

# On the Calculus of Centrifugal Moments for Plane Plates and Plane Bars

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*Abstract: This paper is focused on the calculus of centrifugal moments for plane plates and bars, starting from the definition. General cases of plane plates and bars are studied. General formulae of calculus for centrifugal moments are established. These formulae are based on the positions of the mass centers of the rotation surfaces and rotation bodies generated by the bars and plates in rotation, respectively.*

*Keywords: centrifugal moments; plane plate; plane bar*

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## 1 Introduction

A number of problems on the dynamics of rigid bodies [1] are solved by the application of the theorem of the angular momentum, or d’Alembert’s principle [2], [3], [4], [5].

In order to solve the problems on the dynamics of plates and bars by using this theorem, it is necessary to find the centrifugal moments by a calculus, which can sometimes be difficult. In the technical literature [4], [5] this is done by integration, starting from the definition.

In this paper, the authors propose two general formulae for the calculus of the centrifugal moments for plane plates and bars. The formulae proposed are original and are based on the positions of the mass centers of the rotation bodies and rotation surfaces generated by plates and bars in rotation, respectively.

## 2 Centrifugal Moments for Plates

Let us consider a homogeneous plane plate with the mass  $m$ , area  $A$  and surface density  $\rho$ . We relate the plate to a Cartesian reference system (Figure 1a) so that the plate will be situated in the  $xOz$  plane. The  $Ox$  and  $Oz$  axes do not cut the plate. The center of mass,  $C$ , has the coordinates  $C(\xi, 0, \zeta)$ .

We isolate an element with infinite little area  $dA = dx dz$ , with the mass  $dm$ . Starting from the definition of the centrifugal moment, we can write:

$$J_{xz} = \int_{(D)} xz dm = \rho \iint_{(S)} xz dA = \rho \iint_{(S)} xz dx dz = \frac{\rho}{2\pi} \iint_{(S)} z 2\pi x dx dz = \frac{\rho}{2\pi} \iiint_{(V)} z dV, \quad (1)$$

where  $dV = 2\pi x dA = 2\pi x dx dz$  is the volume of an infinite little element generated by the plate in rotation around the  $Oz$  axis (Figure 1b).

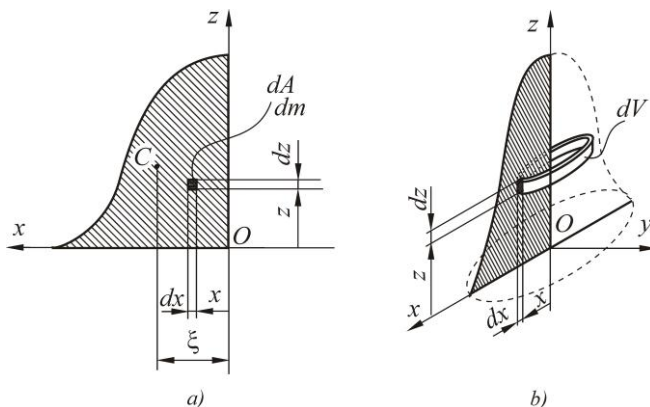


Figure 1

Plane plate completely situated on the same part of the rotation axis ( $Oz$ ): a) calculus of centrifugal moment, starting from definition; b) rotation body generated by plate in rotation

If we consider that the mass center coordinate  $z_{rc}$  of the rotation body generated by the plate rotating around the  $Oz$  axis is given by the relation:

$$z_{rc} = \frac{\iiint_{(V)} z dV}{\iiint_{(V)} dV}, \quad (2)$$

the relation (1) becomes:

$$J_{xz} = \frac{\rho}{2\pi} \iiint_{(V)} z dV = \frac{\rho}{2\pi} z_{rc} \iiint_{(V)} dV = \frac{\rho}{2\pi} z_{rc} \iint_{(S)} 2\pi x dA. \quad (3)$$

Taking into account that the mass center coordinate  $\xi$  of the plane plate on the axis which is perpendicular on the rotation axis  $Oz$  (used to generate the rotation body) is given by the relation:

$$\xi = \frac{\iint_{(S)} x dA}{\iint_{(S)} dA}, \quad (4)$$

the relation (3) becomes:

$$J_{xz} = \frac{\rho}{2\pi} z_{rc} \iint_{(S)} 2\pi x dA = \rho z_{rc} \iint_{(S)} x dA = \rho z_{rc} \xi \iint_{(S)} dA = \rho z_{rc} \xi A = m z_{rc} \xi, \quad (5)$$

where  $m = \rho A$  represents the mass of the plate.

So, we obtain the following formula for the centrifugal moment:

$$J_{xz} = m \xi z_{rc}, \quad (6)$$

or

$$J_{xz} = \rho A \xi z_{rc}. \quad (7)$$

In conclusion, the centrifugal moment is equal to the product of the mass of the plate, the mass center coordinate of the rotation body generated by the plate, and the mass center coordinate of the plate on the axis which is perpendicular on the rotation axis.

If we consider the second Guldin's law,  $V_{Oz} = 2\pi \xi A$  (where  $V_{Oz}$  is the volume of the rotation body generated by the plate in rotation around the  $Oz$  axis), the relation (7) becomes:

$$J_{xz} = \frac{\rho z_{rc} V_{Oz}}{2\pi}. \quad (8)$$

The geometric centrifugal moment will be:

$$I_{xz} = A \xi z_{rc}, \quad (9)$$

or

$$I_{xz} = \frac{z_{rc} V_{Oz}}{2\pi}. \quad (10)$$

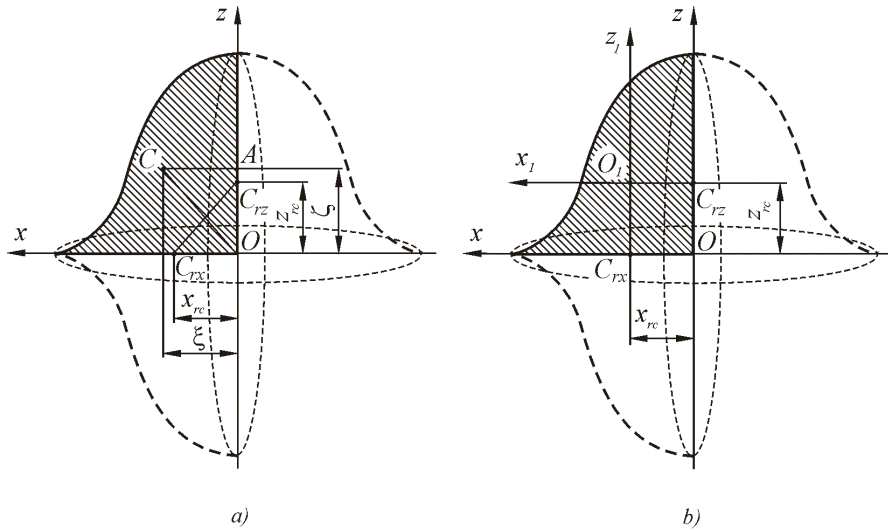


Figure 2

Plane plate rotated around two orthogonal axes: a) rotation bodies generated by the plate in rotation around two orthogonal axes; b) principal axes of inertia

Let us consider a plate like in Figure 2a. With the axes as in the figure, when we rotate the plate around the  $Oz$  axis, we obtain  $J_{xz} = \rho A \xi z_{rc}$ , and when we rotate the plate around the  $Ox$  axis, we obtain  $J_{xz} = \rho A \zeta x_{rc}$ . It follows that:

$$\xi z_{rc} = \zeta x_{rc}, \tag{11}$$

or

$$\frac{\xi}{\zeta} = \frac{x_{rc}}{z_{rc}}. \tag{12}$$

From the relation  $\frac{\xi}{\zeta} = \frac{x_{rc}}{z_{rc}}$  and Figure 2a it results that the triangles  $OAC$  and  $C_{rx}OC_{rz}$  are similar.

From the relation  $J_{xz} = m \xi z_{rc}$  it results that, if  $z_{rc} = 0$ , then  $J_{xz} = 0$ . It follows that axis  $C_{rz}x_I$ , which crosses the mass center of the rotation body generated by the plate, is a principal axis of inertia (Figure 2b). Also, from the relation  $J_{xz} = \rho A \zeta x_{rc}$  it results that axis  $C_{rx}z_I$  is a principal axis of inertia.

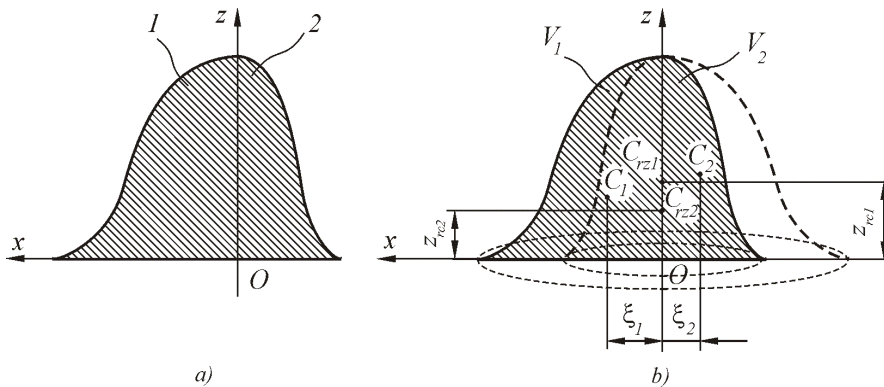


Figure 3

The case when the rotation axis cuts the plane plate: a) plane plate cut by an axis; b) rotation bodies generated by plate in rotation around the axis that cuts the plate

The relations (6), (7), (8), (9), (10) are available if the plate is fully situated on the same part of the rotation axis. In the case when the rotation axis cuts the plate (Figure 3a), the latter is split into two parts by areas  $A_1$ ,  $A_2$  and mass center coordinates  $\xi_1$  and  $\xi_2$ , respectively (Figure 3b). By the rotation of these two parts, two rotation bodies are generated, with volumes  $V_{1Oz}$  and  $V_{2Oz}$ , respectively. In this case the centrifugal moment is:

$$\begin{aligned}
 J_{xz} &= J_{1xz} + J_{2xz} = \rho A_1 \xi_1 z_{rc1} + \rho A_2 \xi_2 z_{rc2} = \frac{\rho(2\pi A_1 \xi_1 z_{rc1} + 2\pi A_2 \xi_2 z_{rc2})}{2\pi} = \\
 &= \frac{\rho}{2\pi} [2\pi A_1 \xi_1 z_{rc1} - 2\pi A_2 (-\xi_2) z_{rc2}] = \frac{\rho(z_{rc1} V_{1Oz} - z_{rc2} V_{2Oz})}{2\pi}. \quad (13)
 \end{aligned}$$

The geometric centrifugal moment will be:

$$I_{xz} = A_1 \xi_1 z_{rc1} + A_2 \xi_2 z_{rc2} = \frac{(z_{rc1} V_{1Oz} - z_{rc2} V_{2Oz})}{2\pi}. \quad (14)$$

Let us consider the example of a homogeneous plane plate  $OAB$  (Figure 4a), quarter of disk of radius  $R$ . The surface density of the material is  $\rho$  (kg/m<sup>2</sup>). With the axes as in the figure, we want to determinate the centrifugal moment  $J_{xz}$ .

First, the “classic way” will be used, the calculus by integration. Let  $dA$  be the area of an element of the plate, with the mass  $dm$ , which corresponds to an angle  $d\theta$  and a radius  $r$  (Figure 4a). For this element we can write:

$$dm = \rho dA = \rho r dr d\theta; \quad x = r \cos \theta; \quad y = r \sin \theta.$$

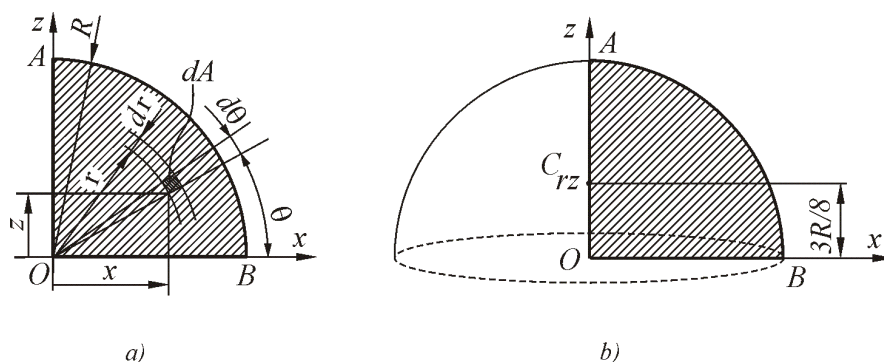


Figure 4

Determination of centrifugal moment for a plane plate quarter of disk: a) calculus of centrifugal moment for a plane plate quarter of disk, starting from definition; b) rotation body generated by plane plate in rotation

$$\begin{aligned}
 J_{xz} &= \int_{(D)} xz dm = \iint_{(S)} r^2 \sin\theta \cos\theta \rho r dr d\theta = \rho \int_0^R \int_0^{\pi/2} r^3 \sin\theta \cos\theta d\theta dr = \\
 &= \rho \int_0^R r^3 dr \int_0^{\pi/2} \sin\theta \cos\theta d\theta = \rho \frac{R^4}{8} \int_0^{\pi/2} \sin\theta \cos\theta d\theta.
 \end{aligned}$$

With the change of variable  $v = \sin\theta$ ,  $dv = \cos\theta d\theta$  it follows that:

$$J_{xz} = \rho \frac{R^4}{8} \int_0^1 v dv = \rho \frac{R^4}{8} \cdot \frac{1}{2} = \rho \frac{R^4}{8}.$$

The same result can be obtained quickly using the relation (8). During the rotation, the plate describes a semi sphere whose volume is  $\frac{2}{3}\pi R^3$  (Figure 4b). The mass center of the semi sphere is on the axis of symmetry ( $Oz$ ),  $z_{rc} = \frac{3}{8}R$ . So, for the centrifugal moment it results that:

$$J_{xz} = \rho \frac{\frac{3}{8}R \cdot \frac{2}{3}\pi R^3}{2\pi} = \rho \frac{R^4}{8}.$$

### 3 Centrifugal Moments for Bars

This study is similar to the one for plates presented above. A homogeneous plane curve bar is considered, with the mass  $m$ , length  $l$  and density  $\rho$ . We relate the bar to a Cartesian reference system (Figure 5a) so that the bar should be situated in the  $xOz$  plane. The  $Ox$  and  $Oz$  axes do not cut the bar. The center of mass  $C$  has the coordinates  $C(\xi, 0, \zeta)$ .

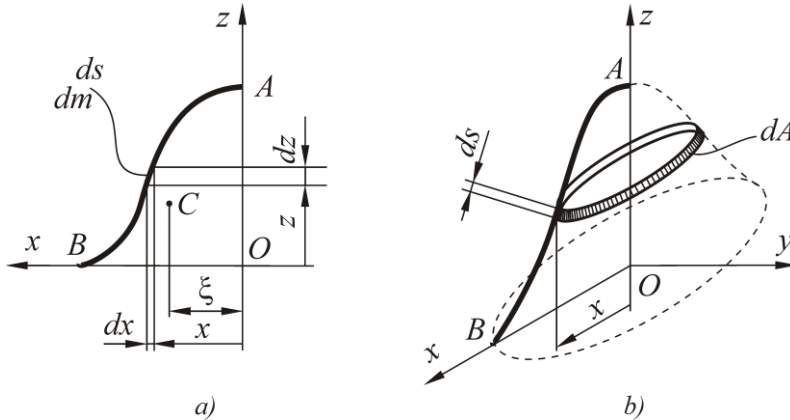


Figure 5

Determination of centrifugal moment for a plane bar: a) calculus of centrifugal moment, starting from definition; b) rotation surface generated by bar in rotation

We isolate an element with infinite little length  $ds = \sqrt{1+(z')^2} dx$ , with the mass  $dm$ . Starting from the definition of the centrifugal moment, we can write:

$$J_{xz} = \int_{(D)} xz dm = \rho \int_{(L)} xz ds = \frac{\rho}{2\pi} \int_{(L)} z 2\pi x ds = \frac{\rho}{2\pi} \iint_{(S)} z dA, \quad (15)$$

where  $dA = 2\pi x ds = 2\pi x \sqrt{1+(z')^2} dx$  is the area of an infinite little element generated by the bar in rotation around the  $Oz$  axis (Figure 5b).

Taking into account the fact that the mass center coordinate  $z_{rc}$  of the rotation body (surface) generated by the bar in rotation around the  $Oz$  axis is given by the relation:

$$z_{rc} = \frac{\iint_{(S)} z dA}{\iint_{(S)} dA}, \quad (16)$$

the relation (15) becomes:

$$J_{xz} = \frac{\rho}{2\pi} \iint_{(S)} z dA = \frac{\rho}{2\pi} z_{rc} \iint_{(S)} dA = \frac{\rho}{2\pi} z_{rc} \int_{(L)} 2\pi x ds. \quad (17)$$

Taking into consideration that the mass center coordinate  $\xi$  of the plane bar on the axis which is perpendicular on the rotation axis  $Oz$  (used to generate the rotation surface) is given by the relation:

$$\xi = \frac{\int_{(L)} x ds}{\int_{(L)} ds}, \quad (18)$$

the relation (17) becomes:

$$J_{xz} = \frac{\rho}{2\pi} z_{rc} \int_{(L)} 2\pi x ds = \rho z_{rc} \int_{(L)} x ds = \rho z_{rc} \xi \int_{(L)} ds = \rho z_{rc} \xi l = m z_{rc} \xi, \quad (19)$$

where  $m = \rho l$  represents the mass of the bar.

So we obtain the following formula for the centrifugal moment:

$$J_{xz} = m \xi z_{rc}, \quad (20)$$

or

$$J_{xz} = \rho l \xi z_{rc}. \quad (21)$$

In conclusion, the centrifugal moment is equal to the product of the mass of the bar, the mass center coordinate of the rotation surface generated by the bar, and the mass center coordinate of the bar on the axis which is perpendicular on the rotation axis.

Considering the first Guldin's law,  $A_{Oz} = 2\pi \xi l$  (where  $A_{Oz}$  is the area of the rotation surface generated by the bar in rotation around the  $Oz$  axis), the relation (21) becomes:

$$J_{xz} = \frac{\rho z_{rc} A_{Oz}}{2\pi}. \quad (22)$$

Let us consider the example of a homogeneous plane straight bar  $AB$  (Figure 6a). We know the angle  $\alpha$  between the bar and the  $Oz$  axis. The linear density of the material is  $\rho$  (kg/m). With the axes as in the figure, we want to determinate the centrifugal moment  $J_{xz}$ .



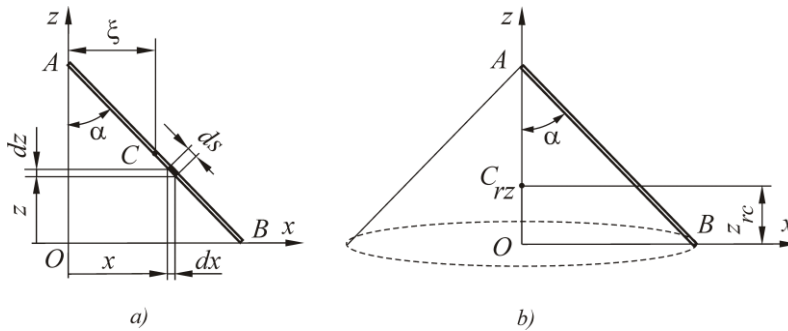


Figure 6

Determination of centrifugal moment for a plane straight bar: a) calculus of centrifugal moment for straight bar, starting from definition; b) rotation surface generated by straight bar in rotation

First the “classic way” will be used, the calculus by integration. Let  $ds$  be the length of an infinite little element of the bar, with the mass  $dm$ , which corresponds, on the axes, to the infinite little distances  $dx$  and  $dz$ , respectively (Figure 6a). For this element we can write:

$$z = -xctg\alpha + l \cos\alpha; \quad z' = \frac{dz}{dx} = -ctg\alpha; \quad dm = \rho ds = \rho \sqrt{1 + (z')^2} dx = \frac{\rho}{\sin\alpha} dx.$$

$$\begin{aligned} J_{xz} &= \int_{(D)} xz dm = \frac{\rho}{\sin\alpha} \int_{(L)} xz dx = \frac{\rho}{\sin\alpha} \int_0^{l \sin\alpha} x(l \cos\alpha - xctg\alpha) dx = \\ &= \frac{\rho}{\sin\alpha} \left( l \cos\alpha \int_0^{l \sin\alpha} x dx - ctg\alpha \int_0^{l \sin\alpha} x^2 dx \right) = \frac{ml^2}{6} \sin\alpha \cos\alpha \end{aligned}$$

This result can be obtained faster by using the relation (20). During the rotation, the bar describes a cone surface whose centre of mass is situated on the  $(Oz)$  axis (Figure 6b),  $z_{rc} = \frac{1}{3} l \cos\alpha$ . The mass center coordinate of the bar is  $\xi = \frac{l}{2} \sin\alpha$ .

So, for the centrifugal moment it results that:

$$J_{xz} = m \frac{l}{2} \sin\alpha \cdot \frac{1}{3} l \cos\alpha = \frac{ml^2}{6} \sin\alpha \cos\alpha.$$

## Conclusions

In this paper the authors have proposed formulae for the calculus of the centrifugal moments for plane plates and bars, based on the positions of the centers of mass.

To sum up, for a plane plate the centrifugal moment is equal to the product of the mass of the plate, the mass center coordinate of the rotation body generated by the

plate, and the mass center coordinate of the plate on the axis which is perpendicular on the rotation axis. Also, for a plane bar the centrifugal moment is equal to the product of the mass of the bar, the mass center coordinate of the rotation surface generated by the bar, and the mass center coordinate of the bar on the axis which is perpendicular on the rotation axis.

Taking into consideration the fact that in the technical literature it is easy to find the positions of the mass centers for a lot of bodies (necessary in statical calculus), the formulae proposed here are accessible because they replace the integral calculus with arithmetical calculus.

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