Robust Optimization of the Steel Single Story Frame

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Abstract: In contemporary design practices, building structures are expected to not only meet safety requirements but also be optimized. However, optimal designs can be highly sensitive to random variations in model parameters and external actions. Solutions that appear effective under nominal conditions may prove inadequate when parameter randomness is considered. To address this challenge, the concept of robust optimization has been introduced, which extends deterministic optimization formulations to incorporate the random variability of parameter values. In this study, we demonstrate the applicability of robust optimization in the design of building structures using a simple orthogonal frame as an example. The static-strength analysis is conducted based on the displacement method, utilizing second-order theory. To assess the safety level of the steel frame, a preliminary evaluation is performed by determining the reliability index and failure probability using the Monte Carlo Method. Robust optimization is then employed, leveraging the second-order response surface. Experimental designs are generated following an optimal Latin hypercube plan. The proposal of a mathematical-numerical algorithm for solving the optimization problem while considering the random nature of design parameters constitutes the innovative aspect of this research.

Keywords: reliability; robust optimization; second order theory; displacement method

1 Introduction

The design of complex structures places a dual responsibility on engineers: ensuring building safety while simultaneously minimizing construction costs and structural weight. This trend has led to an increasing interest in optimization methods to achieve efficient material utilization, making optimization an indispensable tool for rational structural design. While optimization modules are commonly incorporated in design approaches based on the Finite Element Method, they are typically employed in a deterministic manner. In this traditional approach, the random nature of design parameters is considered by incorporating partial safety factors into the optimization formulation. These factors, defined by relevant design standards (codes) [N1-N3], are often calibrated to suit a broad range of design tasks. However, this approach frequently yields overly conservative solutions, as the partial safety factors do not directly account for the random variations in design variables. Consequently, optimal structures may not achieve the desired level of reliability. If ensuring structural safety is a primary design requirement, it is worth considering a reliability-based design optimization (RBDO) formulation [1-7]. RBDO-based design constraints are formulated using probabilities of failure, which represent the likelihood of exceeding specific permissible states related to load capacity or allowable displacement. These permissible states are incorporated into the formulation through relevant limit functions in reliability analysis, commonly known as 'performance functions.'

Random fluctuations in structural response degrade its quality, leading to deviations from intended functionality and increased maintenance costs (e.g., inspections, maintenance, and repairs). An effectively designed building should minimize such costs, ensure proper functioning, and be less sensitive to the random nature of design parameters. To meet the required level of quality, design methods and procedures have been developed, collectively referred to as 'robust design' [8-10]. This methodology aims to design structures, devices, and production processes that maintain high functionality within a system. The objective is to find solutions that are as resistant as possible to variations in design parameters. Limiting or eliminating the variability of input parameters can reduce variance in quantities characterizing the structural state, such as displacements and stress. However, implementing such procedures often incurs unacceptable costs. A more effective approach is to reduce parameter variability without altering the variance of the structure's input parameters. This approach is known as robust optimization.

Robust optimization provides solutions that are less sensitive to model parameters that are challenging to control. This paper focuses on highlighting this significant aspect of optimal design for bar structures using robust optimization. The analysis centers on an orthogonal steel frame, with specific emphasis on columns subjected to high axial forces. The analysis incorporates the second-order theory, and the preliminary assessment of the steel frame's safety level involves determining the reliability index and failure probability using the Monte Carlo Method. Robust optimization is performed using a second-order response surface, which effectively mitigates the impact of uncontrollable model parameters. The second-order method accounts for non-linearity and interactions between variables in the model, enabling the identification of optimal solutions with a reduced need for experimental analysis. Reliable results are generated using an optimal Latin cubes plan for conducting experiments, ensuring even and balanced sampling of the parameter space. This approach accelerates the analysis process while maintaining result reliability. The proposal of a mathematical-numerical algorithm for optimization that accounts for the random nature of design parameters contributes to the innovative nature of this research.

2 Materials and Methods

2.1 Displacement Method and Second-Order Beam Theory

To reduce computation time, we employed explicit forms of the performance function and constraints. The calculations are based on the classical displacement method and the second-order theory. This method involves several simplifying assumptions, including assuming small curvatures of the member axis in the current configuration, assuming flat cross-sections (known as Bernoulli's geometric hypothesis), assuming longitudinal non-deformability, assuming a linear-elastic physical law, and assuming material continuity, homogeneity, and isotropy. However, the second-order theory does not account for the principles of stiffness (small displacements) and superposition. The degrees of freedom consist of translational and rotational displacements of nodes. By imposing constraints on unknown displacements, the level of geometric indeterminacy can be determined, resulting in the derivation of additional equilibrium equations known as canonical equations. The right-side vector in the system of canonical equations represents the reactions of constraints determined by the external load.

These formulas establish a relationship between the forces and displacements associated with the local system of a given member. Our analysis focuses on a perfectly elastic prismatic rod with a length of 'l' and a bending stiffness of 'EJ' (see Figure 1). This rod serves as an illustration of the behavior of the columns in the frame, where significant axial force is considered alongside bending. Transformation formulas are derived through analytical solutions to the relevant boundary problem.

To perform the computations, it is necessary to derive a differential equation representing the equilibrium of forces on two planes perpendicular to the axis, located at distances x and $x + dx$ from the origin of the coordinate system. Figure 2 depicts a cut section of the member with a length of dx in a deformed configuration, subjected to external load and two systems of cross-sectional forces that represent the interaction with the remaining portions of the member.

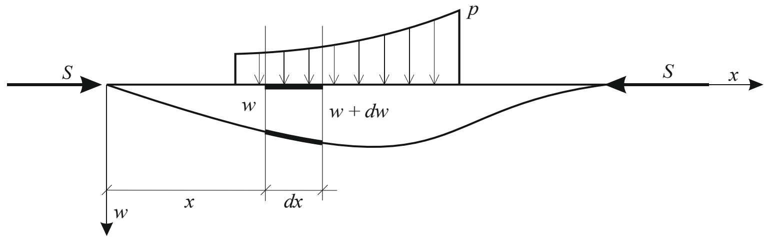

Figure 1
Rod in deformed configuration

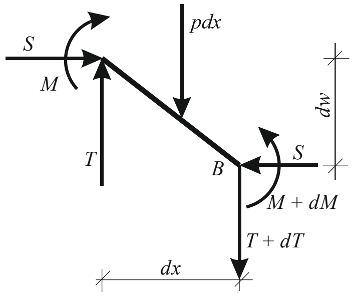

Figure 2
Section of the rod dx long

The section of the rod must be in equilibrium, so the equilibrium must be met:

$$\begin{cases} \Sigma W = -T + T + dT + pdx = 0 \\ \Sigma M_B = -M + M + dM - Tdx - Sdw + pdx \cdot \frac{dx}{2} = 0 \end{cases} \quad (1)$$

The quantity *dx* in the equation $\frac{pdx^2}{2}$ is infinitesimally small, so we can skip it.

$$\begin{cases} \frac{dT}{dx} + p = 0 \\ \frac{dM}{dx} - T - S\frac{dw}{dx} = 0 \to T = \frac{dM}{dx} - S\frac{dw}{dx} \end{cases} \quad (2)$$

$$\begin{cases} \frac{dT}{dx} = -p \\ \frac{dT}{dx} = \frac{d^2M}{dx^2} - S\frac{d^2w}{dx^2} \end{cases} \quad (3)$$

$$\frac{d^2M}{dx^2} - S\frac{d^2w}{dx^2} = -p \quad (4)$$

$$M = -EJ\frac{d^2w}{dx^2} \quad (5)$$

$$\frac{d^2M}{dx^2} = -EJ\frac{d^4w}{dx^4} \quad (6)$$

$$-EJ\frac{d^4w}{dx^4} - S\frac{d^2w}{dx^2} = \frac{-p}{\cdot(-1)} \quad (7)$$

$$EJ\frac{d^4w}{dx^4} + S\frac{d^2w}{dx^2} = p \tag{8}$$

Transformating into the dimensionless space, we can write: $\xi = \frac{x}{l}$

$$\frac{EJ}{l^4} \cdot \frac{d^4w}{d\xi^4} + \frac{S}{l^2} \cdot \frac{d^2w}{d\xi^2} = \frac{p}{\cdot\frac{l^4}{EJ}} \tag{9}$$

$$\frac{d^4w}{d\xi^4} + \frac{Sl^2}{EJ} \cdot \frac{d^2w}{d\xi^2} = \frac{pl^4}{EJ} \tag{10}$$

$\sigma^2 = \frac{Sl^2}{EJ}$, dimensionless parameter σ^2

$$w^{\mathrm{IV}} + \sigma^2 w^{\mathrm{II}} = 0 \tag{11}$$

The solution of the homogeneous differential equation is an exponential function:

$$w = e^{k\xi} \tag{12}$$

where: k – the coefficient that we determine by substituting into the equation (11) the corresponding derivatives:

$$k^4 e^{k\xi} + \sigma^2 k^2 e^{k\xi} = 0 \tag{13}$$

$$(k^2 + \sigma^2)k^2 e^{k\xi} = 0 \tag{14}$$

$$\begin{array}{ccc} k^2 = 0 & or & k^2 + \sigma^2 = 0 \\ \Downarrow & & \Downarrow \\ k_1 = k_2 = 0 & & k_3 = i\sigma \quad k_4 = -i\sigma \end{array} \tag{15}$$

The general integral of the equation (11) is therefore a function

$$w_o = \tilde{C}_1 + \tilde{C}_2\sigma\xi + \tilde{C}_3 e^{i\sigma\xi} + \tilde{C}_4 e^{-i\sigma\xi} \tag{16}$$

After using the Euler formula

$$e^{\pm i\alpha} = cos\,\alpha \pm i\,sin\,\alpha \quad \alpha \in R \tag{17}$$

Get:

$$\begin{aligned} w_o &= C_1 + C_2\sigma\xi + C_3\,cos\,\sigma\,\xi + C_4\,sin\,\sigma\,\xi \\ w_o^I &= C_2\sigma - C_3\sigma\,sin\,\sigma\,\xi + C_4\sigma\,cos\,\sigma\,\xi \\ w_o^{II} &= -C_3\sigma^2\,cos\,\sigma\,\xi - C_4\sigma^2\,sin\,\sigma\,\xi \\ w_o^{III} &= C_3\sigma^3\,sin\,\sigma\,\xi - C_4\sigma^3\,cos\,\sigma\,\xi \\ w_o^{IV} &= C_3\sigma^4\,cos\,\sigma\,\xi + C_4\sigma^4\,sin\,\sigma\,\xi \end{aligned} \tag{18}$$

The cross-sectional forces: the bending moment and the transverse force after switching to a dimensionless variable $\xi = \frac{x}{l}$ will be as follows:

$$T = -\frac{EJ}{l^3}(w^{\mathrm{III}} + \sigma^2 w^I) = -\frac{EJ}{l^3}\sigma^3 C_2$$

$$M = -\frac{EJ}{l^2} w^{\mathrm{II}} = \frac{EJ}{l^2} \sigma^2 (C_3 \cos \sigma \xi + C_4 \sin \sigma \xi) \tag{19}$$

The integration constants present in the above formulas depend on the way the member is supported. For a fixed rod with support on both ends, as shown in Figure 3, the following boundary conditions can be written:

$$\begin{cases} w(0)=w_i \\ w^I(0)=l\varphi_i \\ w(1)=w_j \\ w^I(1)=l\varphi_j \end{cases} \qquad \begin{cases} C_1+C_3=w_i \\ C_2\sigma+C_4\sigma=l\varphi_i \\ C_1+C_2\sigma+C_3 \cos \sigma + C_4 \sin \sigma = w_j \\ C_2\sigma - C_3\sigma \sin \sigma + C_4\sigma \cos \sigma = l\varphi_j \end{cases} \tag{20}$$

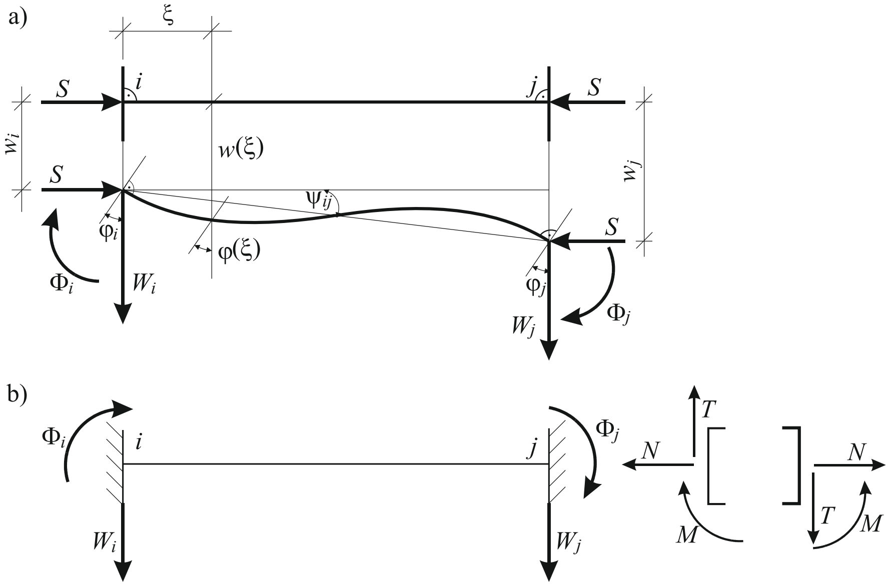

Figure 3
Fixed rod fixed on both sides

$$\Phi_i = M(0); \quad \Phi_j = -M(1); \quad W_i = -T(0); \quad W_j = T(1)$$

$$\Phi_i = \frac{EJ}{l^2} \sigma^2 C_3; \quad \Phi_j = -\frac{EJ}{l^2} \sigma^2 (C_3 \cos \sigma + C_4 \sin \sigma) \tag{21}$$

After some rearrangement, we obtain:

$$\Phi_i = \frac{EJ}{l} \left(\alpha\varphi_i + \beta\varphi_j - \vartheta\psi_{ij} \right)$$

$$\Phi_j = \frac{EJ}{l} \left(\beta\varphi_i + \alpha\varphi_j - \vartheta\psi_{ij} \right)$$

$$W_i = \frac{EJ}{l^2} \left(\vartheta\varphi_i + \vartheta\varphi_j - \delta\psi_{ij} \right) \tag{22}$$

$$W_j = -\frac{EJ}{l^2}\left(\vartheta\varphi_i + \vartheta\varphi_j - \delta\psi_{ij}\right)$$

The coefficients $\alpha, \beta, \delta, \vartheta$ are not numbers, but the complex trigonometric functions of the parameter σ.

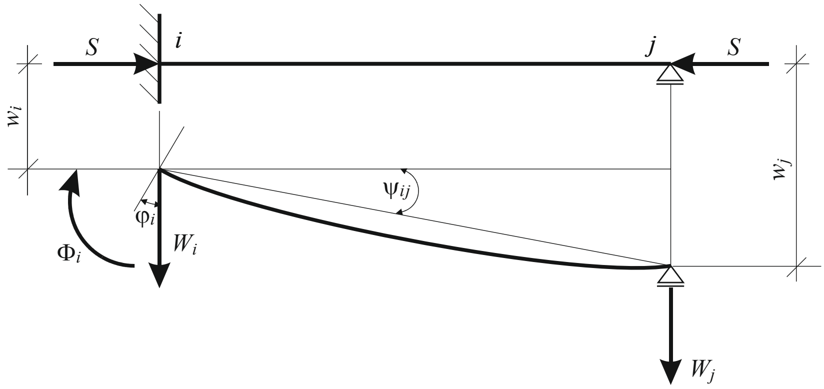

Figure 4

Rod fixed at one end and freely supported at the other

For a rod of the type fixed-joint (Figure 4), on the simply supported side (right), the bending moment is equal to zero $\Phi_j = 0$

$$\Phi_j = 0 \Rightarrow \beta\varphi_i + \alpha\varphi_j - \vartheta\psi_{ij} = 0$$

$$\Phi_i = \frac{EJ}{l}\left[\alpha\varphi_i + \beta\left(-\frac{\beta}{\alpha}\varphi_i + \frac{\vartheta}{\alpha}\psi_{ij}\right) - \vartheta\psi_{ij}\right] = \frac{EJ}{l}\left[\underbrace{\left(\alpha - \frac{\beta^2}{\alpha}\right)}_{\alpha_I}\varphi_i - \underbrace{\left(\vartheta - \frac{\beta\vartheta}{\alpha}\right)}_{\alpha_I}\psi_{ij}\right]$$

$$W_i = \frac{EJ}{l^2}\left[\vartheta\varphi_i + \vartheta\left(-\frac{\beta}{\alpha}\varphi_i + \frac{\vartheta}{\alpha}\psi_{ij}\right) - \delta\psi_{ij}\right] = \frac{EJ}{l^2}\left[\underbrace{\left(\vartheta - \frac{\vartheta\beta}{\alpha}\right)}_{\alpha_I}\varphi_i - \underbrace{\left(\delta - \frac{\vartheta^2}{\alpha}\right)}_{\delta_I}\psi_{ij}\right]$$

$$W_i = \frac{EJ}{l^2}\left(\alpha_I\varphi_i - \delta_I\psi_{ij}\right)$$

$$W_j = -\frac{EJ}{l^2}\left(\alpha_I\varphi_i - \delta_I\psi_{ij}\right)$$

2.2 The Reliability Assessment using Monte Carlo Method

According to the codes [N4], the assessment of structural reliability is based on the concept of limit states, and their verification is conducted using a semi-probabilistic method with the use of partial safety factors. The purpose of these factors is to ensure the desired level of structural reliability. Approximate methods such as FORM [11-15] and SORM [16-18], as well as simulation techniques such as Monte

Carlo [19-23] and Importance Sampling [24-26], are extensions of the semi-probabilistic limit state method. The probabilistic approach allows for a more accurate and realistic modeling of structural materials, geometric parameters, and loads. In the present study, the Monte Carlo method was employed to determine the probability of failure.

The classic Monte Carlo simulation method involves generating realizations of the random vector $\boldsymbol{X}$ according to the joint probability distribution density function $f(x)$. In the next step, for each realization of the random vector $\boldsymbol{X}$, the performance function is computed. The ratio of the number of 'hits' in the failure area to the total number of simulations provides an estimator of the probability of failure (see Figure 5). The above idea can be expressed by defining the characteristic function of the failure area set as:

$$X_{\Omega_f}(\boldsymbol{x})=\begin{cases}1 \text{ if } & \boldsymbol{x}\in\Omega_f\\ 0 \text{ if } & \boldsymbol{x}\notin\Omega_f\end{cases} \tag{24}$$

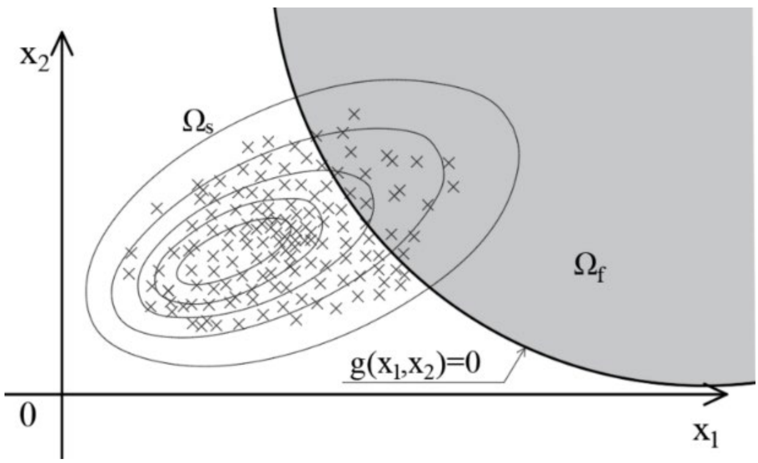

Figure 5
Idea of Monte Carlo method

$X_{\Omega_f}(\mathbf{x})$ is therefore a random variable with a two-point distribution:

$$P\left[X_{\Omega_f}(\boldsymbol{X})=1\right]=P_f \qquad P\left[X_{\Omega_f}(\boldsymbol{X})=0\right]=1\text{-}P_f \tag{25}$$

where: $P_f=P[\boldsymbol{X}\in\Omega_f]$

$$Var\left[X_{\Omega_f}(\boldsymbol{X})\right]=E\left[\left(X_{\Omega_f}(\boldsymbol{X})\right)^2\right]-\left(E\left[X_{\Omega_f}(\boldsymbol{X})\right]\right)^2=P_f-P_f^2=P_f(1-P_f) \tag{26}$$

In the Monte Carlo method, an estimator of the mean value of the characteristic function of the set of the form:

$$\widetilde{X}^{0}_{\Omega_f} = \frac{1}{K}\sum_{k=1}^{K} X_{\Omega_f}(\boldsymbol{X_k}) = \widetilde{P}_f \tag{27}$$

where: $\boldsymbol{X_k}$ - independent random vectors with a probability distribution defined by the density function $f_x(\mathbf{x})$, K - the number of simulations.

The mean value and variance of the estimator are given as:

$$\begin{aligned} \widetilde{P^{0}_f} &= E\left[\widetilde{P}_f\right] = \frac{1}{K}\sum_{k=1}^{K} X^{0}_{\Omega_f}(\boldsymbol{X_k}) = \frac{1}{K}K \cdot P_f = P_f \\ \sigma^2_{\widetilde{P}_f} &= Var\left[\widetilde{P}_f\right] = \frac{1}{K^2}\sum_{k=1}^{K} Var[X_{\Omega_f}(\boldsymbol{X_k})] = \frac{1}{K^2}K \cdot P_f\left(1 - P_f\right) = \frac{1}{K}P_f\left(1 - P_f\right) \end{aligned} \tag{28}$$

The coefficient of variation of the estimator is of the form:

$$v_{\widetilde{P}_f} = \frac{\sigma_{\widetilde{P}_f}}{\widetilde{P^{0}_f}} = \sqrt{\frac{1 - P_f}{K \cdot P_f}} \tag{29}$$

The formula above indicates that in order to obtain a coefficient of variation of the estimator of 0.1 along with the expected probability of failure, which typically ranges from10^{-7} to 10^{-4}, for real structures, it requires conducting K = 10^{9} - 10^{6} simulations.

2.3 The Robust Optimization

Robust optimization is a non-deterministic optimization method that takes into account the random nature of parameters, which leads to scattering of the response. It typically enhances the reliability of the structure. In robust optimization, the objective function commonly includes the variance of the selected structural response quantity. Constraints can be deterministic or expressed through statistical moments. The optimal structure achieved through robust optimization is more resilient to fluctuations in parameter values. Unlike other types of optimization (e.g., reliability optimization), the precise determination of probability distribution types is not crucial. The values of the first statistical moments of the structural response primarily depend on the first moments of random variables. In the absence of adequate data, a uniform or normal distribution of variables is often assumed.

The goal of robust optimization is to minimize both the mean value and the variation (standard deviation) of the target function. Consequently, the task of robust optimization can be formulated as follows:

Find values for variables: $\boldsymbol{X_d}, \boldsymbol{\mu}_x$ (30)

Minimizing: $\{E\,[f(\boldsymbol{X_d}, \boldsymbol{X}, \boldsymbol{P})], \sigma[f(\boldsymbol{X_d}, \boldsymbol{X}, \boldsymbol{P})]\}$ (31)

Subjected to:

$$E\,[g_i(\boldsymbol{X_d}, \boldsymbol{X}, \boldsymbol{P})] - \widetilde{\beta_i}\,\sigma[g_i(\boldsymbol{X_d}, \boldsymbol{X}, \boldsymbol{P})] \geq 0, \quad i=1,\dots, k_g, \tag{32}$$

$$\sigma\,[c_k\,(\boldsymbol{X_d}, \boldsymbol{X}, \boldsymbol{P})] \leq \sigma_k^u, \qquad k=1,\ldots,k_c, \tag{33}$$

$$X_{d\,j}^{\;l} \leq X_{d\,j} \leq X_{d\,j}^{\;u} \qquad j = 1,\ \ldots,\ n_d, \tag{34}$$

$$\mu_{x_r}^{l} \leq \mu_{x_r} \leq \mu_{x_r}^{u}, \qquad r = 1,\ \ldots,\ n_x, \tag{35}$$

where: f – objective function, $\boldsymbol{X_d}$ – vector of deterministic design variables, $\boldsymbol{X}, \boldsymbol{P}$ – vectors of random variables with expected values $\boldsymbol{\mu_x}$ and $\boldsymbol{\mu_p}$ respectively, g_i – constraint functions, c_k – functions whose standard deviations cannot exceed the permissible values σ_k^u, $\widetilde{\beta_i} > 0$ – the coefficients corresponding to the constraints, $g_i \geq 0$ represents the safety margins.

The vectors of random variables have been distinguished due to their different nature. Random variables $\boldsymbol{X}$ can be defined as random design variables because their expected values $\boldsymbol{\mu_x}$ change during the optimization process (where $\boldsymbol{\mu_x}$ represents the design variable). This leads to a shift in the probability density function $f_X(\boldsymbol{x})$. On the other hand, the probability distribution of the vector $\boldsymbol{P}$ remains unchanged during optimization, making these variables pure random parameters.

The concept of robustness is illustrated in Figure 6. The objective is to ensure that at the optimal point, the mean values of the constraint function g_i , $i = 1, \ldots, k_g$, are greater than or equal to their corresponding standard deviations. The natural consequence of robust optimization is an increase in the reliability of the structure.

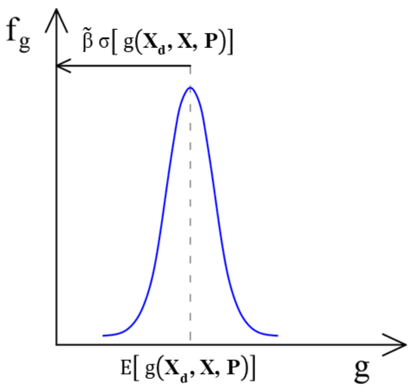

Figure 6

The concept of robust optimization limitations

In the above formulation, we are dealing with a multi-objective optimization problem: the mean value of $E\ [f(\boldsymbol{X_d}, \boldsymbol{X}, \boldsymbol{P})]$ and the standard deviation of $\sigma[f(\boldsymbol{X_d}, \boldsymbol{X}, \boldsymbol{P})]$.

A widely used approach for identifying points in the Pareto set is scalarization of the multi-objective optimization problem, where a linear combination of the objectives is used as the objective function. By adjusting the coefficients (weights) assigned to each component of the vector, one can obtain points in the Pareto set.

The values of these points are also influenced by the designer's preferences, which focus on minimizing both the average value and variance. Therefore, the task can be reformulated as the following scalar optimization problem:

Find values for variables: $\boldsymbol{X_d}, \boldsymbol{\mu}_x$ (36)

Minimizing: $\widetilde{F} = \frac{1-\gamma}{\mu^*} E[f(\boldsymbol{X_d}, \boldsymbol{X}, \boldsymbol{P})] + \frac{\gamma}{\sigma^*} \sigma[f(\boldsymbol{X_d}, \boldsymbol{X}, \boldsymbol{P})]$ (37)

subjected to:

$E[g_i(\boldsymbol{X_d}, \boldsymbol{X}, \boldsymbol{P})] - \widetilde{\beta}_i \sigma[g_i(\boldsymbol{X_d}, \boldsymbol{X}, \boldsymbol{P})] \geq 0, \quad i=1, \ldots, k_g,$ (38)

$\sigma[c_k(\boldsymbol{X_d}, \boldsymbol{X}, \boldsymbol{P})] \leq \sigma_k^u, \quad k=1, \ldots, k_c,$ (39)

$X_{d\,j}^{\,l} \leq X_{d\,j} \leq X_{d\,j}^{\,u} \quad j = 1, \ldots, n_d,$ (40)

$\mu_{x_r}^l \leq \mu_{x_r} \leq \mu_{x_r}^u, \quad r = 1, \ldots, n_x,$ (41)

The weighting factor $\gamma \in [0, 1]$ in the formula determines the meaning of each criterion, while μ^* and σ^* are normalizing constants. Assuming $\gamma = 0$, the optimization problem is transformed into a simple task of minimizing the mean value, while for $\gamma = 1$ to the task of minimizing the variance of the target function in the other words increasing structural robustness.

In the algorithm of robust optimization, we can distinguish 7 stages:

1) Definition of the permissible and choice of weighting factor γ.

2) Generating N implementation of the vector of design variables, which are evenly distributed in the current permissible area, according to the plan of optimal Latin Hypercubes.

3) Determination of statistical moments of the objective function and the constraint function for each of the N realizations of the vector $\{\boldsymbol{X_d}, \boldsymbol{\mu}_x\}$.

4) Structure of the response surface, e.g. by kriging, directly for individual statistical moments: $\hat{\mu}_f, \hat{\sigma}_f, \hat{\mu}_{g_i}, \hat{\sigma}_{g_i}, \hat{\sigma}_{c_k}$.

5) Solving the task of deterministic optimization

Find the values of the variables: $\boldsymbol{X_d}, \boldsymbol{\mu}_x$ (42)

Minimizing : $\tilde{f}^{DRS} = \frac{1-\gamma}{\mu^*} \hat{\mu}_f(\boldsymbol{X_d}, \boldsymbol{\mu}_x) + \frac{\gamma}{\sigma^*} \hat{\sigma}_f(\boldsymbol{X_d}, \boldsymbol{\mu}_x)$ (43)

Subjected to:

$\hat{\mu}_{g_i}(\boldsymbol{X_d}, \boldsymbol{\mu}_x) - \widetilde{\beta}_i \hat{\sigma}_{g_i}(\boldsymbol{X_d}, \boldsymbol{\mu}_x) \geq 0, \quad i=1, \ldots, k_g,$ (44)

$\hat{\sigma}_{c_k}(\boldsymbol{X_d}, \boldsymbol{\mu}_x) \leq \sigma_k^u, \quad k=1, \ldots, k_c,$ (45)

$\widehat{X}_{d\,j}^{\,l} \leq \widehat{X}_{d\,j} \leq \widehat{X}_{d\,j}^{\,u}, \quad j=1, \ldots, n_d,$ (46)

$$\hat{\mu}_{x_r}^{\ l} \leq \mu_{x_r} \leq \hat{\mu}_{x_r}^{u}, \qquad r=1,\ \ldots,\ n_x, \tag{47}$$

where: $\widehat{X_d}_{\,j}^{\,l}$, $\widehat{X_d}_{\,j}^{\,u}$, $\hat{\mu}_{x_r}^{\,l}$, $\hat{\mu}_{x_r}^{\,u}$ - the current boundaries of the permissible area, μ* and σ* – the normalization factors determined on the basis of the extreme values of the relevant moments obtained in point 3.

6) Check the terminate convergence condition.

7) Moving the permissible area over the optimal point determined in step 5. Reduction of the permissible area and return to 2.

2.3.1 Determination of the Response Surface by Polynomial Approximation

A crucial component of the algorithm employed in robust optimization is an efficient method for estimating the mean values and deviations of objective functions and constraints. To achieve this, various techniques have been utilized to approximate implicit functions of design variables using metamodels, also known as response surfaces. These surfaces are constructed by fitting suitable approximating functions to a set of experimental data points (Li et al. [26], Tang, Xu [27], Vining, Myers [28], Yeniay et al. [29]).

To approximate the response function, a commonly used method is to employ a low-degree polynomial. The purpose of the polynomial is not to precisely describe the entire response surface of the structure, but rather to provide the closest possible approximation near the limit state.

If the response of the structure is confined to a narrow region, it is possible to approximate the response surface using a linear function of independent random variables, which corresponds to a polynomial of the first-degree. Such a model is referred to as a first-order model and can be represented as:

$$\hat{y}(\boldsymbol{x}) = B_0 + \sum_{i=1}^{n} B_i x_i + \varepsilon \tag{48}$$

where: B_i, $i = 0,\ 1,\ \ldots,\ n$ – dimension of the boundary state surface, – design variables, $x_i \varepsilon$ – error taking into account the scatter of y_i values.

In cases where there is interaction between the realizations of random variables x_i, the first-order model can be extended to include second-order interaction terms, which take into account the curvature of the surface (taking into account the curvature of the surface) (Box, Wilson [30]):

$$\hat{y}(\boldsymbol{x}) = B_0 + \sum_{i=1}^{n} B_i x_i + \sum_{i=1}^{n} B_{ii} x_i^2 + \sum_{i<j}^{n} \sum_{j=2}^{n} B_{ij} x_i x_j + \varepsilon \tag{49}$$

In the strategy described in the paper, response surfaces are constructed not for the target function *f* or the constraint functions g_i or c_k, directly, but for their statistical moments. The approximate surfaces of the mean value and the standard deviation of these respective functions are taken into consideration. Typically, these surfaces are represented in polynomial form:

$$\hat{y}_{\mu}(z) = b_0^{\mu} + \sum_{i=1}^{n_{dX}} b_i^{\mu} z_i + \sum_{i=1}^{n_{dX}} b_{ii}^{\mu} z_i^2 + \sum_{i=1}^{n_{dX}} \sum_{j=2,\, j>i}^{n_{dX}} b_{ij}^{\mu} z_i z_j \tag{50}$$

$$\hat{y}_{\sigma}(z) = b_0^{\sigma} + \sum_{i=1}^{n_{dX}} b_i^{\sigma} z_i + \sum_{i=1}^{n_{dX}} b_{ii}^{\sigma} z_i^2 + \sum_{i=1}^{n_{dX}} \sum_{j=2,\, j>i}^{n_{dX}} b_{ij}^{\sigma} z_i z_j \tag{51}$$

where: $\hat{y}_{\mu}$ – approximation of the mean value of the function (f, g_i or c_k); $\hat{y}_{\sigma}$ – approximation of the standard deviation of the function ($\hat{y}_{\sigma}$ f, g_i or c_k); z_i, $i=1, ..., n_{dX}$ – design variables ($\boldsymbol{X_d}$ and μ_x); $n_{dX} = n_d + n_x$ – number of design variables.

3 Numerical Example

In order to illustrate the advantages of robust optimization, a steel single-storey frame with dimensions L=600cm and B=550cm (Figure 7) was analyzed. Both the bolt and the columns of the frame were modeled with steel square tubes with dimensions D = 39cm and d = 33cm, Young's module E = 210GPa, Poisson coefficient v = 0.3, yield strength f_y=235MPa. The structure was loaded with two forces with values, S = 1200kN and P = 120kN.

Static Analysis

According to the method of displacement, only horizontal displacement of the bolt is possible. Therefore, adopting the basic scheme of the displacement method, an additional bond blocking the movement was introduced. The reaction R in the bond was determined from the equilibrium equation of $\sum X$ floor 1-2 of the frame (Figure 8).

$$\sum X = 0$$

$$P + R - W_1 - W_2 = 0 \tag{52}$$

$$R = W_1 + W_2 - P = 0$$

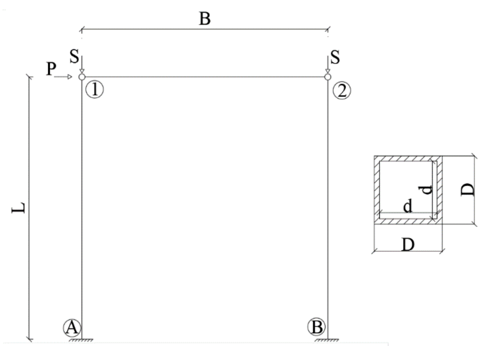

Figure 7

Geometry and cross-section of the steel single-storey frame

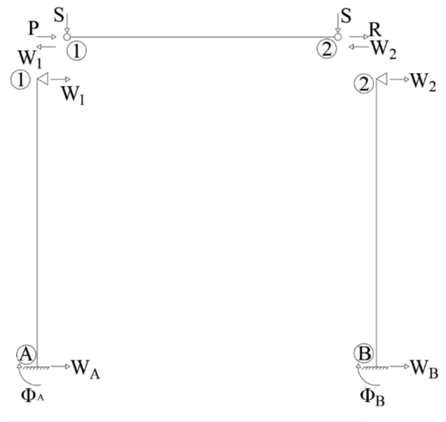

Figure 8
Basic scheme of the frame according to the method of displacement

The values of the parameter σ on both columns are the same ($\sigma=\sqrt{\frac{SL^2}{EI}}$). In the case of a unit displacement Δ, the angles of rotation of the columns 1-A and 2-B are equal to $\Psi_{1A}=\Psi_{1B}=\frac{1}{L}$. Using the transformation formulas, the equilibrium equation can be written:

$$2\delta_1(\sigma)\cdot\Delta = \frac{P\cdot L^3}{EI} \tag{53}$$

The horizontal displacement of the frame bolt is:

$$\Delta = \frac{P\cdot L^3}{2EI\delta_1(\sigma)} \tag{54}$$

$$M=\frac{EI}{L^2}\left(-\alpha_1(\sigma)\right)\frac{PL^3}{2EI\cdot\delta_1(\sigma)} = -\frac{PL}{2\sigma}\ tg\ \sigma \tag{55}$$

Reliability Analysis

The reliability analysis of the structure was carried out using the Monte Carlo method (random sample size: 10^6). The geometrical characteristics of the cross-sections of the members were adopted as random variables (D – the external dimension of the cross-section, d – the internal dimension of the cross-section). The random variables are described in Table 1. Random variables are not correlated. The initial mass of the modeled structure is M = 5935.36 kg. The value of the coefficient of variation was set at 2% for the external dimension D and 1% for the internal dimension d of the cross-section.

The example assumes two boundary functions describing the serviceability limit state SGU and the ultimate limit state SGN, respectively.

Table 1
Description of random variables

Random variables x_i	Average values [cm]	Standard deviation [cm]	Coefficient of variation [%]
D	39	0.78	2
d	33	0.33	1

$$f_{SGU} = \Delta_{max} - \Delta \quad (56)$$

where: Δ – horizontal displacement of the frame bolt, Δ_{max} – maximum horizontal displacement equal to L/150=4 cm.

$$f_{SGN} = 0.87 \cdot W_y \cdot f_y - M \quad (57)$$

where: W_y – the section modulus.

The reliability index for SGU and SGN were $\beta^{SGU} = 2.29$ and $\beta^{SGN} = 3.82$, respectively.

Deterministic Optimization

In the next step, we look for optimal cross-section dimensions, using the classic deterministic optimization algorithm.

The objective function is the mass of the structure:

$$f_C = minimum\ (\rho \cdot A \sum_{i=1}^{3} L_i\ ) = min\ (Mass) \quad (58)$$

where: L_i – length of the i^{th} member, A – cross-sectional area, ρ – volumetric density of steel.

Simple bounds are described in Table 2. They are the upper and lower limits of the searched design variables.

Table 2
Simple bounds of design variables

Design variable	Lower limit [cm]	Upper limit [cm]
D	36.66	41.34
d	32.34	33.66

Simple bounds were imposed on the basis of literature [31, 32, N5, N6]. For this case 2% tolerance of the cross-sectional dimensions of the pipe was assumed. Inequality limitations are formulated as conditions for not exceeding the permissible displacement of the horizontal bolt of the frame and not exceeding 87% of the load capacity for bending:

$$f_{SGU} = \Delta_{max} - \Delta = 4 - \Delta \tag{59}$$

$$f_{SGN} = 0.87 \cdot W_y \cdot f_y - M \tag{60}$$

The resulting cross-sectional dimensions are summarized in Table 3. The value of the objective function was 3862.916 kg.

Table 3
Values of design variables obtained in deterministic optimization

Design variable	Optimal value [cm]
D	37.61
d	33.66

The probability of failure and the reliability index were also verified. For SGU and SGN functions, respectively: $p^{SGU} = 0.495$, $\beta^{SGU} = 0.011$, $p^{SGN} = 0.065$, $\beta^{SGN} = 1.51$.

Robust Optimization

The objective function is mass of the structure, but assuming that it takes into account the weighting factor γ determining the meaning of each of the criteria. Design variables are the expected values of the external and internal dimensions of the cross-section: μ_D, μ_d. The value of the coefficient of variation was set at 2% and 1%.

The task of robust optimization takes the form of:

Find the values of the variables: μ_D, μ_d (61)

Minimizing: $f_C = \frac{1-\gamma}{\eta^*} E\,[Mass] + \frac{\gamma}{\sigma^*} \sigma\,[Mass]$ (62)

Subjected to:

$$E[4 - \Delta] - \widetilde{\beta^{SGU}} \cdot \sigma[4 - \Delta] \geq 0 \tag{63}$$

$$E\left[0.87 \cdot W_y \cdot f_y - M\right] - \widetilde{\beta^{SGN}} \cdot \sigma\left[0.87 \cdot W_y \cdot f_y - M\right] \geq 0 \tag{64}$$

$$36.66 \leq \mu_D \leq 41.34 \tag{65}$$

$$32.34 \leq \mu_d \leq 33.66 \tag{66}$$

where: $\gamma \in [0, 1]$ – weighting factor determines the importance of each of the criteria, η^*, σ^* - normalizing constants,

Robust optimization was performed using the second-order response surface. Experiments are generated according to the plan of optimal Latin cubes. The parameters: $\gamma = 0.5$, $\widetilde{\beta^{SGU}} = 2.0$, $\widetilde{\beta^{SGN}} = 3.0$.

The values of the design variables obtained as a result of robust optimization are summarized in Table 4. The weight of the structure in this case was 5150.32 kg.

Table 4

Mean values of random variables obtained in robust optimization

Random variable	Optimal value [cm]
D	38.03
d	32.34

The probability of failure and reliability index were verified for SGU and SGN functions, respectively: $p^{SGU} = 0.04452$, $\beta^{SGU} = 1.700$, $p^{SGN} = 0.00035$, $\beta^{SGN} = 3.390$.

Impact of the γ Weighting Factor on Optimization Results

The weighting factor $\gamma \in [0, 1]$ determines the importance of each of the criteria of the objective function. If $\gamma = 0$, the optimization problem is transformed into a regular average value minimization task, while for $\gamma = 1$ into a variance value minimization task. The influence of the weighting factor on the values of the optimized design variables is presented on Figure 9.

The higher the weight of the average value, the more optimal the structure should be. And when weights of the standard deviation is increased, the robustness of the structure is increasing, but optimality is reduced.

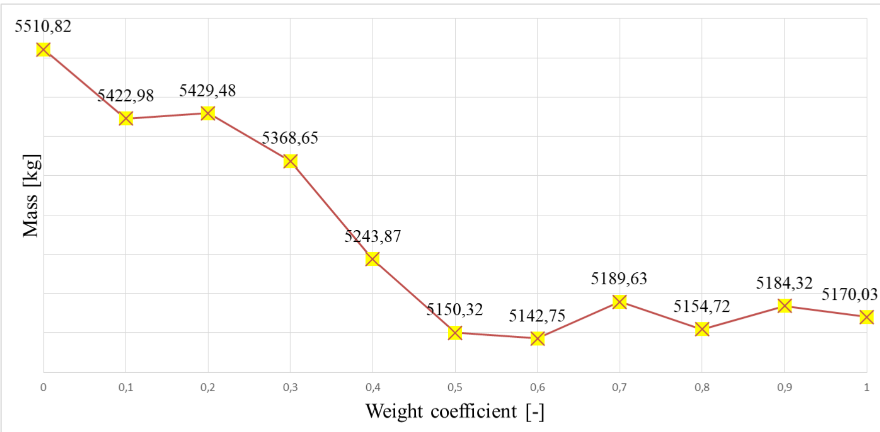

Figure 9

Mass of optimal structure depending on the weight of the average value

Conclusions

An indispensable element of rational structural design should include both deterministic optimization and robust optimization. Through robust optimization, we obtain designs that are slightly less optimal in terms of weight but significantly safer, as indicated by the reliability indices. Optimal structures are sensitive to imperfections in design parameters. Optimal solutions located near the limit state surface can easily become infeasible if parameter values deviate even slightly from the assumed nominal values. Incorporating the uncertainty of design parameters in the formulation of robust optimization effectively addresses this issue by providing the designer with control over the level of structural safety. By controlling the value of the weighting factor, we can consciously decide whether to prioritize minimizing the average value or the variance of the target function.

Furthermore, it is recommended to integrate robust optimization as a standard practice in structural design processes. This approach allows engineers to account for the random nature of design parameters and provides a more realistic and accurate representation of the structure's behavior under uncertain conditions. By considering both the mean value and variance of the target function, designers can achieve designs that are not only efficient but also resilient to parameter variations.

Future research efforts should focus on refining the methodologies and techniques used in robust optimization, such as improving the accuracy of response surface models and exploring advanced optimization algorithms. Additionally, investigations into the impact of different probability distribution assumptions and the development of techniques for handling correlated random variables would further enhance the robustness and reliability of structural designs.

In conclusion, incorporating robust optimization into the design process enhances the overall safety and performance of structures, ensuring their functionality and minimizing the risks associated with parameter uncertainties. By embracing robust optimization as a standard approach, engineers can achieve designs that strike a balance between efficiency and resilience, meeting the demands of modern structural engineering.

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N 3. IN 1991-1-4. Eurocode 1: Actions on structures – Part 1–4: General actions – Wind actions.

N 4. EN 1990: 2002. Eurocode – Basis for structural design.

N 5. PN-EN 10210-2. Hot finished steel structural hollow sections – Part 2: Tolerances, dimensions and sectional properties.

N 6. PN-EN 1090-2. Execution of steel structures and aluminium structures. Part 2: Technical requirements for steel structures.