On Two Modifications of $E_r/E_s/1/m$ Queuing System Subject to Disasters

Michal Dorda, Dusan Teichmann

VSB − Technical University of Ostrava, Faculty of Mechanical Engineering, Institute of Transport, 17. listopadu 15, 708 33 Ostrava-Poruba, Czech Republic, e-mail: michal.dorda@vsb.cz, dusan.teichmann@vsb.cz

Abstract: The paper deals with modelling a finite single-server queuing system with the server subject to disasters. Inter-arrival times and service times are assumed to follow the Erlang distribution defined by the shape parameter r or s and the scale parameter rλ or sμ respectively. We consider two modifications of the model − server failures are supposed to be operate-independent or operate-dependent. Server failures which have the character of so-called disasters cause interruption of customer service, emptying the system and balking incoming customers when the server is down. We assume that random variables relevant to server failures and repairs are exponentially distributed. The constructed mathematical model is solved using Matlab to obtain steady-state probabilities which we need to compute the performance measures. At the conclusion of the paper some results of executed experiments are shown.

Keywords: $E_r/E_s/1/m$; queuing; disasters; method of stages; balking; Matlab

1 Introduction

Queuing theory is a useful tool which enables us to find characteristics of queuing systems. We can meet queuing systems in many sectors, for example in informatics, telecommunications, transport or economics. In general, a queuing system represents a system which serves customers coming into the system. There are a lot of possible ways to classify queuing systems − for example according to the type of the input process (Poisson, *k* Erlang etc.), the service discipline (FCFS, LCFS, priority queues etc.), the capacity of the queue intended for waiting customers (systems which do not permit waiting for the service, finite capacity or the infinity capacity of the queue) and so on. Another criterion of the queuing system's classification is whether failures of servers are considered in the model. For a lot of queuing systems which have already been modelled it is assumed that no failures of servers can occur. Such queuing systems represent the first group of queuing systems often called reliable queuing systems. The second group of queuing systems is represented by the so-called unreliable queuing systems or queuing systems subject to server breakdowns.

Failures of the server have an obvious impact on the performance measures of the studied queuing system. It is clear, for example, that the mean number of the customers in the service for an unreliable queuing system should be less than the value of the same performance measure for a corresponding queuing system which is not subject to breakdowns. A lot of policies have been developed; the policies determine what happens with the customer being served when the server breaks down. In this paper we consider that failures of the server cause emptying of the queuing system; meaning that all customers in the system leave it without being served. Moreover, each customer coming into the system when the server is broken down is not willing to wait and leaves the system (or is rejected). Such type of server failures are often called disasters or catastrophes.

Some authors have already modelled queuing systems with server failures having the character of disasters or catastrophes. Krishna Kumar et al. [1] solved an M/M/1 queuing system. Catastrophes of the server occur according to the Poisson process when the server is busy. Whenever a catastrophe occurs the system empties instantly and all newly arriving customers are lost during the server repair. Sudhesh [2] studied a similar M/M/1 queue which differs in the fact that customers entering the system become impatient when the server is down. The authors of papers [1] and [2] executed transient analysis of the systems, which means they derived formulas for system state probabilities as functions of the time t. Yechiali [3] examined an M/M/c queue with random disastrous failures which cause all present customers to be lost. Customers entering the system during repairing of the server are considered to be impatient.

Queuing systems in which inter-arrival and service times are considered to follow the Erlang distribution have already been studied in the past. But in comparison with queues assuming exponential or general distributed inter-arrival and service times, the models of queuing systems under the assumption of the Erlang distribution are not so common, especially in the case of being subject to breakdowns.

Let us look at some models of reliable queuing systems in which the Erlang distribution is assumed. Plumchitchom and Thomopoulos [4] studied a single-server queuing system with Erlang distributed inter-arrival and service times. Wang and Huang [5] modelled a finite $M/E_k/1$ queuing system with a removable server (the server is turned off and turned on depending on the number of customers in the system). The authors further presented a cost function to determine the optimal policy. Cost and profit analysis for an $M/E_k/1$ queuing system with removable service station was carried out by Mishra and KumarYadav [6]. Yu et al. [7] developed a model of an $M/E_k/1$ queuing system with no damage service interruptions − it is assumed that after the first phase of service the service process can be interrupted with given probability. Binkowski and McCarragher [8] employed an $E_r/E_k/1/N$ queuing system to model the operation of a mining stockyard. An optimal management problem of the N-policy $M/E_k/1$ queuing system with a removable service station under steady-state condition was solved by Pearn and Chang [9]. In the paper [10] written by El-Paoumy and Ismail a solution of a finite $M^X/E_k/1/N$ with bulk arrivals, balking and reneging is demonstrated. The matrix-geometric solution of the $M/E_k/1$ queue with balking and state-dependent service was demonstrated by Yue et al. [11]. Shawky [12] considered a single-channel service time Erlangian queue with finite source of customers, one server, finite storage capacity and balking and reneging. Adan et al. [13] analysed an $E_k/E_r/c$ queuing system.

Some authors solved queues subject to failures under the assumption of the Erlang distribution which was most often applied to model service times. An $M/E_k/1$ queue with server vacation was studied by Jain and Agrawal [14]. The authors further assumed that the server can break down when it is busy and the Poisson arrival rate is state dependent. Wang and Kuo [15] solved an $M/E_k/1$ machine repair problem − several identical machines operating under the care of an unreliable service station. The authors employed matrix geometric method to derive the steady-state probabilities and developed the steady-state profit function to find out the optimum number of machines. Kumar et al. [16] considered an $M^X/E_k/1$ two-phase queuing system with a single removable server and with gating, server start-up and unpredictable breakdowns.

In this paper we will focus on finite single-server queuing systems with the server subject to disasters, where inter-arrival times and service times follow Erlang distribution. Further we will assume that times between failures and times to repair are exponentially distributed. We employed a "direct" approach to model the queuing systems consisting of creating a state transition diagram, on the basis of the diagram we derive a linear equation system describing the system and the equation system is solved numerically using suitable software; in the paper we give a hint for solving in Matlab [17]. We hope that this paper makes the solving of finite Erlang distributed queuing systems possible primarily to non-mathematicians who are not able to employ the most advanced mathematical methods used for analytical solving of queuing systems.

The rest of the paper is organized as follows. In Section 2 we will discuss the necessary assumptions. In Section 3 we will present the mathematical model and its solving using Matlab. In Section 4 we will present results of some numerical experiments we did with the proposed model. Please note that the paper is an extended version of our conference paper presented at the conference Mathematical Methods in Economics 2012 [18].

2 General Assumptions and Notations

Let us consider a single-server queuing system with a finite capacity equal to m, where $m>1$; that means the system has the capacity of m places for customers − single place in the service and $m-1$ places intended for the waiting of customers.

Customers waiting in the queue are served one by one according to the FCFS service discipline.

Let inter-arrival times follow the Erlang distribution with the shape parameter $r \geq 2$ and the scale parameter $r\lambda$; therefore the mean inter-arrival time is then equal to $\frac{r}{r\lambda} = \frac{1}{\lambda}$. Service times are also Erlang distributed with the shape parameter $s \geq 2$ and the scale parameter $s\mu$; thus the mean service time is equal to $\frac{s}{s\mu} = \frac{1}{\mu}$. We apply the Erlang distribution because it is able to model time duration of a lot of practical processes in comparison with the exponential distribution, which is often used. On the other hand, using the Erlang distribution brings some complications in modelling the system. However, single-server queues with the Erlang distribution of inter-arrival times or service times are still solvable using conventional methods.

Let us assume that the server is successively failure-free (or available) and under repair. We will assume two modifications. For the first modification we consider that failures of the server can occur when the server is idle or busy − we say that server failures are operate-independent. In the case of the second modification we assume that failures of the server are so-called operate-dependent, which means the server can break down only when it is servicing a customer. Let us assume for both modifications that repair of the server is started immediately after breakdowns, and it immediately starts to operate when repaired.

Now it is necessary to make some assumptions about failure frequency. Due to the fact that our modifications differ in assumptions about the occurrence of server failures we have to make different suppositions for individual modifications of the studied system. Assuming ergodicity (the system has the finite capacity), all of the possible states of the system can be summarized into three states:

- The idle state − no customer is in the system (the system is empty) and the server is not broken down (is in working condition). Let us denote the equilibrium probability that the server is idle P_{idle}.
- The busy state − i customers are in the system, where $i \in \{1, \ldots, m\}$; that means a customer is in the service and $(i-1)$ customers are waiting in the queue. The equilibrium probability that the system is found in the busy state is denoted with P_{busy}.
- The down state − no customer is in the system (as we are considering disasters) and the server is broken down and under repair; let us denote the equilibrium probability of this state P_{down}.

As these states are mutually exclusive and exhaustive, the sum of these probabilities has to be equal to 1:

$$P_{idle} + P_{busy} + P_{down} = 1. \quad (1)$$

Let us start with the first modification. The server is successively available and broken down. Let the times the server is available be exponentially distributed with the parameter η meaning that the mean time the server is available equals the reciprocal value of the parameter η. Times to repair are exponentially distributed as well, but with the parameter ζ; the mean time to repair therefore equals to $\frac{1}{\zeta}$. It is clear that the server's steady-state availability A (the ratio of time the server is available in expected value) is equal to:

$$A = \frac{\frac{1}{\eta}}{\frac{1}{\eta} + \frac{1}{\zeta}} = \frac{\zeta}{\eta + \zeta} = P_{idle} + P_{busy} \qquad (2)$$

and the server's steady-state unavailability U (the ratio of time the server is broken down in expected value) is:

$$U = 1 - A = \frac{\eta}{\eta + \zeta} = P_{down} . \qquad (3)$$

And now we have to use similar assumptions about the second modification. Let us assume that times of overall server working between failures are exponentially distributed with the parameter η; the mean time of overall server working between failures is then equal to the reciprocal value of the parameter η. Times to repair are exponentially distributed as well, but with the parameter ζ; the mean time to repair is therefore equal to $\frac{1}{\zeta}$ as well.

To express the server's steady-state availability it is necessary to realize that in this modification the server failures are not as frequent as in the first modification for the same value of the parameter η. The value η has to be multiplied by a coefficient expressed by ratio $\frac{P_{idle} + P_{busy}}{P_{busy}}$ which takes into account the fact that the ratio of time the server is idle has no impact on failure frequency. Now, for the server's steady-state availability we can write:

$$A = \frac{\frac{P_{idle} + P_{busy}}{P_{busy}} \cdot \frac{1}{\eta}}{\frac{P_{idle} + P_{busy}}{P_{busy}} \cdot \frac{1}{\eta} + \frac{1}{\zeta}} = \frac{\zeta \cdot \left(P_{idle} + P_{busy}\right)}{\eta \cdot P_{busy} + \zeta \cdot \left(P_{idle} + P_{busy}\right)} = P_{idle} + P_{busy} . \qquad (4)$$

Realizing that $P_{down} = 1 - \left(P_{idle} + P_{busy}\right)$ and substituting it into (4) we can derive an expected formula in the form:

$$P_{down} = \frac{\eta}{\zeta} \cdot P_{busy} = U . \qquad (5)$$

As far as the behaviour of customers at the moment of the failure is concerned, we will assume that the system empties after every breakdown of the server; and the system is empty when the server is down − i.e. failures represent so-called disasters (or catastrophes) in the system.

Let us mention an example of such queuing system from railway transport. Marshalling yards represent important nodes on each railway net because they carry out inbound freight trains classification according to directions of individual wagons and form new outbound freight trains. Such yards are usually equipped with corresponding infrastructure consisting of reception sidings, a hump, sorting sidings and so on. The process of freight trains classification is carried out via the hump − a train of wagons is shunted from an arrival track over the hump and individual wagons (or set of wagons) are classified onto sorting sidings according to their directions.

The hump can be considered to be the server, the inbound freight trains represent customers and the classification process is their service. However, the infrastructure belonging to the hump can break down from time to time. For example, some switches of the ladder below the hump can be broken down so wagons cannot be classified over the hump. In such cases we must carry out the classification process in a different way without using the hump.

Another example of such queuing system could be a gas station that is open non-stop. The station is equipped with one gas pump. Drivers arrive at the gas station in order to pump and pay. However, the gas station is a technical device so it can be subject to breakdowns. Because the gas station has only one gas pump, no driver can be served and therefore drivers do not arrive at the station when the gas pump is closed (under repair). Also all drivers who are at the station when the gas pump breaks down leave the station to pump somewhere else.

As we stated before, the failures of the server have an impact on performance measures, therefore it is important to incorporate the failures in mathematical models of such queuing systems in order to get non-biased results.

3 Mathematical Model

To model the studied queuing system we applied the method of stages (see for example Kleinrock [19]). The method utilizes the fact that the Erlang distribution with the shape parameter r or s and the scale parameter $r\lambda$ or $s\mu$ is a sum of r or s independent exponential distributions with the same parameter $r\lambda$ or $s\mu$. The process of each customer's arrival consists of r exponential phases and the customer enters the system (or is rejected when the system is full) after finishing the last phase. Analogously, the service of each customer consists of s exponential phases and the customer leaves the system after finishing the last phase. Because the duration of all phases is exponentially distributed, the queue can be modelled by a Markov chain. The reason for using the Erlang distribution is that this distribution is more general than the exponential distribution, which cannot be used in many practical examples; the Erlang distribution can be used to model random variables with a coefficient of variation less than 1.

Let us consider a random variable $K(t)$ being the number of the customers found in the system, a random variable $I(t)$ being the number of finished phases of customer's arrival, a random variable $J(t)$ being the number of finished phases of customer's service and a random variable $F(t)$ being the number of broken servers at the time t. On the basis of the assumptions established in Section 2 it is clear that $\{K(t), I(t), J(t), F(t)\}$ constitutes a Markov chain with the state space

$$\Omega = \{(k,i,j,f), k=0, i=0,\ldots,r-1, j=0, f=0,1\} \cup$$
$$\cup \{(k,i,j,f), k=1,\ldots,m, i=0,\ldots,r-1, j=0,\ldots,s-1, f=0\}.$$

Let us note that the first subset contains all the idle and down states and the second subset the busy states. The system is found in the state (k,i,j,f) at the time t if $K(t)=k$, $I(t)=i$, $J(t)=j$ and $F(t)=f$; let us denote the corresponding probability $P_{(k,i,j,f)}(t)$. Complex information about Markov chains can be found for example in Bolch et al. [20].

Now we would like to set up the mathematical model of the system. At first, let us establish a group of variables α_k, where $k=0,1,...,m$. The variable α_k for $k=0,1,...,m$ can take its value from the set $\{0,1\}$. The variables enable us to create the general model for both modifications (we can even create other modifications of the studied system using the variables, for example a modification in which the server breaks down only when the system is full). The variables will be used as a multiplier of the failure rate η. For the first modification we have $\alpha_k=1$ for $k=0,1,...,m$; that means the server can break down when idle ($k=0$) or busy ($k=1,...,m$). For the second modification we have $\alpha_0=0$ (the server can not break down when idle) and $\alpha_k=1$ for $k=1,...,m$ (the server can break down when busy).

Now we can illustrate the queuing model graphically on a state transition diagram (Figure 1). The vertices represent the particular states of the system and the directed edges indicate the possible transitions with the corresponding rates.

Please note that in Figure 1 only those states are depicted which are necessary for the formation of an equation system. Due to the fact that some edges lead from nowhere or point to nowhere in Figure 1, let us comment on such examples:

- The all red edges should point to states (0,0,0,1) up to $(0,r-1,0,1)$ (the down states). But we did not draw all of them to these states because it would make the diagram more chaotic.
- Some green and blue edges are duplicated because the states to which the edges point or from which they lead are not depicted in Figure 1. For example, the green edge exiting the state (0,0,0,0) leads to the state (0,1,0,0), which is not depicted in Figure 1. On the other hand, the green edge leading to the state $(0,i,0,0)$ exits the state $(0,i-1,0,0)$, which is not

depicted either. The blue edge exiting the state (1,0,0,0) leads to the state (1,0,1,0) (not depicted) and the blue edge leading to the state (1,0,*j*,0) exits the state (1,0,*j*−1,0) (not depicted too).

- Some diagonal green edges lead from nowhere or point to nowhere for the same reason. For example the green edge exiting the state (1,*r*−1,0,0) leads to the state (2,0,0,0) (not depicted) and the green arc leading to the state (*k*,0,0,0) exits the state (*k*−1,*r*−1,0,0) (not depicted).

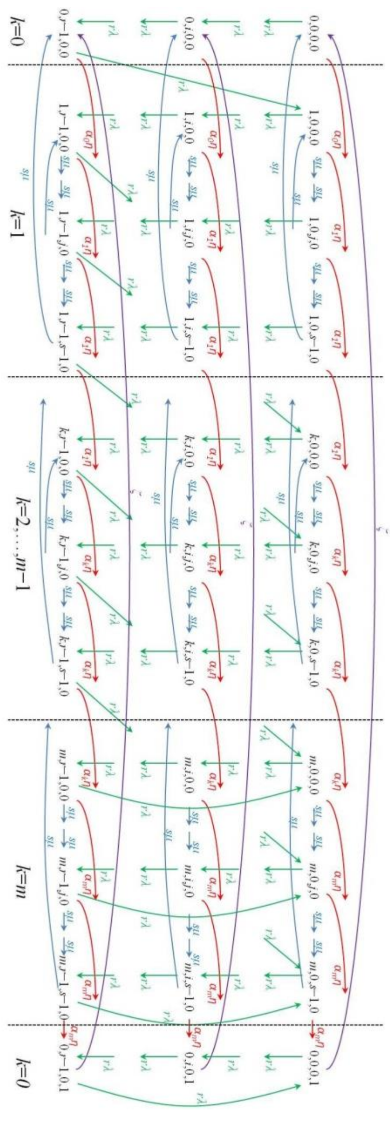

Figure 1
The state transition diagram

An original file containing the diagram can be downloaded from a web-page with the supplementary material − see [21].

Now we apply the global balance principle, which states that for each set of states X the flow out of the set X is equal to the flow into the set X (see Adan and Resing [22]). On the basis of the state transition diagram we are able to write the finite linear equation system of the steady-state balance equations in the form:

$$\left(r\lambda+\alpha_0\eta\right)\cdot P_{(0,0,0,0)}=s\mu\cdot P_{(1,0,s-1,0)}+\zeta\cdot P_{(0,0,0,1)}, \tag{6}$$

for $i=1,\ldots,r-1$:

$$\left(r\lambda+\alpha_0\eta\right)\cdot P_{(0,i,0,0)}=r\lambda\cdot P_{(0,i-1,0,0)}+s\mu\cdot P_{(1,i,s-1,0)}+\zeta\cdot P_{(0,i,0,1)}, \tag{7}$$

for $k=1,\ldots,m-1$:

$$\left(r\lambda+s\mu+\alpha_k\eta\right)\cdot P_{(k,0,0,0)}=r\lambda\cdot P_{(k-1,r-1,0,0)}+s\mu\cdot P_{(k+1,0,s-1,0)}, \tag{8}$$

for $k=1,\ldots,m-1, i=1,\ldots,r-1$:

$$\left(r\lambda+s\mu+\alpha_k\eta\right)\cdot P_{(k,i,0,0)}=r\lambda\cdot P_{(k,i-1,0,0)}+s\mu\cdot P_{(k+1,i,s-1,0)}, \tag{9}$$

for $j=1,\ldots,s-1$:

$$\left(r\lambda+s\mu+\alpha_1\eta\right)\cdot P_{(1,0,j,0)}=s\mu\cdot P_{(1,0,j-1,0)}, \tag{10}$$

for $k=1,\ldots,m, i=1,\ldots,r-1, j=1,\ldots,s-1$:

$$\left(r\lambda+s\mu+\alpha_k\eta\right)\cdot P_{(k,i,j,0)}=r\lambda\cdot P_{(k,i-1,j,0)}+s\mu\cdot P_{(k,i,j-1,0)}, \tag{11}$$

for $k=2,\ldots,m-1, j=1,\ldots,s-1$:

$$\left(r\lambda+s\mu+\alpha_k\eta\right)\cdot P_{(k,0,j,0)}=r\lambda\cdot P_{(k-1,r-1,j,0)}+s\mu\cdot P_{(k,0,j-1,0)}, \tag{12}$$

$$\left(r\lambda+s\mu+\alpha_m\eta\right)\cdot P_{(m,0,0,0)}=r\lambda\cdot P_{(m-1,r-1,0,0)}+r\lambda\cdot P_{(m,r-1,0,0)}, \tag{13}$$

for $j=1,\ldots,s-1$:

$$\left(r\lambda+s\mu+\alpha_m\eta\right)\cdot P_{(m,0,j,0)}=r\lambda\cdot P_{(m-1,r-1,j,0)}+r\lambda\cdot P_{(m,r-1,j,0)}+s\mu\cdot P_{(m,0,j-1,0)}, \tag{14}$$

for $i=1,\ldots,r-1$:

$$\left(r\lambda+s\mu+\alpha_m\eta\right)\cdot P_{(m,i,0,0)}=r\lambda\cdot P_{(m,i-1,0,0)}, \tag{15}$$

$$\left(r\lambda+\zeta\right)\cdot P_{(0,0,0,1)}=r\lambda\cdot P_{(0,r-1,0,1)}+\alpha_0\eta\cdot P_{(0,0,0,0)}+\eta\cdot\sum_{k=1}^{m}\sum_{j=0}^{s-1}\alpha_k\cdot P_{(k,0,j,0)}, \tag{16}$$

for $i=1,\ldots,r-1$:

$$\left(r\lambda+\zeta\right)\cdot P_{(0,i,0,1)}=r\lambda\cdot P_{(0,i-1,0,1)}+\alpha_0\eta\cdot P_{(0,i,0,0)}+\eta\cdot\sum_{k=1}^{m}\sum_{j=0}^{s-1}\alpha_k\cdot P_{(k,i,j,0)}. \tag{17}$$

Subtracting the probability on the left side of each equation (6) − (17) we got an equation system which can be written in the matrix form:

$$\mathbf{0} = \mathbf{Q}^T \cdot \mathbf{P},$$

where $\mathbf{Q}^T$ is the transposed infinitesimal generator matrix containing the transition rates of the Markov process and **P** is the unknown steady-state probability vector.

Because the matrix $\mathbf{Q}^T$ is singular (the equation set is not linearly independent and one equation is redundant), it is necessary to incorporate the normalization condition in the form:

$$\sum_{i=0}^{r-1}\sum_{f=0}^{1} P_{(0,i,0,f)} + \sum_{k=1}^{m}\sum_{i=0}^{r-1}\sum_{j=0}^{s-1} P_{(k,i,j,0)} = 1. \qquad (18)$$

3.1 Solving Equation System using Matlab and Performance Measures

We got the equation system of $m \cdot r \cdot s + 2r + 1$ linear equations formed by equations (6) up to (18). The number of the unknown stationary probabilities is equal to $m \cdot r \cdot s + 2r$.

To solve the corresponding equation system we can omit an equation, for example equation (6). Numerical solving of the system can be performed using Matlab. However, the applied state description in the form of (k,i,j,f) is four-dimensional and is very good for the formation of the equation system but is absolutely unsuitable for the computations in Matlab. Therefore we are obliged to establish an alternative one-dimensional state description in the following form:

- The states (k,i,j,f) for $k=1,...,m$, $i=0,...,r-1$, $j=0,...,s-1$ and $f=0$ can be denoted using a single value $(k-1) \cdot r \cdot s + j \cdot r + i + 1$,
- The states (k,i,j,f) for $k=0$, $i=0,...,r-1$, $j=0$ and $f=0,1$ can be denoted using a single value $m \cdot r \cdot s + f \cdot r + i + 1$.

Applying the alternative one-dimensional state description we are able to transform the equation system in the form we need for using Matlab (we have to work with matrices). In Matlab we solve the linear system in the form:

$$\mathbf{B} = \mathbf{A} \cdot \mathbf{P},$$

where $\mathbf{B} = [0;\ldots;1;\ldots;0]^T$, where the value 1 is in the row $m \cdot r \cdot s + 1$ (in the case that we omit the equation corresponding to the steady-state probability $P_{(0,0,0,0)}$), **A** we get from the matrix $\mathbf{Q}^{\mathrm{T}}$ in which the row $m \cdot r \cdot s + 1$ is substituted by the row matrix $[1;1;\ldots;1]$ and **P** is the unknown steady-state probability vector.

After numerical solving of the equation system rewritten in the matrix form we obtain the stationary probabilities we need in order to compute performance measures of the studied system.

On the basis of the known stationary probability vector **P**, the steady-state probability that the server is idle is equal to:

$$P_{idle} = \sum_{i=0}^{r-1} P_{(0,i,0,0)} \,, \tag{19}$$

the steady-state probability that the server is busy can be expressed by the formula:

$$P_{busy} = \sum_{k=1}^{m} \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} P_{(k,i,j,0)} \tag{20}$$

and for the equilibrium probability that the server is down it holds:

$$P_{down} = \sum_{i=0}^{r-1} P_{0,i,0,1} \,. \tag{21}$$

Now let us consider three performance measures − the mean number of the customers in the service *ES*, the mean number of the customers waiting in the queue *EL* and the mean number of the broken servers *EF*. All of them can be computed according to the formula for the mean value of discrete random variable, where the random variable $S \in \{0,1\}$ is the number of customers in the service, $L \in \{0, m-1\}$ the number of waiting customers and $F \in \{0,1\}$ the number of broken servers.

For the performance measures we can write following formulas:

$$ES = P_{busy} \,, \tag{22}$$

$$EL = \sum_{k=2}^{m} (k-1) \cdot \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} P_{(k,i,j,0)} \,, \tag{23}$$

$$EF = P_{down} \,. \tag{24}$$

The Matlab script (m.file) with defined function enabling computation of equilibrium probabilities (19), (20) and (21) and performance measures (22), (23) and (24) is published online − see [21].

4 Results of Experiments

We performed several experiments with both modifications to demonstrate solvability of the presented model and to obtain some graphical dependencies. Applied values of the model parameters are summarized in Table 1.

Table 1

Summary of applied values of the model parameters

Parameter	m [-]	r [-]	$r\lambda$ [h^{-1}]	s [-]	$s\mu$ [h^{-1}]	η [h^{-1}]	ζ [h^{-1}]
Value	5	2	18	2	20	0.01 up to 0.1 with step 0.01	0.1 up to 1.0 with step 0.1

Substituting the values summarized in Table 1 into the model rewritten in Matlab we are able to compute the steady-state probabilities of the system states and on the basis of them we get the performance measures *ES*, *EL* and *EF* using formulas (22), (23) and (24).

The values of the mean number of the customers in the service *ES* are listed in Table 2. The upper value corresponds to the queuing system with operate-independent server failures, the lower value to the queue with operate-dependent server failures. The data from Table 2 are further shown in Figure 2, the left graph corresponds to the operate-independent modification and the right graph to the operate-dependent modification of the studied queuing system.

Comparing the values with each other we can see that increasing value of the parameter η causes the decrease of the performance measure *ES*. It is an expected fact because with increasing value of the parameter η server failures are more frequent. On the other hand, increasing value of the parameter ζ brings about the increase of the measure *ES*. This dependency could also be logically expected because increasing value of ζ causes shorter times to server repair. Let us note that the values of *ES* are greater for the second modification than for the first modification; it is also logical because it has to hold that the failure frequency of the operate-dependent modification is lower than the failure frequency of the operate-independent modification. The failure frequency of the operate-dependent modification was equal to the failure frequency of the operate-independent modification only in the case that the P_{idle} would be equal to zero.

Table 2

The mean number of the customers in the service *ES*

η / ζ	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
0.01	0.769	0.806	0.819	0.825	0.829	0.832	0.834	0.835	0.837	0.837
	0.780	0.812	0.823	0.828	0.832	0.834	0.836	0.837	0.838	0.839
0.02	0.704	0.768	0.792	0.804	0.812	0.817	0.821	0.824	0.826	0.828
	0.722	0.779	0.799	0.810	0.817	0.821	0.825	0.827	0.829	0.830

0.03	0.648	0.733	0.766	0.784	0.795	0.803	0.808	0.812	0.816	0.818
	0.673	0.748	0.777	0.793	0.802	0.809	0.814	0.817	0.820	0.822
0.04	0.601	0.701	0.743	0.765	0.779	0.789	0.796	0.801	0.806	0.809
	0.630	0.720	0.757	0.776	0.788	0.797	0.803	0.808	0.811	0.814
0.05	0.560	0.672	0.720	0.747	0.764	0.775	0.784	0.791	0.796	0.800
	0.592	0.694	0.737	0.760	0.775	0.785	0.793	0.798	0.803	0.806
0.06	0.524	0.645	0.699	0.729	0.749	0.762	0.772	0.780	0.786	0.791
	0.558	0.670	0.718	0.745	0.762	0.774	0.782	0.789	0.794	0.799
0.07	0.493	0.620	0.679	0.713	0.734	0.750	0.761	0.770	0.777	0.782
	0.528	0.648	0.700	0.730	0.749	0.763	0.773	0.780	0.786	0.791
0.08	0.464	0.597	0.660	0.697	0.721	0.738	0.750	0.760	0.768	0.774
	0.501	0.626	0.684	0.716	0.737	0.752	0.763	0.771	0.778	0.784
0.09	0.439	0.576	0.642	0.681	0.707	0.726	0.739	0.750	0.759	0.766
	0.477	0.607	0.667	0.703	0.726	0.742	0.754	0.763	0.770	0.776
0.10	0.417	0.555	0.625	0.666	0.694	0.714	0.729	0.741	0.750	0.757
	0.455	0.588	0.652	0.690	0.714	0.732	0.745	0.755	0.763	0.769

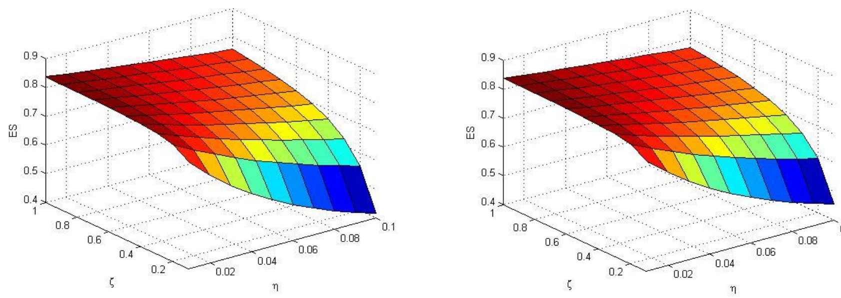

Figure 2

The dependence of *ES* on the parameters η and ζ (first modification on the left, second modification on the right)

Table 3

The mean number of the customers waiting in the queue *EL*

η / ζ	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
0.01	1.180	1.236	1.256	1.267	1.273	1.277	1.280	1.282	1.284	1.285
	1.197	1.246	1.263	1.271	1.277	1.280	1.283	1.285	1.286	1.287
0.02	1.076	1.173	1.210	1.229	1.241	1.249	1.255	1.259	1.263	1.265
	1.104	1.190	1.222	1.239	1.249	1.255	1.260	1.264	1.267	1.269
0.03	0.987	1.116	1.167	1.194	1.211	1.222	1.231	1.237	1.242	1.246
	1.024	1.139	1.184	1.207	1.222	1.232	1.239	1.244	1.248	1.252
0.04	0.912	1.064	1.126	1.160	1.182	1.196	1.207	1.215	1.222	1.227
	0.955	1.092	1.148	1.177	1.196	1.209	1.218	1.225	1.230	1.235

0.05	0.846	1.015	1.088	1.128	1.154	1.171	1.184	1.194	1.202	1.209
	0.894	1.049	1.113	1.149	1.171	1.186	1.197	1.206	1.213	1.218
0.06	0.789	0.971	1.052	1.097	1.127	1.147	1.162	1.174	1.183	1.191
	0.840	1.008	1.081	1.121	1.147	1.164	1.177	1.187	1.195	1.202
0.07	0.738	0.930	1.018	1.068	1.101	1.124	1.141	1.154	1.164	1.173
	0.791	0.971	1.050	1.095	1.123	1.143	1.158	1.169	1.178	1.186
0.08	0.693	0.891	0.985	1.040	1.076	1.101	1.120	1.135	1.146	1.156
	0.748	0.935	1.021	1.069	1.101	1.123	1.139	1.152	1.162	1.170
0.09	0.653	0.856	0.955	1.013	1.052	1.079	1.100	1.116	1.128	1.139
	0.709	0.902	0.993	1.045	1.079	1.103	1.121	1.135	1.146	1.155
0.10	0.617	0.823	0.926	0.987	1.029	1.058	1.080	1.097	1.111	1.122
	0.673	0.871	0.966	1.022	1.058	1.084	1.103	1.118	1.130	1.140

The values of the mean number of the customers waiting in the service *EL* are listed in Table 3 and graphically shown in Figure 3. We can see the same character of dependencies as in the case of the performance measure *ES*.

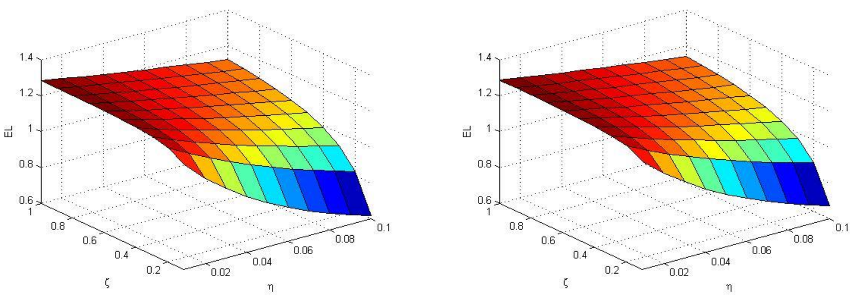

Figure 3

The dependence of *EL* on the parameters η and ζ (first modification on the left, second modification on the right)

The values of the mean number of the broken servers *EF* are listed in Table 4 and graphically shown in Figure 4. It is logical to expect that the measure *EF* should increase with increasing value η and decrease with increasing value of ζ − both expectations were confirmed by reached results. Furthermore, we can check the correctness of reached results using formulas (3) and (5).

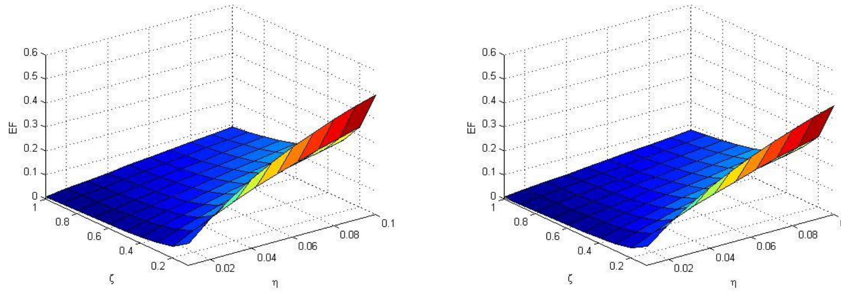

Figure 4

The dependence of *EF* on the parameters *η* and *ζ* (first modification on the left, second modification on the right)

Table 4

The mean number of the broken servers *EF*

η / *ζ*	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
0.01	0.091	0.048	0.032	0.024	0.020	0.016	0.014	0.012	0.011	0.010
	0.078	0.041	0.027	0.021	0.017	0.014	0.012	0.010	0.009	0.008
0.02	0.167	0.091	0.063	0.048	0.038	0.032	0.028	0.024	0.022	0.020
	0.144	0.078	0.053	0.041	0.033	0.027	0.024	0.021	0.018	0.017
0.03	0.231	0.130	0.091	0.070	0.057	0.048	0.041	0.036	0.032	0.029
	0.202	0.112	0.078	0.059	0.048	0.040	0.035	0.031	0.027	0.025
0.04	0.286	0.167	0.118	0.091	0.074	0.062	0.054	0.048	0.043	0.038
	0.252	0.144	0.101	0.078	0.063	0.053	0.046	0.040	0.036	0.033
0.05	0.333	0.200	0.143	0.111	0.091	0.077	0.067	0.059	0.053	0.048
	0.296	0.174	0.123	0.095	0.078	0.065	0.057	0.050	0.045	0.040
0.06	0.375	0.231	0.167	0.130	0.107	0.091	0.079	0.070	0.063	0.057
	0.335	0.201	0.144	0.112	0.091	0.077	0.067	0.059	0.053	0.048
0.07	0.412	0.259	0.189	0.149	0.123	0.104	0.091	0.080	0.072	0.065
	0.370	0.227	0.163	0.128	0.105	0.089	0.077	0.068	0.061	0.055
0.08	0.444	0.286	0.211	0.167	0.138	0.118	0.103	0.091	0.082	0.074
	0.401	0.251	0.182	0.143	0.118	0.100	0.087	0.077	0.069	0.063
0.09	0.474	0.310	0.231	0.184	0.153	0.130	0.114	0.101	0.091	0.083
	0.429	0.273	0.200	0.158	0.131	0.111	0.097	0.086	0.077	0.070
0.10	0.500	0.333	0.250	0.200	0.167	0.143	0.125	0.111	0.100	0.091
	0.455	0.294	0.217	0.172	0.143	0.122	0.106	0.094	0.085	0.077

Conclusions

In the paper we discussed two modifications of an $E_r/E_s/1/m$ queuing system subject to disasters which cause loss of all customers in the system and balking all customers incoming to the system while the server is under repair. To solve the proposed model using Matlab, we developed the one-dimensional system state description, which enabled us to rewrite the linear equation system into the matrix form. After numerically solving the equation system we got the steady-state probabilities we need for computing the performance measures. We focused on three performance measures – *ES*, *EL* and *EF*.

In the experimental part of the paper we presented the dependencies of the performance measures on the parameters of η and ζ defining the failure frequency and the repair rate. Our experiments confirmed that the presented mathematical model can be successfully applied for solving such queuing system. The experiments showed the expected dependencies:

- The increasing value of η decreases the value of *ES* and the increasing value of the parameter ζ increases the value of *ES*. For the same values of η and ζ, the value of *ES* is lower for the operate-independent modification than for the operate-dependent modification.
- The values of *EL* evince the same character of dependency as the values of *ES*.
- The value of *EF* increases with the increasing value of η and decreases with the increasing value of ζ. The values of *EF* are greater for the operate-independent modification than for the operate-dependent modification.

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