Novel Metric to Quantify the Consequences of Modeling Imprecisions in Adaptive Dynamic Control

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Abstract: Control solutions are often based on a dynamic model of important, physically interpreted, but only imprecisely known parameters, e.g., in control-based treatment of patients suffering from diabetes, certain cancerous illnesses, and in anesthesia control. Besides making efforts to identify the model parameters that normally have serious inter-patient deviations, an alternative approach can be the Fixed Point Iteration-based Adaptive Control that evades the complicated task of parameter identification, though it also applies an approximate model. However, for qualifying the control process, besides the tracking precision, quantitative evaluation of the consequences of the model's imprecision also has practical significance. In this paper, a particular metric is introduced for this purpose that is based on the specialties of this adaptive approach by considering the differences between the actually needed and the purely model-based control efforts. It measures the consequences of the errors without revealing the errors themselves. It allows a wide field of applications in which not only are the parameters of the model uncertain, but even the analytical structure of the model can be questionable, too. The use of this metric is illustrated via simulations. An alternative possible use of this metric in Multidimensional Scaling is also illustrated.

Keywords: fixed point iteration-based adaptive control; error metrics; abstract rotations-based adaptation; rotational metric

1 Introduction

Model-based control applications are widely used in the life sciences. For instance, to treat Type 1 diabetes mellitus, various models were developed, e.g., [1]. Model predictive control is a natural approach in anesthesia, e.g., [2]. Model identification plays an important role in cancer research, too [3]. Chemical reactions also represent an interesting field for modeling approaches, e.g., [4]. For relatively simple systems such as particular mechanical constructions or electric motors, direct measurement procedures can be invented, as in [5]. The use of particular model structure as Linear Parameter Varying form in cruise control of cars (e.g., [6]) or even the "Model-Free Approaches", actually also use some simple universal model form, as in [7] can be mentioned, too.

With respect to parameter estimation efforts, it is expedient to note that in practical control applications, coping with the aftermath and consequences of the estimation errors is much more significant than obtaining precise information on the model parameters. These consequences manifest themselves in the variation of several variables of important physical interpretation, and depend on the complex circumstances of the whole control task. In certain segments of the controlled motion, the estimation errors can have quite insignificant effects, while in other segments they may become quite serious. For this reason, the introduction of a single, time-dependent scalar variable that measures this significance during the execution of the control task can be practically useful. Also, if we are not in the possession of a satisfactory, analytically precise model form, quantifying the consequences of its modeling inadequacy can be reasonable, though in this case it cannot be stated that it originates from the estimation error of certain parameter(s).

The latter example in [8] belongs to the special class of adaptive controllers, the Fixed Point Iteration (FPI)-based controllers that transform the control task into iteratively finding the fixed point of a contractive map in a Banach space based on Banach's fixed point theorem [9]. In the next section, the operation of these controllers is briefed. Following that, the particular metric that was inspired by this control structure is defined and formally analyzed. For illustrating the applicability of this metric, the effects of particular parameter imprecision are demonstrated via simulations. Simulations will be given for an example when the analytical model form used by the controller does not cover the "reality", i.e., when the controlled system is precisely described by a different model form. Finally, the application of this metric in Multidimensional Scaling will be presented.

2 Operational Principle of the FPI-based Adaptive Controllers

The whole computational structure developed for the FPI-based adaptive control of a fully driven second-order system, announced in 2009 in [10], is described in Figure 1. This figure also contains the complementary elements that are used for the calculation of the novel metric.

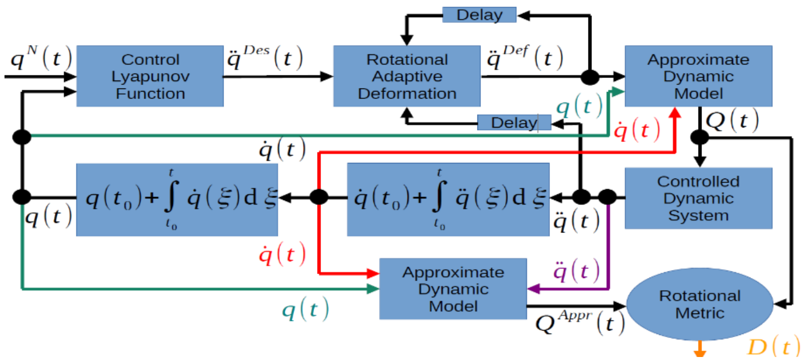


Figure 1
The Control Lyapunov Function-based Adaptive Controller with Rotational Metric for a fully driven 2nd order dynamic system

Its main function is prescribing how the trajectory tracking error should converge to zero and realizing this error decay. This description happens by the use of purely kinematic terms, using the nominal trajectory $q^N(t)$ to be tracked, and the actually realized trajectory $q(t)$ as inputs into the block "Control Lyapunov Function." Its output is the desired 2nd time-derivative of the coordinates of the system $\ddot{q}$. It must be stressed that within this block numerous different kinematic descriptions can be present based on the needs of, e.g., the "Computed Torque Control," "Backstepping Control," "Variable Structure / Sliding Mode Control," etc. By defining the main components of the so called PID feedback terms that successfully were used for automatic steering of ships in the forties of the past century [11] the definitions in (1) can be used: the error, the integrated error, and the time-derivative of the error

$$e(t) := q^N(t) - q(t)\ , \quad e_{\text{int}}(t) = \int_{t_0}^{t} e(\xi)d\xi\ , \quad \dot{e}(t) = \frac{de(t)}{dt} \tag{1}$$

In the mainstream of adaptive control design based on Lyapunov's 2nd Method [12] a quadratic metric composed of the array $x = [e_{\text{int}}^T(t), e^T(t), \dot{e}^T(t)]^T$

$M = M^T$ is introduced as a Lyapunov function $V(t) := x^T Mx$. The design purpose is to guarantee that $\dot{V} < 0$, or more precisely, $\dot{V}$ must be negative enough for concluding $x(t) \longrightarrow 0$ as $t \longrightarrow \infty$. In [10] the main criticism of using Lyapunov's technique was summarized as follows:

i) From $\dot{V} < 0$ it cannot be concluded that the time-derivative of the absolute value of any component of x would be negative. However, e.g., in life sciences, such a property would be desirable.

ii) The method concentrates on the asymptotic stability of the controller. However, again, e.g., in life sciences, the behavior of the details of the initial transients would be of great interest.

To evade these problems in the FPI-based Adaptive Control definite behavior was prescribed for the components of the tracking error $e_i(t)$. In the case of a 2nd order nonlinear physical system the existence of an exact dynamic model $\ddot{q}(t) = F(q(t), \dot{q}(t), Q(t))$ was assumed, where Q refers to the exerted forces (including the control and disturbance forces, too). Even in the case of dynamic modeling of robots even in the nineties it became clear that it is impossible to develop a very precise model [13]. Therefore, it was assumed that there is an approximate inverse dynamic model available for the controller for computing the control forces as $Q(t) = \tilde{F}^{-1}(q(t), \dot{q}(t), \ddot{q}^{Des}(t))$. Evidently, the use of this approximate model in the Computed Torque Control [13] will not precisely realize the desired $\ddot{q}^{Des}(t)$ value due to the modeling imprecision:

$$\ddot{q}(t) = F\left(q(t), \dot{q}(t), \tilde{F}^{\ 1}\left(q(t), \dot{q}(t), \ddot{q}^{Def}(t)\right)\right) \neq \ddot{q}^{Des}(t) \quad . \tag{2}$$

The main idea was that for realizing the desirable $\ddot{q}(t) = \ddot{q}^{Des}(t)$ situation, instead of $\ddot{q}^{Des}(t)$ its deformed version $\ddot{q}^{Def}(t)$ must be used as the input of the approximate model as in (3)

$$\ddot{q}(t) = F\left(q(t), \dot{q}(t), \tilde{F}^{-1}\left(q(t), \dot{q}(t), \ddot{q}^{Def}(t)\right)\right) \approx \Im\left(\ddot{q}^{Def}(t)\right) \quad . \tag{3}$$

where it was assumed that the controller can abruptly or in a very fast manner modify the control force $Q(t)$ $q(t)$ and $\dot{q}(t)$ can vary only slowly. In the case of a digital controller in Fig. 1 the delay time exactly corresponds to the δt cycle time of the controller, and during one digital control step only one step of the adaptive iteration can be done, in which the slow drift of $q(t)$ and $\dot{q}(t)$ is neglected. In the block called "Rotational Adaptive Deformation" in a more general framework various deformation functions

$\ddot{q}^{Def}(t_{i+1}) = G(\ddot{q}^{Def}(t_i), \ddot{q}(t_i), \ddot{q}^{Des}(t_{i+1}))$ can be present. In this paper we use the Abstract Rotations-based solution published in [14].

3 Details of the Present Design of the Controller

In the present simulations in the Kinematic Block the Control Lyapunov Function was present. In this approach a simple diagonal metric can be used for the Lyapunov function containing the error feedback gains as in (4), with the desired convergence to zero as $\dot{V}(t) = -KV(t)$ where $K > 0$ is constant:

$$V(t) = \frac{1}{2} e_{\text{int}}^T e_{\text{int}} K_I + \frac{1}{2} e^T e K_P + \frac{1}{2} \dot{e}^T \dot{e} K_D \tag{4}$$

The derivative of $V(t)$ easily can be computed as in (5)

$$\dot{V}(t) = e^T e_{\text{int}} K_I + \dot{e}^T e K_P + \ddot{e}^T \dot{e} K_D \tag{5}$$

in which the identity $1 \equiv \frac{\dot{e}^T \dot{e}}{\|\dot{e}\|^2} \cong \dot{e}^T \frac{\dot{e}}{\varepsilon + \|\dot{e}\|^2}$ can be used with the use of a small $\varepsilon > 0$ by the use of which the numerically inconvenient "division by zero" situation can be avoided. The common multiplicative array $\dot{e}^T$ can be placed to the left-hand side and an equation

$$\dot{e}^T \left[(K_I e^T e_{\text{int}} + KV) \frac{\dot{e}}{\|\dot{e}\|^2 + \varepsilon} + K_P e + K_D \ddot{e} \right] = 0 \tag{6}$$

is obtained that can be valid for an arbitrary array $\dot{e}^T(t)$ if the term within the square brackets is zero. In this manner $\ddot{q}^{Des}(t)$ can be obtained via purely kinematic considerations as

$$\ddot{q}^{Des}(t) = \ddot{q}^N(t) + \frac{1}{K_D} \left[(K_I e_{\text{int}}^T e + KV) \frac{\dot{e}}{\|\dot{e}^T\|^2 + \varepsilon} + K_P e \right] \tag{7}$$

In the Adaptive Deformation block we applied the abstract rotations that were introduced in [14]. Figure 2 illustrates the simple idea for two dimensional vectors.

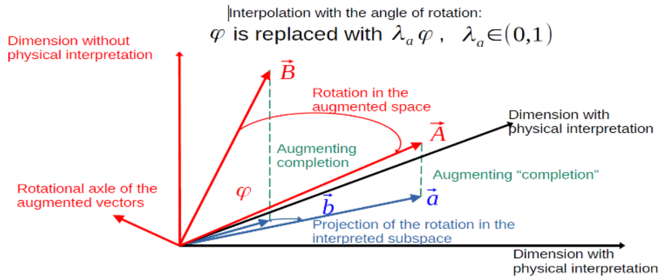


Figure 2

The idea representing the abstract rotations-based adaptive deformations for two dimensional vectors

Assume that we have two nonzero vectors $\vec{a},\vec{b} \in \Re^n$ and we wish to deform vector $\vec{b}$ toward $\vec{a}$. Generally, this means the modification of the direction and the Frobenius norm of vector $\vec{b}$. A simultaneous modification can be achieved by so adding a new, physically not interpreted dimension to these vectors as $\Re^{n+1} \ni \vec{A} = [\vec{a}, D_a]$ and $\Re^{n+1} \ni \vec{B} = [\vec{b}, D_b]$ with $D_a, D_b > 0$ that $\|\vec{A}\| = \|\vec{B}\| = R$, i.e., the augmented vectors obtain a common Frobenius norm R.

The part of vector $\vec{B}$ that is orthogonal to vector $\vec{A}$ can be computed as $B_{\perp A} := \vec{B} - \frac{A^T B}{A^T A}\vec{A}$. In this manner two orthogonal unit vectors $\vec{e} := \frac{\vec{A}}{\|\vec{A}\|}$, and $\vec{f} = \frac{B_{\perp A}}{\|B_{\perp A}\|}$ can be obtained that generate rotations in $\Re^{n+1}$ with the skew symmetric matrix $G = ef^T - fe^T$ having the special property $G^3 = -G$.

In this manner the rotations with angle φ in $\Re^{n+1}$ leaving the orthogonal subspace of vectors $\vec{e}, \vec{f}$ invariant have the closed analytical form given in (8)

$$O = e^{\varphi G} = \sum_{l=0}^{\infty} \varphi^l \frac{G^l}{l!} = I + \sin(\varphi) G + [1 - \cos(\varphi)] G^2 \qquad (8)$$

The angle of rotation that rotates $\vec{B}$ $\vec{A}$ $\sin(\varphi) = \frac{\|\vec{B}_{\perp A}\|}{R}$ φ then the physically interpreted part $\vec{b}$ will be exactly deformed into $\vec{a}$. If only a fragment of φ is used, then $\vec{b}$ will be deformed toward $\vec{a}$. In the iterative adaptive application a G matrix and an angle φ_n is calculated that deforms the realized value $\ddot{q}(t_n)$ into $\ddot{q}^{Des}(t_{n+1})$, and the $\ddot{q}^{Def}(t_n)$ value is transformed with G and a fragment of φ_n to obtain

$\ddot{q}^{Def}(t_{n+1})$. It is expected that in the case of convergence near identity rotations will be applied as the fixed point slowly drifts with $q(t)$ and $\dot{q}(t)$.

4 Introduction of the Rotational Metric

Consider a subset of $\Re^n$ for the vector $\vec{v}$ the Frobenius norm of which is smaller than $R > 0$: $S := \left\{\vec{v} \in \Re^n : \|\vec{v}\| < R\right\}$. Three of such vectors, say $\vec{a}, \vec{b}$, and $\vec{c}$ can be augmented into $\Re^{n+1}$ vectors as $\vec{A}, \vec{B}$, and $\vec{C}$ according to the above detailed procedure, i.e., $\|\vec{A}\| = \|\vec{B}\| = \|\vec{C}\| = R$. It is stated that the absolute value of the angles between these augmented vectors behave as a metric.

This statement can be proved in the following manner. In $\Re^{n+1}$ consider the points as follows: the origin O, and the endpoints of the above vectors as A, B and C. These three points determine a 3-dimensional subspace of $\Re^{n+1}$ that is equivalent to $\Re^3$. The plain determined by these points A, B, and C corresponds to a 2-dimensional plane, while the location of O determines the direction of the third dimension as it is illustrated in Fig. 3.

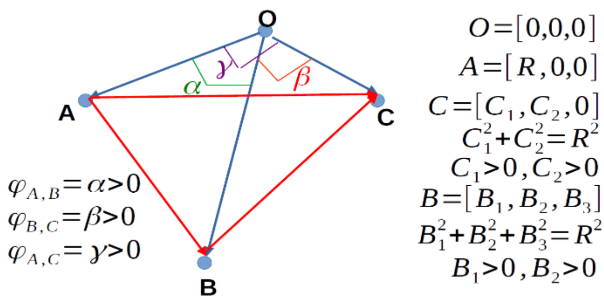


Figure 3
Illustration of the points A, B, C and O in $\Re^{n+1}$ determining a 3-dimensional orthogonal subspace in $\Re^{n+1}$. The identical signs of C_1 and C_2 determine the orientation of the two unit vectors in the 3D subspace.

Since $\|\vec{OA}\| = \|\vec{OB}\| = R$, and $\|\vec{OB}\| = \|\vec{OC}\| = R$, i.e., we have special triangles, it can be written as

$$\left\|\vec{A}B\right\| = 2R\sin\left(\frac{\alpha}{2}\right),$$

$$\left\|\vec{B}C\right\| = 2R\sin\left(\frac{\beta}{2}\right), \tag{9}$$

$$\left\|\vec{A}C\right\| = 2R\sin\left(\frac{\gamma}{2}\right) .$$

The consequence of (9) is that the following two equations are valid:

$$C_1^2 + C_2^2 = R^2 , \tag{10a}$$

$$\left(C_1 - R\right)^2 + \left(C_2 - 0\right)^2 = 4R^2 \sin^2\left(\frac{\gamma}{2}\right) \text{ i.e.,} \tag{10b}$$

$$C_1^2 + C_2^2 - 2C_1R + R^2 = 4R^2 \sin^2\left(\frac{\gamma}{2}\right) , \tag{10c}$$

From (10a) and (10c) it follows that

$$C_1 = -R\left(2\sin^2\left(\frac{\gamma}{2}\right) - 1\right) > 0 . \tag{11}$$

Now apply the formula

$$\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v) ,$$

$$\cos^2(u) + \sin^2(u) = 1 \tag{12}$$

That for $u = v = \frac{\gamma}{2}$ yields

$$2\sin^2\left(\frac{\gamma}{2}\right) - 1 = -\cos(\gamma) . \tag{13}$$

Substitute that into (11) yielding

$$C_1 = R\cos(\gamma) . \tag{14}$$

Then from (10a) it follows that a positive C_2 value can be obtained as

$$C_2 = R\sin(\gamma) . \tag{15}$$

In similar manner, it can be written that

$$\left(B_1 - R\right)^2 + B_2^2 + B_3^2 = 4R^2 \sin^2\left(\frac{\gamma}{2}\right), \tag{16}$$

$$B_1^2 + B_2^2 + B_3^2 = R^2 .$$

That with the analogy of C_1 yields that

$$B_1 = R\cos(\alpha) . \tag{17}$$

Now take it into account that because of (9)

$$4R^2 \sin^2\left(\frac{\beta}{2}\right) = \left(B_1 - C_1\right)^2 + \left(B_2 - C_2\right)^2 + \left(B_3 - C_3\right)^2 , \tag{18}$$

$$B_1^2 + B_2^2 + B_3^2 = R^2 \quad , C_1^2 + C_2^2 + C_3^2 = R^2 .$$

This results in

$$-2B_1C_1 - 2B_2C_2 - 3B_3C_3 + R^2 + B_1^2 + B_2^2 + B_3^2 = 4R\sin^2\left(\frac{\beta}{2}\right) \quad \text{i.e.,}$$

(19)

$$-B_1C_1 - B_2C_2 - B_3C_3 = R^2\left[2\sin^2\left(\frac{\beta}{2}\right) - 1\right]$$

By substituting here $C_1 = R\cos(\gamma)$, $C_2 = R\sin(\gamma)$, $C_3 = 0$, and $B_1 = R\cos(\alpha)$ it is obtained that

$$-R\cos(\alpha)\cos(\gamma) - B_2 R\sin(\gamma) = -R^2\cos(\beta), \tag{20}$$

i.e.,

$$B_2 = R\frac{\cos(\beta) - \cos(\alpha)\cos(\gamma)}{\sin(\gamma)} > 0 . \tag{21}$$

Substituting here that $\cos(\gamma)\cos(\alpha) = \cos(\alpha + \gamma) + \sin(\gamma)\sin(\alpha)$, and taking into account that $R > 0$ and for $\alpha, \gamma \in [0, \frac{\pi}{2}]$, $\sin(\alpha), \sin(\gamma) \geq 0$ this means that

$$\cos(\beta) > \cos(\alpha + \gamma) + \sin(\alpha)\sin(\gamma) \geq \cos(\alpha + \gamma) , \tag{22}$$

i.e.,

$$\cos(\beta) \geq \cos(\alpha + \beta) \ . \tag{23}$$

Since for $x \in [0, \pi]$ the function $\cos(x)$ is monotonic decreasing, this means that $\beta \leq \alpha + \gamma$ if $0 < \beta$, $\alpha, \gamma \leq \pi/2$, that is for *acute angles* **the triangle inequality of the metric spaces** has been proved for the rotational metric. For this purpose, great enough common $R >> \|\vec{a}\|, \|\vec{b}\|$ norm has to be chosen.

The other metric properties that $\rho(a,b) = \rho(b,a)$, $\rho(a,b) \geq 0$, $\rho(a,a) = 0$, and from $\rho(a,b) = 0$ it follows that $a = b$ are trivially valid. **Therefore, these angles can be used as metrics.**

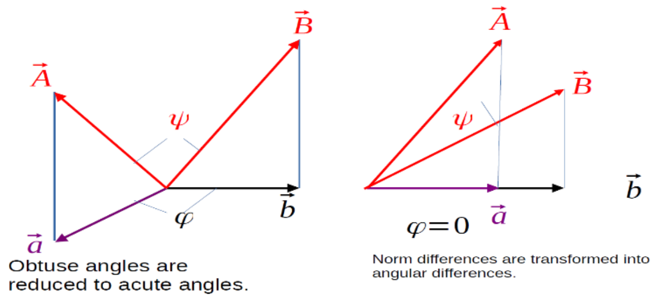


Figure 4

Certain trivial but interesting properties of the new "Rotational Metric"

This new metric has certain trivial but interesting properties listed as follows:

i) In contrast to the usual quadratic metrics in which the matrix elements of a positive definite symmetric matrix of the metric tensor have numerous free, arbitrary parameters, *this special nonlinear metric* has only a single real free parameter, i.e., R , the common norm of the augmented vectors.

ii) According to Fig. 4, obtuse angle between two vectors of originally identical norm is transformed into acute angle.

iii) Again, according to Fig. 4, the norm difference between two vectors of originally identical direction is transformed into angular differences.

In the sequel, application possibilities are illustrated.

5 Application Possibilities in Adaptive Dynamic Control

With the presented computational results, we wish to illustrate the fact that the consequences of modeling imprecision depend on the combination of the independent factors as follows:

a) the imprecision of the dynamic model itself;

b) the nominal trajectory to be tracked that determines the force need of the control in the case of available exact dynamic model and zero initial tracking error for $q(t)$ and $\dot{q}(t)$;

c) the initial tracking error for $q(t)$ and $\dot{q}(t)$;

d) the kinematic tracking strategy that prescribes how the components of the integrated tracking error, the tracking error and its time-derivative have to converge to zero; for instance, in the case of a PID-type CTC controller the feedback gains with these error terms will serve as a basis for computing the necessary control forces on the basis of the available imprecise system model; in similar manner, in the case of other kinematic prescriptions the appropriate control parameters play the same role;

e) if the controller has some adaptation ability the parameters of the adaptation mechanism will determine the finally exerted control forces.

In general, the "significance of modeling imprecisions" cannot be considered or defined independently of the above factors. This simple fact explains the experience when for observing the model parameters evolutionary methods as Genetic Algorithms or Particle Swarm Optimization are used real-time. Instead of firm convergence some fluctuation in the identified parameters can be observed since the model parameter values have different importance in the various segments of the motion, and this time-varying significance cannot be generally well-balanced.

However, in the case of the adaptive controllers the above context can be simplified if the adaptation mechanism is successful and consequently the integrated tracking error, the error and its time-derivatives already achieved are very small (practically zero) value: in this case the error terms fed back are very small, $q^N(t)$, $\dot{q}^N(t)$ and $\ddot{q}^N(t)$ are well approximated by $q(t)$, and $\dot{q}(t)$ respectively. In this case the above-mentioned items c) and d) can be neglected in the context. The significance of modeling imprecision remains a typically time-dependent data determined by the properties of the "nominal trajectory," the reality, and the imprecise model under consideration.

In the simulations two typical cases were considered:

a) when the precise analytical form of the model of the controlled system is available, only the parameter values are imprecisely known: in this case the question of the significance of the estimation error of a certain parameter or set of parameters can be quantified by this metric;

b) when even the precise analytical form of the available model is dubious, i.e., it cannot be expected that a given parameter of the model itself represents something in the reality: then the inadequacy of the whole model can be measured by the new metric. The simulations were made using Julia Language Version 1.10.4 (2024-06-04).

5.1 Simulations for the Motion of Mass Points Hanging on Nonlinear Springs-The precise Analytical Model Form is Available

The controlled system consisted of a *mass point hanging from the ceiling* with a distance *generalized coordinate* $q_1 > 0$, and a *second mass point hanging on the first one* with a distance from the ceiling as *generalized coordinate* $q_2 > q_1$. The mass points are connected to *nonlinear springs* having *nonzero force lengths*. The motion of each spring is damped by viscous friction. The equations of motion are given (24). Let $\langle m_1, m_2 \rangle$ be a *diagonal matrix* of size 2×2,

$s_1 \overset{def}{=} \text{sign}(q_1 - L_{01})$, $s_2 \overset{def}{=} \text{sign}(q_2 - q_1 - L_{02})$, and

$$h_1 = -m_1 g + k_1 s_1 |q_1 - L_{01}|^{\sigma_1} - k_2 s_2 |q_2 - L_{02}|^{\sigma_2} + b_1 \dot{q}_1 \quad , \tag{24a}$$

$$h_2 = -m_2 g + k_2 s_2 |q_2 - q_1 - L_{02}|^{\sigma_2} + b_2 \dot{q}_2 \quad , \tag{24b}$$

$$\ddot{q} = H^{-1}(Q - h) . \tag{24c}$$

in which L_{01} and L_{02} are zero force springs lengths, $k_1[\text{N} \cdot \text{m}^{-\sigma_1}]$ and $k_2[\text{N} \cdot \text{m}^{-\sigma_2}]$ are nonlinear spring stiffness values, σ_1 and σ_2 are non-dimensional stiffness exponents, and $m_1[\text{kg}]$ and $m_2[\text{kg}]$ are masses, $b_1, b_2[\text{N} \cdot \text{s} \cdot \text{m}^{-1}]$ are viscous damping coefficients, $g[\text{m} \cdot \text{s}^{-2}]$ is the gravitational acceleration, $q_1[m]$ and $q_2[m]$ denote the vertical positions of the appropriate mass-points, and $Q \in R^2$ denote the active control forces exerted on the appropriate mass points. The dynamic parameters of the *exact model* are given in Table 1. (If the exact parameters are known, the output of the box "*Controlled Dynamic System*" in Fig. 1 must be measured real-time.)

The *novel metric* is used for the force components $[Q_1(q,\dot{q},\ddot{q});Q_2(q,\dot{q},\ddot{q})]$ and $[Q_1^{Appr}(q,\dot{q},\ddot{q});Q_2^{Appr}(q,\dot{q},\ddot{q})]$ *telling us to what extent the force need of the approximate model must be deformed to achieve the really necessary force components in the given control task*. **This is a reasonable question when the capacities of the controllers are designed on the basis of an available dynamic model and control task is formulated by the use of purely kinematic terms**. The appropriate nominal trajectory to be tracked and the initial conditions are the same in each simulation example. The control parameters for the Control Lyapunov Function approach were $\Lambda = 6.0$ $[\mathrm{s}^{-1}]$, $K = 12.0$ $[\mathrm{s}^{-1}]$, $\varepsilon = 10^{-3}$ $[\mathrm{m}^2 \cdot \mathrm{s}^{-2}]$ in (6). In the adaptive deformation the c*ommon norm augmented vectors* was $R_\alpha = 10^6 \mathrm{m}^2 \cdot \mathrm{s}^{-2}$, and the *"interpolation parameter"* was $\lambda_a = 1.3$ in Fig. 2 that in this case realized some "nonlinear extrapolation". The c*ommon norm of the augmented vectors* in the *rotational metric* was $R \equiv \Re = 2.5 \times 10^3$ [N]. To check the adequacy of the adaptive control in the first step t*he parameters of the approximate model were identical with that of the exact one.*

Table 1
The exact Model parameters

Parameter and Measurement Unit	**Exact Value**
Mass $m_1[\mathrm{kg}]$	1.0
Mass $m_2[\mathrm{kg}]$	2.0
Spring stiffness $k_1[\mathrm{N} \cdot \mathrm{m}^{-\sigma}]$	100.0
Spring stiffness $k_2[\mathrm{N} \cdot \mathrm{m}^{-\sigma}]$	150.0
Zero force length $L_{01}[\mathrm{m}]$	2.0
Zero force length $L_{02}[\mathrm{m}]$	2.5
Non-linearity parameter of the spring σ	1.5
Viscous damping friction coefficient $b_1[\mathrm{N} \cdot \mathrm{s} \cdot \mathrm{m}^{-1}]$	0.1
Viscous damping friction coefficient $b_2[\mathrm{N} \cdot \mathrm{s} \cdot \mathrm{m}^{-1}]$	2.0
Viscous damping friction coefficient $g[\mathrm{m} \cdot \mathrm{s}^{-2}]$	9.81

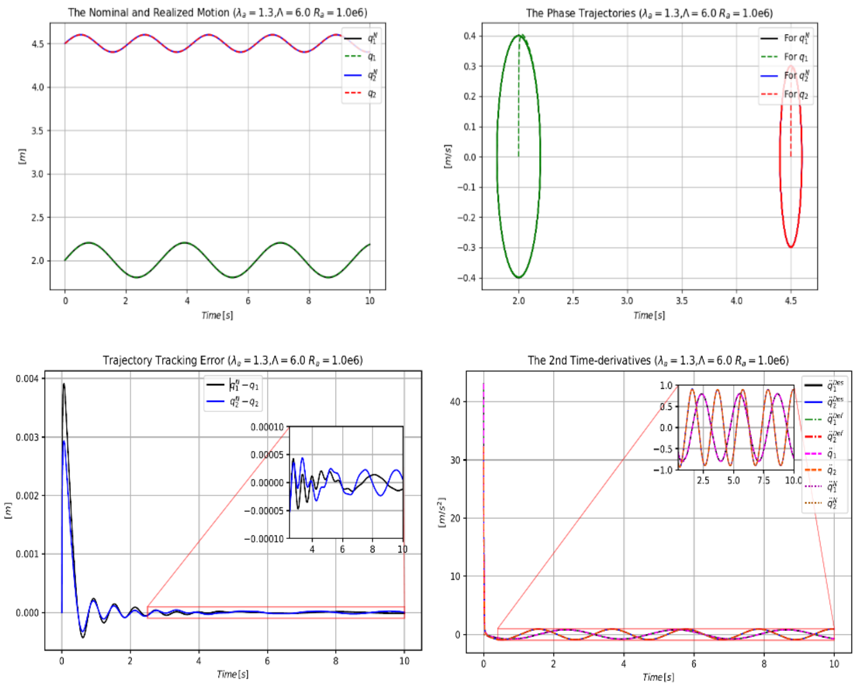
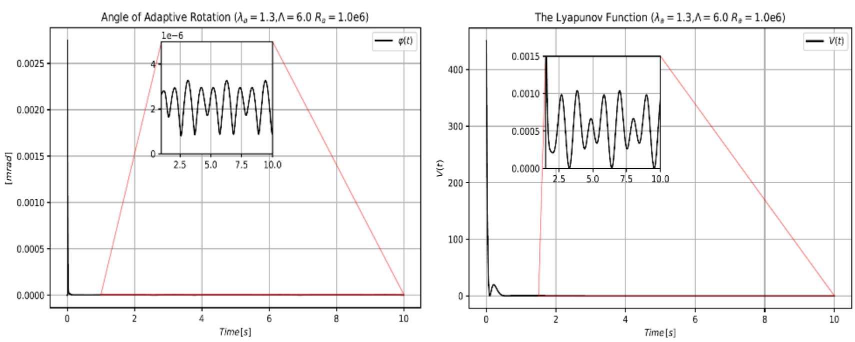


Figure 5

Operation of the adaptive controller for the *exact dynamic model* 1: trajectory tracking, phase trajectory tracking, trajectory tracking error, and the 2nd time- derivatives.

According to the expectations, Fig. 5 reveals precise trajectory and phase trajectory tracking without observable adaptive deformation in the space of the 2nd time-derivatives of the generalized coordinates. Fig. 6 shows adaptive deformation corrections with very little angles that belong to the slow drift of the fixed point of the iteration with the variables $q(t)$, and $\dot{q}(t)$.



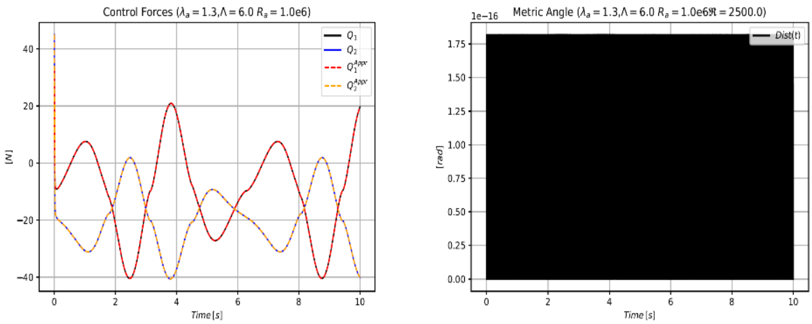


Figure 6

Operation of the adaptive controller for the *exact dynamic model* 2: angle of the corrective adaptive rotations, Lyapunov function, the control force components, and the novel metric for the control forces (the filled in box in the "Metric Angle" graph reveals fluctuation between 0 and a maximal value of order of magnitude 10^{-16} that practically corresponds to zero)

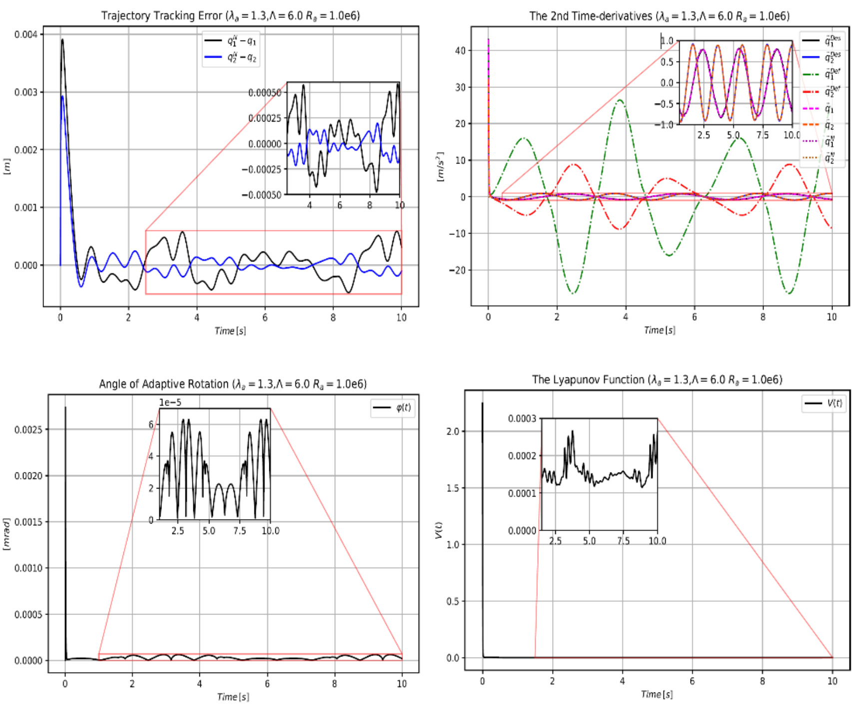


Figure 7

The consequences of using the approximate nonlinearity parameter value $\sigma^{Appr} = 3.0$ instead of the exact value $\sigma = 1.5$ 1: trajectory tracking error, deformation in the space of the 2nd time-derivatives of the generalized coordinates, angle of adaptive rotation, Lyapunov function.

Evidently no modification happened with the control forces, and the order of magnitude 10^{-6} [rad] metric of force deformation practically corresponds to 0 if the floating point representation of numbers is considered. Also, the control Lyapunov function was kept near zero. **Therefore, the simulations passed the test based on the use of the exact model.**

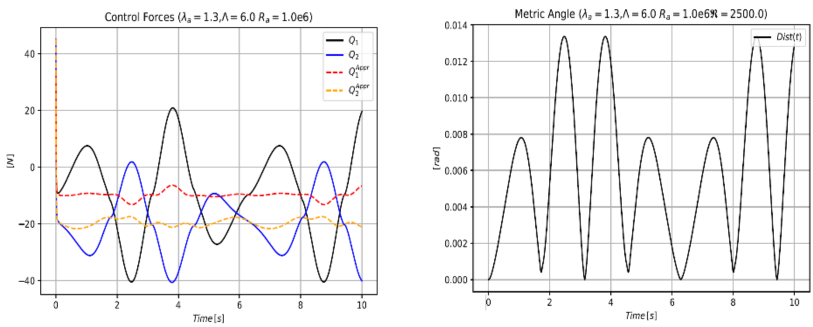
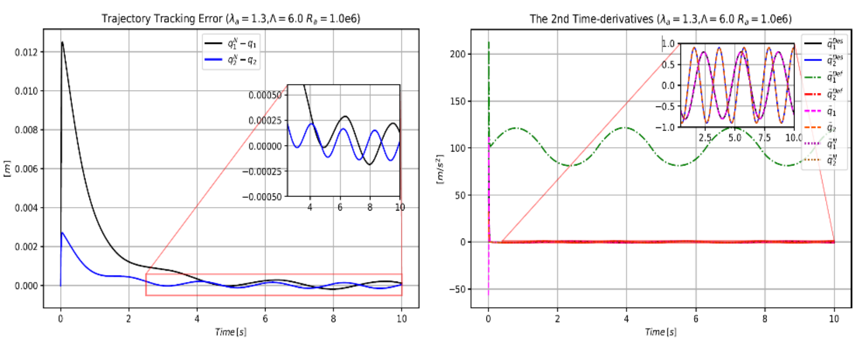


Figure 8

The consequences of using the approximate nonlinearity parameter value $\sigma^{Appr} = 3.0$ instead of the exact value $\sigma = 1.5$ 2: control forces, and the novel metric for the deformation of the control forces

Figures 7 and 8 reveal the consequences of the modeling error in the nonlinearity parameter of the spring $(\sigma = 1.5, \sigma^{Appr} = 3.0)$. The controller produced precise trajectory tracking and near zero Lyapunov function with appropriate, but still very small corrective adaptive rotations that caused drastic deformation in the $\ddot{q}$ values.

By the use of the novel metric the relative significance of the modeling error of this parameter can be tracked as the function of time within this control task. Figure 8 shows that at certain instances this error is quite insignificant while in order it becomes very important.



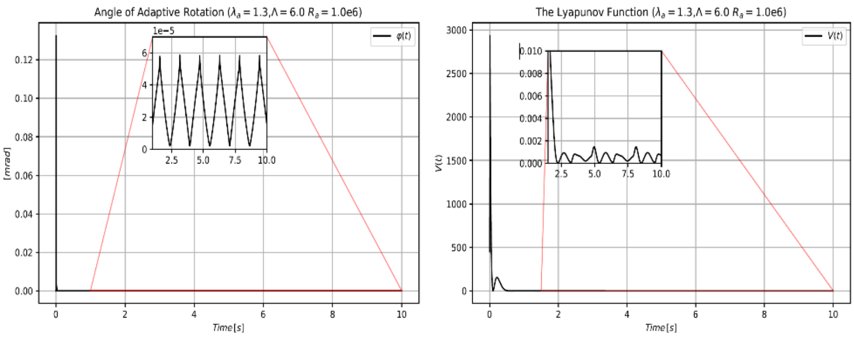


Figure 9

The consequences of using of the approximate zero force length parameter value L_{01}^{Appr} = 3.0 [m] instead of the exact value L_{01} = 2.0 [m] 1: trajectory tracking error, deformation in the space of the 2nd time-derivatives of the generalized coordinates, angle of adaptive rotation, Lyapunov function.

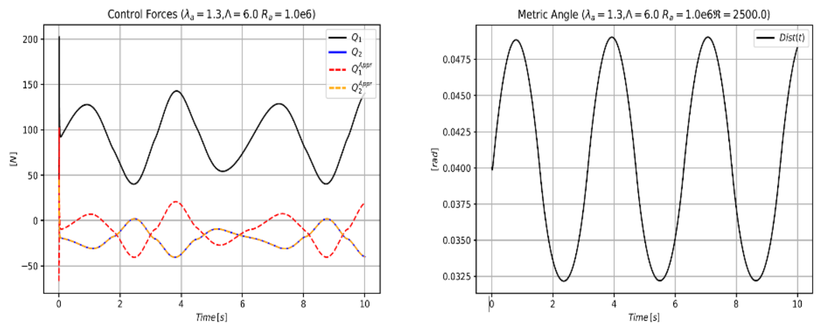


Figure 10

The consequences of using of the approximate zero force length parameter value L_{01}^{Appr} = 3.0 [m] instead of the exact value L_{01} = 2.0 [m] 2: control forces, and the novel metric for the deformation of the control forces.

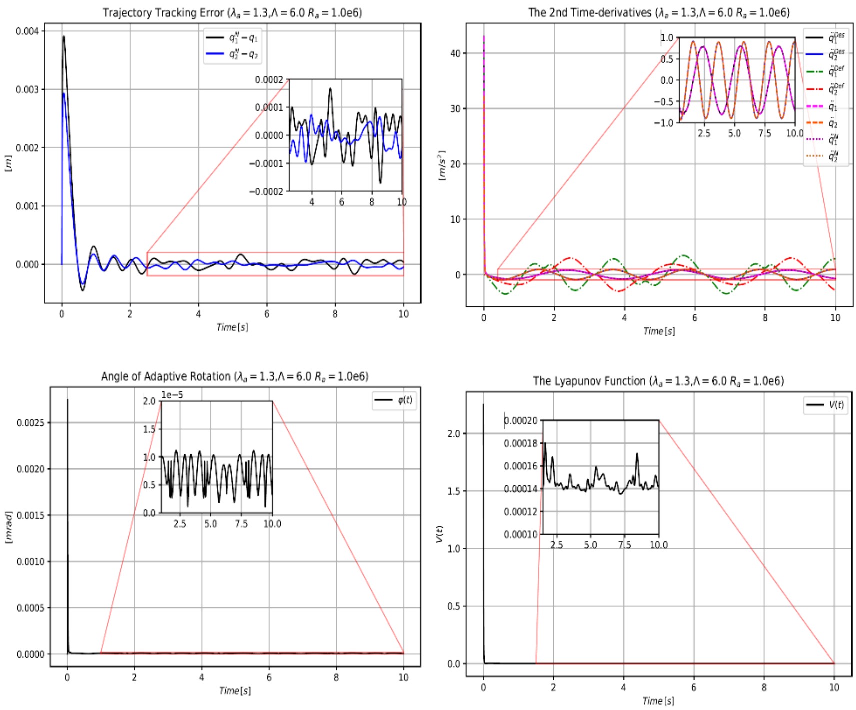


Figure 11
The consequences of swapping the k_1, k_2 spring stiffness parameters 1: trajectory tracking error, deformation in the space of the 2nd time-derivatives of the generalized coordinates, angle of adaptive rotation, Lyapunov.function.

In the next step the significance of the error in the estimation of the *zero force length* spring 1 is considered: $L_{01} = 2.0$ [m], $L_{01}^{Appr} = 3.0$ [m]. Figure 9 reveals ample differences in the details of the controller's operation, however, Fig 10 lucidly summarizes the main consequences by the use of the novel metric that describes the necessary deformation in the adaptive forces.

Finally the consequences of swapping the k_1, k_2 spring stiffness parameters are considered as "modeling errors in parameter groups" in Figs. 11 and 12. The figures testify that the controller worked stably and the proposed rotational metric well comprises the results of the various different effects.

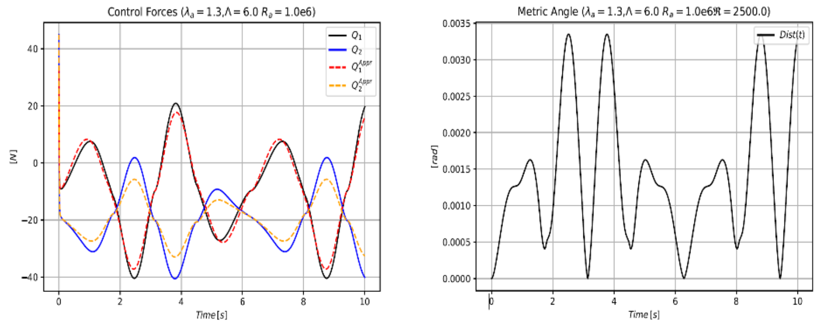


Figure 12
The consequences of swapping the k_1, k_2 spring stiffness parameters 2: control forces, and the novel metric for the deformation of the control forces.

Briefly, the here presented simulations corresponded to the case in which the significance of the estimation errors of a single parameter or a pair of parameters in an analytically exact model form were considered. The adaptive control soon eliminated the tracking error components, therefore, this significance can be related "purely" to the modeling error and the properties of the nominal trajectory. The significance values of the considered parameters were studied in various parts of the trajectory (time), and they varied from zero to the maximum of about 0.0480.

5.2 Simulations for the Motion of Mass Points Hanging on Nonlinear Springs - No precise Analytical Model Form is Applied

In this example, instead of the precise model form given in (24) qualitatively similar but different approximate model form was used as given in (25)

$$Q_1 = m_{1a}\ddot{q}_1 + k_a(q_1 - L_{1a}) - k_a(q_2 - L_{2a}) - m_{1a}g_a \quad , \tag{25}$$

$$Q_2 = m_{2a}\ddot{q}_2 + k_a(q_2 - L_{2a}) - m_{2a}g_a ,$$

where the parameters were: $g_a = 10.0\ [\mathrm{m}\cdot\mathrm{s}^{-2}]$, $m_{1a} = 1.5$ [kg], $m_{2a} = 2.6$ [kg], $k_a = 150.0\ [\mathrm{N}\cdot\mathrm{m}^{-1}]$, $L_{1a} = L_{01}$, and $L_{2a} = L_{01} + L_{02}$. In this case, we cannot speak about the *individual effect of the modeling error of certain parameter.*

To achieve stable controller the control parameters for the Control Lyapunov Function approach were $\Lambda =$ $[\mathrm{s}^{-1}]$, $K = 30.0\ [\mathrm{s}^{-1}]$, and $\varepsilon = 10^{-1}\ [\mathrm{m}^2\cdot\mathrm{s}^{-2}]$ in (6). In adaptive deformation the *common norm of the augmented vectors* was $R_a = 10^6\ [\mathrm{m}^2\cdot\mathrm{s}^{-2}]$, and the "*interpolation parameter*" was reduced to $\lambda_a = 5\times10^{-2}$

in Fig. 2. The *common norm of the augmented vectors in the rotational metric* was $R \equiv \Re = 2.5$ [N].

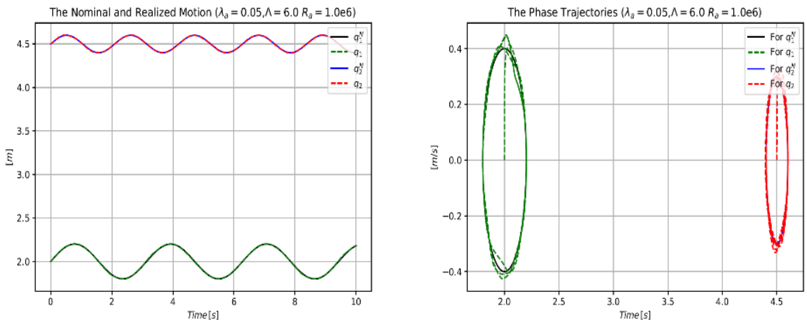


Figure 13
The consequences of using formally not correct model 1: trajectory tracking and phase trajectory tracking

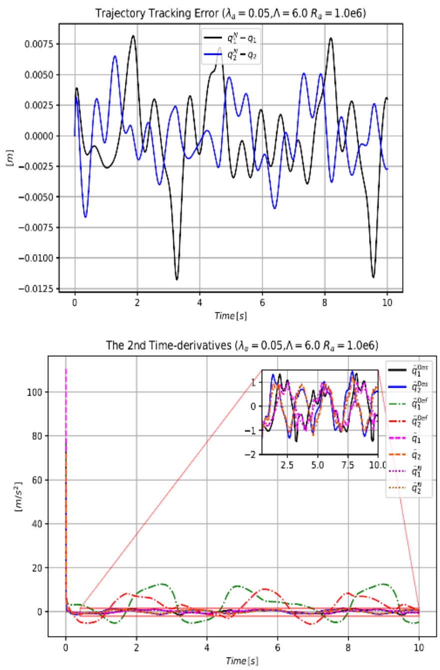


Figure 14
The consequences of using formally not correct model 2: trajectory tracking error and $\ddot{q}$ values

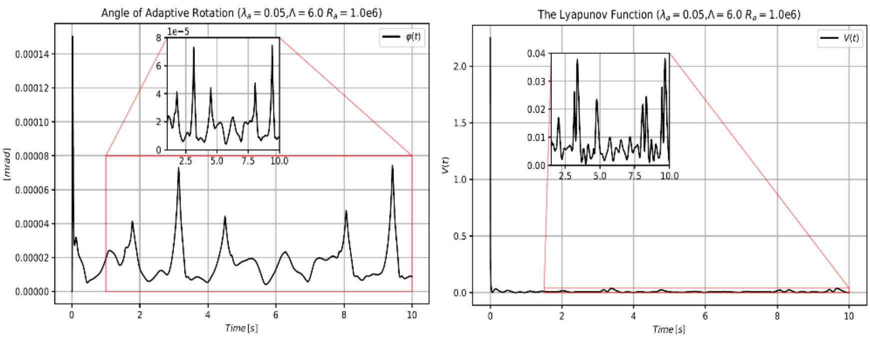


Figure 15

The consequences of using formally not correct model 3: angle of adaptive rotation and the Lyapunov function

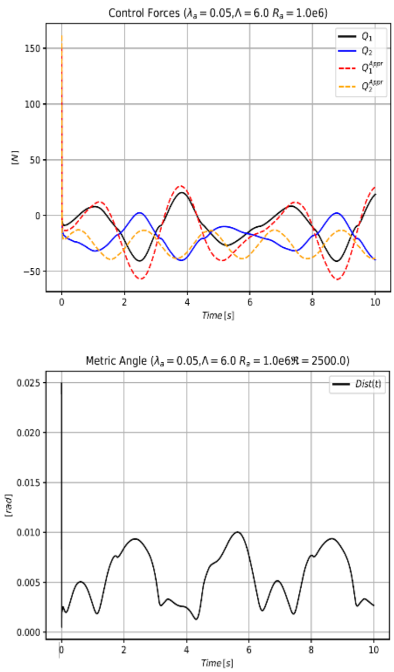


Figure 16

The consequences of using formally not correct model 4: control force components and the novel rotational metric

Figures 13-16 reveal that both the FPI-based Adaptive Controller and the novel rotational metric can work in this case, too. Due to the successful adaptive control only the nominal trajectory and the modeling error play important role. Depending on the phase of the nominal trajectory the error metric varied between [0.01,0.10] is the same scale. The control designer can modify either the parameters of the model of inexact form, or the properties of the nominal trajectory to achieve more acceptable results. Due to the adaptivity no knowledge is needed on the exact model. If the simulation is replaced with measurements and realized experiment, the adaptively produced forces are known and the deformation effort can be compared by the use of the novel metric.

5 Application for Multidimensional Scaling

The idea of "Multidimensional Scaling" – MDS originally was developed by John W. Sammon Jr. in 1969 [15] with the aim of giving lucid visualization of the distribution of point clouds in higher dimensional metric spaces. The higher dimensional points are represented by 3 or 2 dimensional ones so that the "distances between the different points" are maintained in the lower dimension as precisely as possible. Also, for practical purposes the neighboring points can be gathered into clusters to improve lucidity of visualization. For the "distances" various metrics can be applied, and modern software products as, e.g., Julia Language offer useful packages for this purpose, e.g., the package "MultivariateStats", for which the appropriate metric can be conveniently chosen from the program package "Distances".

For visualization purposes the novel metric easily can be added to the Julia's library without serious programming effort. Simply it must be added to the set of already existing metrics in the library by defining it as a new structure as \verb"struct Rotational <: Metric end". Following that the new function \verb"Rotational()" can be defined with a function declaration line as \verb"function (dist::Rotational)(p, q)". (The internal parts of the so defined function make the calculation of the rotational distance in Julia language code between the columns of equal dimension \verb"p", and \verb"q".) For realizing MDS the appropriate metric can be set up as, e.g., \verb"metric=Euclidean()" or \verb"metric=Rotational()".

For application example it is investigated how the original FPI-based adaptive controller defined in [10] improves its tracking precision before losing its stability as one of its adaptive parameter is slowly increased in time. The dynamical system considered was the approximately modeled van der Pol oscillator that was published in [16] in 1927 and later became a popular benchmarking paradigm used for representing nonlinearly oscillating dynamical systems. Its equation of motion with extended 3rd and 5th order terms is given in (26)

$$\ddot{q} = \frac{u + \mu(1-q^2)\dot{q} - \omega^2 q - \alpha q^3 - \lambda q^5}{m}, \tag{26}$$

where u denotes the control force, μ denotes the extent of external excitation for $|q|<1$ and damping for $|q|>1$, ω acts like a spring stiffness, while α and λ belong to further nonlinear contributions in variable q. Parameter m behaves like an inertia. (The physical dimensions are not important for the simulations; one can imagine electrical or instead of that classical mechanical units.) The "exact" and "approximate" parameter values were set as follows: $\mu =$ o.4, $\mu_a = 0.5$, $\omega =$ 0.46, $\omega_a = 0.42$, $\alpha = 1.0$, $\alpha_a = 0.9$, $\lambda = 0.1$, $\lambda_a = 0.09$, $m = 1$ In the simulations Euler integration was applied with discrete time resolution of $\delta t = 1$ ms. The kinematic requirement was formulated similarly to the CTC controllers of robots with PID-type error feedback as

$$\ddot{q}^{Des}(t) = \ddot{q}^{N}(t) + \lambda^3 e_{\text{int}}(t) + 3\lambda^2 e(t) + 3\lambda\dot{e}(t)\,, \tag{27}$$

with $\lambda = 1.0\,[\text{s}^{-1}]$ and a sinusoidal nominal trajectory $q^N(t)$. The adaptive deformation was generated by the function

$$\ddot{q}^{Def}(t+\delta t) = \left(K + \ddot{q}^{Def}(t)\right)\left(1 + B\tanh\left(A(\text{t})(\ddot{q}(t) - \ddot{q}^{Des}(t+\delta t)\right)\right) - K \quad ,$$
(28)

with the adaptive parameters $K = 10^5$, $B = -1$ and the initial value for parameter A was $A_0 = 10^{-5}$. In each digital control step A was increased with $\delta a = 5\times 10^{-9}$. In the control 4999 steps were considered, and the matrix analyzed by the MDS, $\mathbf{X}$, had 13 columns and 4999 rows as follows: 1st column: the $A(t)$, 2nd column: the $q(t)$, 3rd column: the $\dot{q}(t)$, 4th column: the $\ddot{q}(t)$, 5th column: the $q^N(t)$, 6th column: the $\dot{q}^N(t)$, 7th column: the $\ddot{q}^N(t)$, 8th column: the $u(t)$, 9th column: the $\ddot{q}^{Des}(t)$, 10th column: the $\ddot{q}^{Def}(t)$, 11th column: the $e_{\text{int}}(t)$, 12th column: the $e(t)$, and 13th column: the $\dot{e}(t)$ values, respectively. The common radius of the augmented vectors was 200.

Returning to the MDS, the matrix $\mathbf{X}$ was normalized as \verb"X_norm=X/norm(X)", and the distance matrix was created as \verb"D=pairwise(metric, X_norm')". Then, the 2-5 dimensional representations were computed into the appropriate variables as \verb"Y2 = classical_mds(D, 2)", \verb"Y3 = classical_mds(D, 3)", etc. The trajectory tracking function in Fig. 17 reveals how the controller becomes unstable after continuously reducing the tracking error.

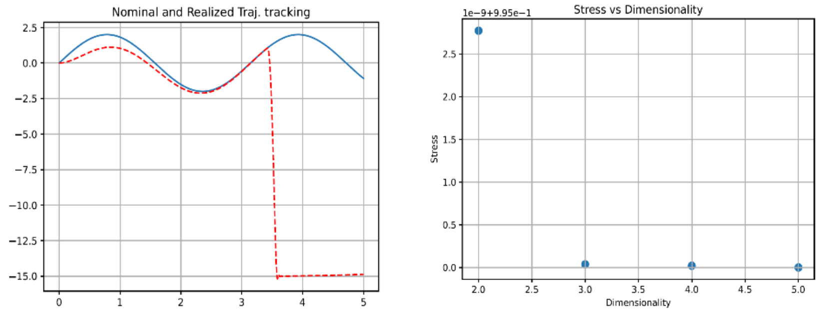


Figure 17

The trajectory tracking and the stress value versus dimensionality function of MDS

The stress function for dimension 2-5 in Fig. 17 reveals that 3 or even 2 dimensions are satisfactory for the visualization of the complex dynamical phenomenon. In Fig. 18 the 3 dimensional result can be seen from two different aspects.

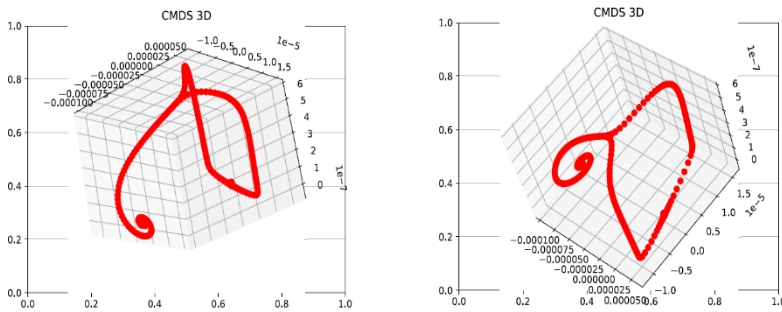


Figure 18

The 3D representation of the results from different aspect using the novel rotational metric

It is evident that this novel metric can be used for the visualization of dynamic phenomena.

Conclusions

In this paper a special, novel, nonlinear metric referred to as "Rotational Metric" was introduced. It was inspired by the particular rotational deformations applied in a special version of the FPI-based adaptive dynamic controller. It is assumed that the task is driving the generalized coordinate of the controlled system $q(t) \in \Re^n$

ovo known nominal trajectory $q^N(t) \in \Re^n$

$q(t_0)$, $\dot{q}(t_0)$ -- in the case of a second order system -- are known, too. For this purpose a purely kinematic tracking strategy prescribing how the integrated tracking error, this error itself and its time-derivative has to converge to zero is available. Also, an approximate dynamic model of the controlled system is available, too. It may be of analytically correct form with approximate parameters or an analytically not precise form with some fictive parameters. Either by measurements or by simulations the adaptive controller can track the nominal trajectory with certain precision. The exerted or simulated control force along the realized trajectory $Q(q(t),\dot{q}(t),\ddot{q}(t))\in\Re^n$ can be compared with the force need of the approximate model $Q^{Appr}(q(t),\dot{q}(t),\ddot{q}(t))\in\Re^n$ in the following manner: the original vectors are augmented with positive complementary components as $[Q(q(t),\dot{q}(t),\ddot{q}(t)),D]\in\Re^{n+1}$, $[Q^{Appr}(q(t),\dot{q}(t),\ddot{q}(t)),D^{Appr}]\in\Re^{n+1}$ so that they have common Frobenius norm. In this case, these vectors can be rotated into each other in $\Re^{n+1}$ with an analytically computable angle that is the numerical value of this metric. (The common norm is a free parameter in this metric, and the orthogonal subspace of these vectors is left invariant.)

Generally, this metric varies in time as the controlled system's state propagates. Beside the modeling inaccuracies it depends on the nature and the parameters of the nominal trajectory, that of the kinematic prescription for the damping of the error components, the initial state of the controlled system, and the parameters of the adaptive controller. If the adaptive controller precisely tracks the nominal trajectory, the situation becomes simpler and the force needs can be attributed only to the dynamic model and the nominal trajectory. The obtained metric provides the designer with information on the significance of the modeling errors in various phases of the nominal trajectory. Also, he/she can consider possible modification of this trajectory.

Following the formal proof of the general properties of the suggested rotational metric particular applications were demonstrated. The motion of two nonlinearly coupled mass points was considered in which the kinematic tracking strategy was formulated by the use of the control Lyapunov function technique. Analytically correct model form with approximate parameters were considered to quantitatively measure the significance of individual parameters and a pair of model parameters. Furthermore, the significance of an analytically incorrect form with fictive parameters was considered.

Finally, it was demonstrated how this rotational metric can be added to the Julia language's library without considerable programming effort for use in MDS. The dynamic process of losing the convergence of a formal version of an FPI-based adaptive controller applied on a nonlinear dynamical system was considered by visualizing a 13-dimensional problem in a 3-dimensional space.

Briefly the result of the research can be formulated as follows: since the identification of the model parameters of strongly nonlinear system normally is very complicated or impossible task, instead of it the significance of the modeling errors in the resulted motion of the controlled system can be easily measured and simulated. In this manner the difficult task of precise identification can be easily evaded.

Acknowledgement

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References

[1] C. Dalla Man, F. Micheletto, D. Lv. and M. Breton, B. P. Kovatchev and C. Cobelli: The UVA/PADOVA Type 1 Diabetes Simulator: New Features, J Diab Scien Techn, Vol. 8, No. 8, 2014, pp. 26-34

[2] Naşcu, R. Oberdieck and E. N. Pistikopoulos: Offset-free explicit hybrid model predictive control of intravenous anaesthesia, Proc. of the 2015 IEEE International Conference on Systems, Man, and Cybernetics, October 9-13, 2015, Hong Kong, 2015, pp. 2475-2480

[3] B. Gergics, M. Puskás, L. Kisbenedek, M. Dömény, L. Kovács and D. A. Drexler: Chemotherapy Optimization and Patient Model Parameter Estimation Based on Noisy Measurements, Acta Polytechnica Hungarica, Vol. 21, No. 10, 2024, pp. 475-494, doi:10.12700/APH.21.10.2024.10.29

[4] Csutak and G. Szederkényi: Optimization-based Parameter Computation for Nonnegative Systems to Achieve Prescribed Dynamic Behaviour, Acta Polytechnica Hungarica, Vol. 21, No. 10, 2024, pp. 457-474, doi:10.12700/APH.21.10.2024.10.28

[5] G. Á. Sziki, A. Szántó and É. Ádámkó: Review of Methods for Determining the Moment of Inertia and Friction Torque of Electric Motors, Acta Polytechnica Hungarica, Vol. 21, No. 10, 2024, pp. 203-218, doi:10.12700/APH.21.4.2024.4.11

[6] B. Németh: Providing Guaranteed Performances for an Enhanced Cruise Control Using Robust LPV Method, Acta Polytechnica Hungarica, Vol. 20, No. 7, 2023, pp. 133-152, doi:10.12700/APH.20.7.2023.7.8

[7] A. Kovacs and I. Vajk: Tuning Parameter-free Model Predictive Control with Nonlinear Internal Model Control Structure for Vehicle Lateral Control, Acta Polytechnica Hungarica, Vol. 20, No. 2, 2023, pp. 185-204, doi:10.12700/APH.20.2.2023.2.10

[8] A. Atinga and J. K. Tar: Application of Abstract Rotations in Data Driven Modeling Supported by Fixed Point Iteration-based Adaptive Control, Proc. of the 2023 European Control Conference (ECC), June 13-16, 2023, Bucharest, Romania, 2023, pp doi:10.23919/ECC57647.2023.10178172

[9] S. Banach: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales (About the Operations in the Abstract Sets and Their Application to Integral Equations), Fund. Math., Vo. 3, 1922, pp. 133-181

[10] J. K. Tar, J. F. Bitó, L. Nádai and J. A. Tenreiro Machado: Robust Fixed Point Transformations in Adaptive Control Using Local Basin of Attraction, Acta Polytechnica Hungarica, Vol. 6, No. 1, pp. 21-37

[11] S. Bennett: Nicholas Minorsky and the automatic steering of ships, IEEE Control Systems Magazine, Vol. 4, No. 4, 1984, pp. 10-15

[12] A. M. Lyapunov: Stability of Motion, Academic Press, New-York and London, 1966

[13] B. Armstrong, O. Khatib and J. Burdick: The Explicit Dynamic Model and Internal Parameters of the PUMA 560 Arm, Proc. IEEE Conf. On Robotic and Automation 1986, pp. 510-518, doi:10.1109/ROBOT.1986.1087644

[14] B. Csanádi, P. Galambos, J. K. Tar, Gy. Györök and A. Serester: A Novel, Abstract Rotation-Based Fixed Point Transformation in Adaptive Control, Proc. of the 2018 IEEE International Conference on Systems, Man, and Cybernetics (SMC) pp. 2577-2582, doi:10.1109/SMC.2018.00441

[15] J. W. Sammon Jr., A Nonlinear Mapping for Data Structure Analysis, IEEE Transactions On Computers, Vol. C-18, No. 5 (May 1969)

[16] B. van der Pol, Forced oscillations in a circuit with non-linear resistance (reception with reactive triode), The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Vol. 7, No. 3, pp. 65-80, 1927