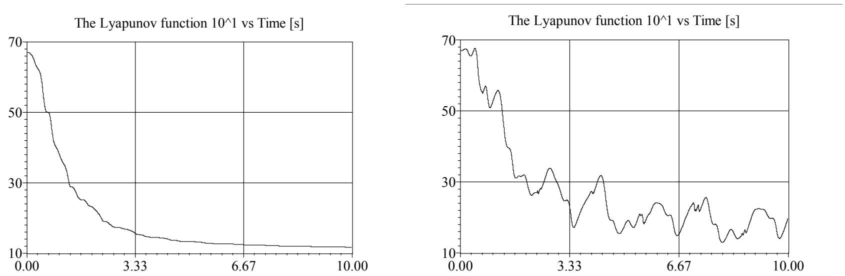


Figure 6

The operation of the adaptive Slotine – Li controller without (left hand side) and with (right hand side) external disturbance forces: the variation of the Lyapunov function versus time

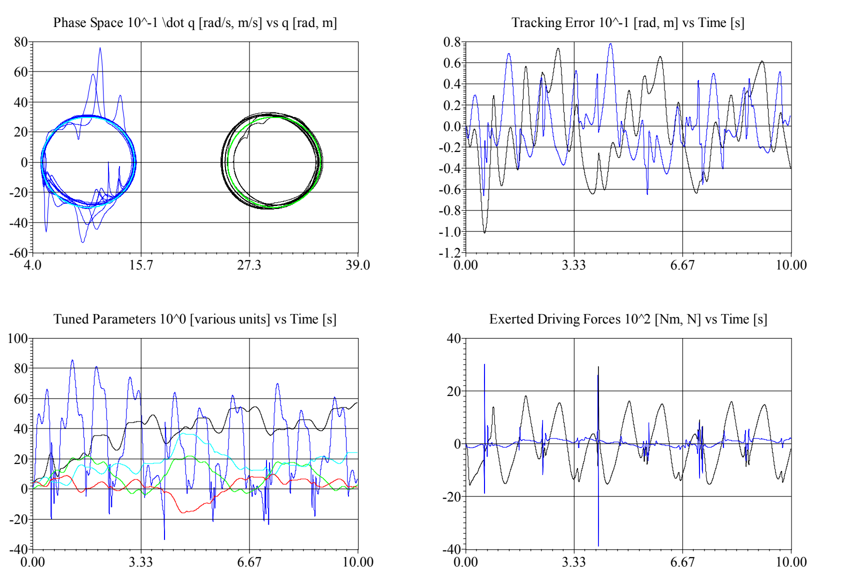


Figure 7

The operation of the adaptive Slotine – Li controller with external disturbance forces

4.3 The Operation of the Fixed Point Transformation based Controller Applied in Coordinate Projections

In this case the same trajectory tracking was prescribed as in the non-adaptive case depicted in connection with Fig. 2. For both projections of the coordinate space (i.e. for q_1 and q_2) the same control parameters were prescribed in harmony with the notation applied in (14): A=10^{-3}, B=0.8, K= -2000.

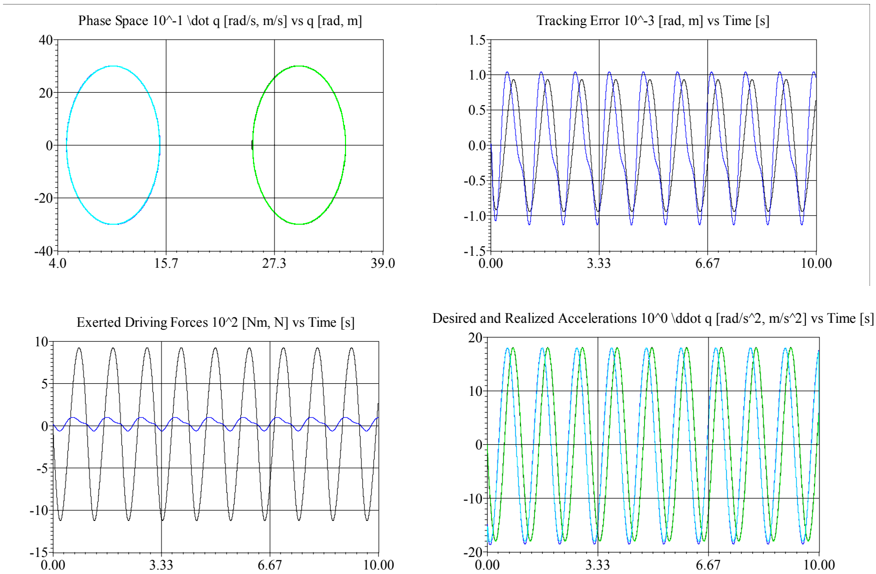


Figure 8
The operation of the fixed point transformation based adaptive controller with simple PD – type prescribed trajectory tracking without external disturbance forces

As it is revealed by Fig. 8 this controller guarantees very precise trajectory and even phase trajectory tracking with nice generalized forces by accurately realizing the kinematically prescribed trajectory error relaxation in the lack of external disturbances.

Figure 9 is the counterpart of Fig. 8 when the same external disturbance forces were applied to the system as that in Fig. 4. It an well be seen that in the kinematic properties of the controlled trajectories only nuances vary, while there is a significant modification in the exerted driving forces that quite accurately compensate the effects of the disturbances.

To demonstrate the robustness of the fixed point transformations based method the counterpart of Fig. 9 is given in Fig. 10 with the considerably modified control parameters as $A=2\times10^{-3}$, $B=0.4$, $K=-1000$. The results of these new settings are almost the same as that of the original ones.

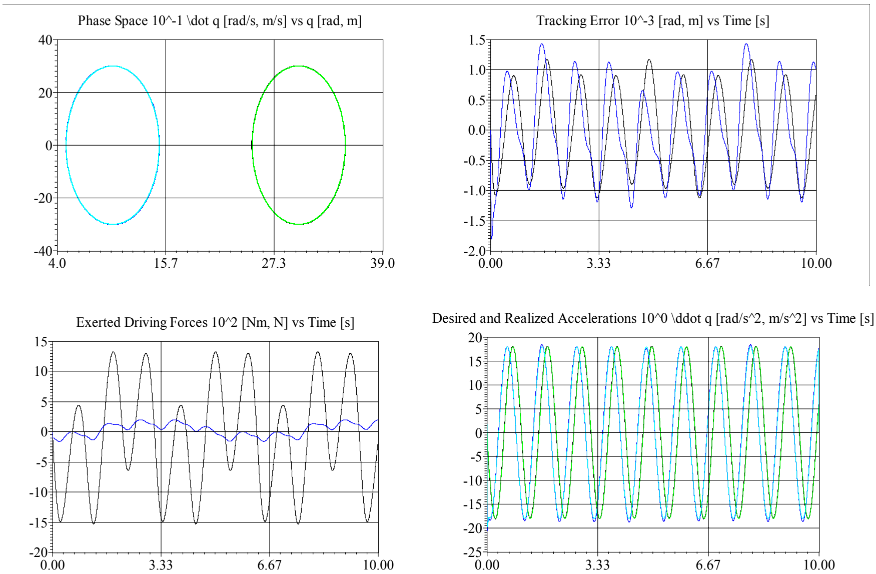


Figure 9

The operation of the fixed point transformation based adaptive controller with simple PD – type prescribed trajectory tracking with external disturbance forces

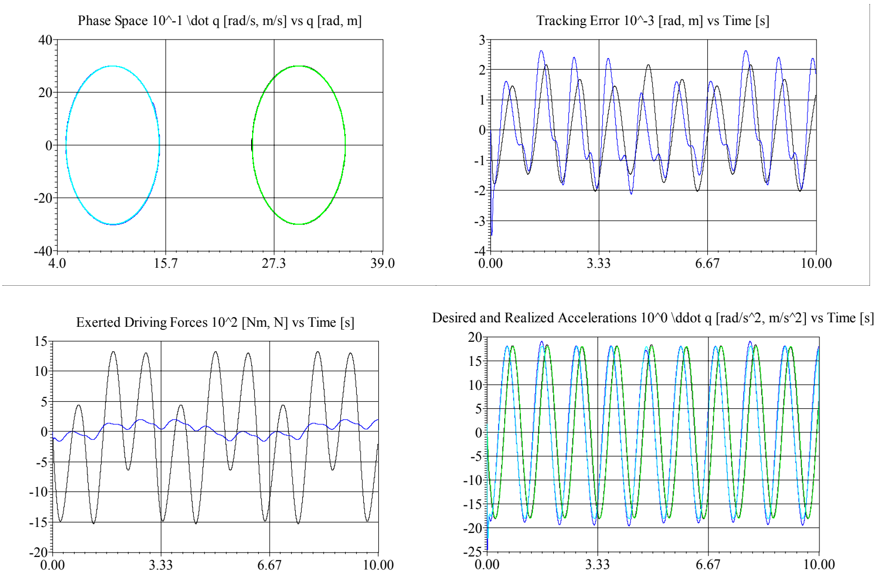


Figure 10

The operation of the fixed point transformation based adaptive controller with simple PD – type prescribed trajectory tracking with external disturbance forces and modified control parameters

Conclusions

In this paper the similarities and the differences between three different control methods, namely the robust VS/SM Controller, the Adaptive Slotine – Li Controller and a novel adaptive controller based on the use of simple, geometrically interpreted fixed point transformations were studied by using a simple paradigm, the wheel plus mass at the ray fixed by an elastic spring. The effects of external disturbances were also studied.

It was found that as far as its ambitions are concerned, though the present approach is a kind of transition between the robust and adaptive controllers, it seems to be superior in terms of simplicity, lucidity, accuracy, and robustness till it remains within its local basin of attraction of convergence. Its main deficiency is that its basin of attraction is bounded so the method cannot guarantee global asymptotic stability as other approaches normally based on the Lyapunow function technique.

Our future aim is to consider the applicability of this novel method for the control of "fractional order systems" the equations of motion of which can be described by fractional order differential equations. It is expected that even in this more complicated case simple convergent Cauchy sequences can be created as in the case of the "integer order systems".

Acknowledgement

The authors gratefully acknowledge the support by the *Hungarian National Research Fund (OTKA)* within the project *No. K063405*. This research was also supported by the *National Office for Research and Technology (NKTH)* in Hungary using the resources of the *Research and Technology Innovation Fund* within the project *No. RET-10/2006* and that of the *Bilateral Hungarian–Portuguese S&T Program No. PT–12/07.*

References

[1] S. V. Emelyanov, S. K. Korovin and L. V. Levantovsky: Higher Order Sliding Regimes in the Binary Control Systems, Soviet Physics, Doklady, 1986, Vol. 31, 291-293

[2] V. I. Utkin: Sliding Modes in Optimization and Control Problems, 1992, Springer Verlag New York

[3] A. Levant: Arbitrary-Order Sliding Modes with Finite Time Convergence, Proceedings of the 6th IEEE Mediterranean Conference on Control and Systems, June 9-11, 1998, Alghero, Sardinia, Italy

[4] Jean-Jacques E. Slotine, W. Li: Applied Nonlinear Control, Prentice Hall International, Inc., Englewood Cliffs, New Jersey 1991

[5] A. M. Lyapunov: A General Task about the Stability of Motion (in Russian), PhD Thesis, 1892

[6] A. M. Lyapunov: Stability of Motion, Academic Press, New York and London, 1966

[7] József K. Tar: Fixed Point Transformations as Simple Geometric Alternatives in Adaptive Control, invited plenary lecture, Proc. of the 5th IEEE International Conference on Computational Cybernetics, October 19-21, 2007, Gammarth, Tunis, ISBN 1-4244-1146-7, pp. 19-34

[8] József K. Tar, Katalin Lőrincz, Roland Kovács: Adaptive Control of an Automatic Convoy of Vehicles, in the Proc of the 11th IEEE International Conference on Intelligent Engineering Systems 2007, June 29-July 2, 2007, Budapest, Hungary, pp. 21-26, ISBN: 1-4244-1148-3

[9] József K. Tar, Katalin Lőrincz, László Nádai, Roland Kovács: Investigation of Various Tracking Rules in Platoons of Unmodeled Loads and Saturated Drives, in the Proc. of the 2nd IEEE International Workshop on Soft Computing Applications, 21-23 August, 2007 Gyula–Hungary, Oradea–Romania (SOFA 2007), pp. 205-210, IEEE Catalog Number: CFP0728DC, ISBN: 978-1-4244-1608-0, Library of Congress: 2007905895

[10] R. Andoga, L. Madarász, L. Főző: Situational Modeling and Control of a Small Turbojet Engine MPM 20, IEEE International Conference on Computational Cybernetics, August 20-22, 2006, Tallinn, Estonia, pp. 81-85

[11] József K. Tar, János F. Bitó, László Nádai, José A. Tenreiro Machado: Preliminary Sketch of Possible Fixed Point Transformations for Use in Adaptive Control, in the Proc. of the 6th International Symposium on Intelligent Systems and Informatics (SISY 2008), September 26-27, 2008, Subotica, Serbia

[12] József K. Tar, János F. Bitó, Imre J. Rudas, Krzysztof R. Kozłowski, José A. Tenreiro Machado: Possible Adaptive Control by Tangent Hyperbolic Fixed Point Transformations Used for Controlling the Φ^6-Type Van der Pol Oscillator, Proc. of the 6th IEEE International Conference on Computational Cybernetics (ICCC 2008), November 27-29, 2008, Hotel Academia, Stará Lesná, Slovakia, pp. 15-20, (CD issue), IEEE Catalog Number: CFP08575-CDR, ISBN: 978-1-4244-2875-5, Library of Congress: 2008907697

[13] B. Van der Pol, Philos. Mag. 7 (3) p. 65, 1927

[14] S. H. Hosseinnia, R. Ghaderi, A. Ranjbar N., J. Sadati, Shaher Momani: Designation of an Adaptive PID Controller to Synchronize Fractional-Order Hard Spring Φ^6 - Van Der Pol Oscillator, Proc. of the 3rd IFAC Workshop on Fractional Differentiation and its Applications, 5 - 7 November, 2008, Ankara, Turkey, file: Control_7_2.PDF, (CD issue)