Exact Linearization of Uncertain Nonlinear Systems with Positive Input

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Abstract: Exact linearization of nonlinear systems enables the application of linear control theory to nonlinear dynamics. However, it relies on exact cancellation of nonlinear terms, which becomes impractical when model uncertainties are present. This paper analyzes the effect of parametric uncertainties on exact linearization and derives equivalent linear perturbed models when the real system deviates from the nominal one. We consider parametric perturbations where the nonlinear drift vector field is linear in the parameters. Necessary and sufficient conditions are given for the relative degree to remain invariant under such perturbations. Furthermore, the case of internal stabilization applied after linearization is analyzed, and the corresponding equivalent perturbed model is provided. Finally, we extend the framework to positive-input systems, derive results on the relative degree of the extended dynamics, and present the associated equivalent linear perturbed model.

Keywords: perturbed model; dynamics extension; internal stabilization

1 Introduction

Exact linearization, also known as feedback linearization, is a classical method used to handle nonlinearities in dynamical systems [1, 2]. It has been widely applied in various fields [3–6]. The main limitation of this approach is its reliance on exact cancellation of nonlinear terms, which makes it sensitive to parametric uncertainties. In [7], this issue was addressed for uncertain systems where the drift vector field depends linearly on the parameters, and the influence of parameter perturbations on the linearization process was examined. A related extension was presented in [8], where a positive input dynamics was incorporated into the framework.

In general, uncertainties in nonlinear systems are managed by adaptive or robust controllers [9, 10]. Classical works, such as Chen [11], analyzed exact linearization under parametric uncertainty by transforming the nonlinear system into a perturbed linear model and providing adaptive control laws that ensure stability under matched uncertainties. Panchal et al. [12] proposed robust control methods for uncertain nonlinear systems without assuming a specific structure of the uncertainty. Sastry and Kokotovic [13] established conditions for convergence of path-tracking controllers when the vector fields are linear in uncertain parameters, and later extended the framework to dynamic uncertainties using singular perturbation theory. Other adaptive schemes were developed for nonlinear systems of different relative degrees, e.g., [14–16].

Most of these studies assume that the relative degree of the system remains invariant under parametric perturbations [11, 13]. In contrast, this paper provides necessary and sufficient conditions for the relative degree of a SISO output to remain robust to parameter changes. The corresponding linear perturbed model is derived for systems without zero dynamics, as first introduced in [7].

Exact linearization is often combined with linear control design methods such as H_2/H_∞ control. While linearization around an operating point provides an approximate model [17], exact linearization yields a model free of approximation errors but results in poles at zero, leading to infinite H_∞-norm. In order to address this, internal stabilization by state feedback is required to shift poles into the left-half plane [18, 19]. The impact of parametric perturbations under such stabilization was studied in [7], and here we generalize those results to derive the equivalent linear perturbed model for stabilized systems.

Furthermore, the framework is extended to systems with positive input dynamics [8], relevant to chemical and physiological applications [20–26], and several other engineering applications [27]. In such systems, the control input must remain nonnegative, which is enforced through a dynamical extension where the physical input becomes a state of the augmented system. We analyze the effect of parametric perturbations on the extended system and provide the corresponding perturbed linear model both with and without stabilizing feedback.

The results include explicit expressions for the perturbed linear models of smooth input-affine nonlinear systems and their positive-input extensions. These formulations can support robust stability analysis [28, 29], fault detection [30], and robust fixed-point transformation-based control [31–33]. Moreover, they can serve as a preprocessing step for tensor-product model transformation and controller synthesis [34–36], or for systems with input saturation constraints [37].

2 Exact Linearization in the Presence of Parametric Uncertainty

2.1 The Original and Perturbed Model

We consider smooth, input-affine nonlinear systems described by

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x(t) \in U \subseteq \mathbb{R}^n$ denotes the system state, $u(t) \in \mathscr{L}^\infty(\mathbb{R},\mathbb{R})$ is the control input, $f \in \mathscr{C}^\infty(U,\mathbb{R}^n)$ is the drift vector field, and $g \in \mathscr{C}^\infty(U,\mathbb{R}^n)$ is the control vector field.

The scalar system output is given by

$$y = h(x) \tag{2}$$

with $h \in \mathscr{C}^\infty(U,\mathbb{R})$ and $y(t) \in \mathbb{R}$ for all $t \geq 0$. Later, in Section 3, we extend the analysis to systems with nonnegative inputs, i.e., $u(t) \geq 0$ for every $t \in \mathbb{R}$.

In this study, the control and output vector fields g and h are assumed to be known exactly, while the drift f depends linearly on uncertain parameters. Specifically,

$$f(x) = \sum_{p=1}^{P} \kappa_p f_p(x), \tag{3}$$

where each $f_p \in \mathscr{C}^\infty(U,\mathbb{R}^n)$ and the coefficients κ_p represent the nominal parameter values. The perturbed drift vector field is defined as

$$\hat{f}(x) = \sum_{p=1}^{P} \hat{\kappa}_p f_p(x), \tag{4}$$

where $\hat{\kappa}_p$ denote the possibly perturbed parameter values. Since g and h are independent of the uncertain parameters, they remain unchanged in the perturbed model ($\hat{g} = g$, $\hat{h} = h$). This formulation follows the structure introduced in [7] and extended in [8].

Example 1. As a representative case, we adopt the minimal tumor growth model from [38], which has been utilized for nonlinear [39, 40], robust [19], and LPV controller design [41]. The system dynamics are given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} ax_1 - bx_1x_2 \\ -cx_2 + u \end{pmatrix} = \underbrace{\begin{pmatrix} ax_1 - bx_1x_2 \\ -cx_2 \end{pmatrix}}_{f(x)} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{g(x)} u. \tag{5}$$

Here, x_1 represents the tumor volume (mm^3), x_2 the drug level ($mg \cdot kg^{-1}$), and u the drug injection rate ($mg \cdot (kg \cdot day)^{-1}$). The output of the system is $y = x_1$. The drift vector field is linear in parameters a, b, and c:

$$f(x) = a\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -x_1x_2 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ -x_2 \end{pmatrix}. \tag{6}$$

Defining $f_1(x) = (x_1\ 0)^\top$, $f_2(x) = (-x_1x_2\ 0)^\top$, and $f_3(x) = (0\ -x_2)^\top$ as the basis vector fields, the nominal parameters are $\kappa_1 = a$, $\kappa_2 = b$, and $\kappa_3 = c$. For the perturbed model with parameters $\hat{\kappa}_1$, $\hat{\kappa}_2$, and $\hat{\kappa}_3$, the perturbed drift vector field becomes

$$\hat{f}(x) = \hat{\kappa}_1\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \hat{\kappa}_2\begin{pmatrix} -x_1x_2 \\ 0 \end{pmatrix} + \hat{\kappa}_3\begin{pmatrix} 0 \\ -x_2 \end{pmatrix} \tag{7}$$

while g and h remain unchanged ($\hat{g} = g$, $\hat{h} = h$).

2.2 Linearization Using Feedback

Feedback linearization transforms a nonlinear system into an equivalent linear one by canceling nonlinearities through appropriately designed feedback [1]. The concept is based on Lie derivatives of smooth scalar fields along smooth vector fields. The Lie derivative of the scalar field h along the vector field f is

$$L_f h := (\partial_x h) f, \tag{8}$$

where ∂_x denotes differentiation with respect to the state vector. Higher-order Lie derivatives are defined recursively as

$$L_f^i h := L_f \left(L_f^{i-1} h \right) = \left(\partial_x \left(L_f^{i-1} h \right) \right) f, \quad i > 0, \quad L_f^0 h := h. \tag{9}$$

The Lie derivative of $L_f^i h$ along g is

$$L_g L_f^i h := \left(\partial_x \left(L_f^i h \right) \right) g. \tag{10}$$

For the system given by (1), the time derivative of the output $y = h(x)$ is

$$\dot{y} = \dot{h}(x) = (\partial_x h) \dot{x} = (\partial_x h)(f(x) + g(x)u) = L_f h + L_g h u. \tag{11}$$

If $L_g h = 0$, differentiation can be continued until the input appears explicitly in the r-th derivative:

$$\ddot{y} = L_f^r h + L_g L_f^{r-1} h u. \tag{12}$$

The smallest integer r for which $L_g L_f^{r-1} h \neq 0$ defines the relative degree of the output $y = h(x)$ [1]. Formally,

$$L_g L_f^k h(x) = 0, \quad k = 0, 1, \ldots, r-2, \quad L_g L_f^{r-1} h(x) \neq 0. \tag{13}$$

If the relative degree r equals the system order n, the system can be exactly linearized using feedback. The corresponding transformed dynamics take the form

$$y = h \tag{14}$$
$$\dot{y} = L_f h \tag{15}$$
$$\ddot{y} = L_f^2 h \tag{16}$$
$$\vdots$$
$$y^{(n-1)} = L_f^{n-1} h \tag{17}$$
$$y^{(n)} = L_f^n h + L_g L_f^{n-1} h u := w. \tag{18}$$

The input u can then be computed from w using (18), where w is the linearized system input.

The expression for u thus becomes

$$u = \frac{w - L_f^n h}{L_g L_f^{n-1} h}. \tag{19}$$

The state transformation defining the linearized coordinates is

$$z = \Phi(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ L_f^2 h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{pmatrix}. \tag{20}$$

Example 2. Using the minimal tumor model from Example 1, the Lie derivatives are computed as follows:

$$L_f h = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} ax_1 - bx_1x_2 \\ -cx_2 \end{pmatrix} = ax_1 - bx_1x_2 \tag{21}$$

$$L_g h = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \tag{22}$$

$$\begin{aligned} L_f^2 h &= \begin{pmatrix} a - bx_2 & -bx_1 \end{pmatrix} \begin{pmatrix} ax_1 - bx_1x_2 \\ -cx_2 \end{pmatrix} \\ &= x_1(a - bx_2)^2 + bcx_1x_2 \end{aligned} \tag{23}$$

$$L_g L_f h = \begin{pmatrix} a - bx_2 & -bx_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -bx_1. \tag{24}$$

Since $L_g h = 0$ and $L_g L_f h \neq 0$ for $x_1 \neq 0$, the output $y = x_1$ has maximal relative degree whenever $x_1 \neq 0$. The corresponding feedback law that linearizes the system is

$$u = \frac{w - x_1(a - bx_2)^2 - bcx_1x_2}{-bx_1}, \quad x_1 \neq 0. \tag{25}$$

The coordinates of the linearized system are

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ ax_1 - bx_1x_2 \end{pmatrix}. \tag{26}$$

2.3 Effect of Parametric Uncertainty on Exact Linearization

Lemma 1. *For any integer $k \geq 0$, we have*

$$L_f^k h = \sum_{p_1=1}^{P} \sum_{p_2=1}^{P} \cdots \sum_{p_k=1}^{P} \kappa_{p_1} \kappa_{p_2} \cdots \kappa_{p_k} L_{f_{p_k}} \ldots L_{f_{p_2}} L_{f_{p_1}} h \tag{27}$$

$$L_g L_f^k h = \sum_{p_1=1}^{P} \sum_{p_2=1}^{P} \cdots \sum_{p_k=1}^{P} \kappa_{p_1} \kappa_{p_2} \cdots \kappa_{p_k} L_g L_{f_{p_k}} \ldots L_{f_{p_2}} L_{f_{p_1}} h. \tag{28}$$

Proof. The result follows by induction using linearity of differentiation, see [7]. □

Example 3. According to Lemma 1, the Lie derivatives in Example 2 can be written as

$$L_f h = aL_{f_1} h + bL_{f_2} h + cL_{f_3} h = ax_1 + b(-x_1 x_2) \tag{29}$$

$$L_g h = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \tag{30}$$

$$\begin{aligned} L_f^2 h &= a^2 L_{f_1} L_{f_1} h + ab \left(L_{f_2} L_{f_1} h + L_{f_1} L_{f_2} h \right) + ac \left(L_{f_3} L_{f_1} h + L_{f_1} L_{f_3} h \right) \\ &\quad + b^2 L_{f_2} L_{f_2} h + bc \left(L_{f_3} L_{f_2} h + L_{f_2} L_{f_3} h \right) + c^2 L_{f_3} L_{f_3} h \\ &= a^2 x_1 - 2abx_1 x_2 + b^2 x_1 x_2^2 + bcx_1 x_2 \\ &= x_1 (a - bx_2)^2 + bcx_1 x_2 \end{aligned} \tag{31}$$

$$L_g L_f h = aL_g L_{f_1} h + bL_g L_{f_2} h + cL_g L_{f_3} h = -bx_1. \tag{32}$$

Theorem 1. *Let the nominal output have relative degree $r > 1$ at x. The perturbed system keeps the same relative degree if: (C1) $L_g L_{f_{p_i}} \cdots L_{f_{p_1}} h(x) = 0$ for all index tuples of length $i = 1, \ldots, r-2$. (C2) There exists $(\bar{p}_1, \ldots, \bar{p}_{r-1})$ such that $L_g L_{f_{\bar{p}_{r-1}}} \cdots L_{f_{\bar{p}_1}} h(x) \neq 0$ and is linearly independent of the others.*

Proof. See [7]. □

Example 4. For the minimal model, $r = 2$ for $x_1 \neq 0$. Since $L_g L_{f_2} h = -x_1 \neq 0$, the perturbed system also has relative degree two if $\hat{\kappa}_2 = \hat{b} \neq 0$.

The following result extends the previous analysis of the perturbed relative degree [7] and complements the positive-input framework [8] by deriving the explicit form of the equivalent linear perturbed model.

Theorem 2. *Assume maximal relative degree $r = n$. Under the nominal feedback* (19)*, the perturbed coordinates satisfy*

$$\dot{\hat{z}}_n = w + \psi_n(x) + \rho(x) \left(w - L_f^n h(x) \right) = w + \varphi(x, w) \tag{33}$$

where $\psi_i = L^i_{\hat{f}}\hat{h} - L^i_f h$ *and*

$$\rho(x) = \frac{L_{\hat{g}}L^{n-1}_{\hat{f}}\hat{h}(x) - L_g L^{n-1}_f h(x)}{L_g L^{n-1}_f h(x)}. \tag{34}$$

The linearized coordinates are

$$\hat{z} = \Phi(x) + \begin{pmatrix} 0 \\ \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_{n-1}(x) \end{pmatrix}. \tag{35}$$

Proof. Since the output of the perturbed system has maximal relative degree, it can be transformed into a series of integrators with $\hat{z}_1 = y$, and the states of the transformed, perturbed system can be written as

$$\begin{aligned}
\hat{z}_1 &= h && (36)\\
\dot{\hat{z}}_1 = \hat{z}_2 &= L_{\hat{f}}h && (37)\\
\dot{\hat{z}}_2 = \hat{z}_3 &= L^2_{\hat{f}}h && (38)\\
&\vdots \\
\dot{\hat{z}}_{n-1} = \hat{z}_n &= L^{n-1}_{\hat{f}}h && (39)\\
\dot{\hat{z}}_n &= L^n_{\hat{f}}h + L_g L^{n-1}_{\hat{f}}hu. && (40)
\end{aligned}$$

If we denote the difference of the ith Lie derivatives of h along $\hat{f}$ and f as $\psi_i = L^i_{\hat{f}}h - L^i_f h$, then the linearized states of the perturbed model expressed in the point $x \in U$ are written as

$$\begin{aligned}
\hat{z}_1 &= z_1 = \Phi_1(x) && (41)\\
\hat{z}_2 &= z_2 + L_{\hat{f}}h(x) - z_2 = z_2 + L_{\hat{f}}h(x) - L_f h(x) = z_2 + \psi_1(x) \\
&= \Phi_2(x) + \psi_1(x) && (42)\\
\hat{z}_3 &= z_3 + L^2_{\hat{f}}h(x) - z_3 = z_3 + L^2_{\hat{f}}h(x) - L^2_f h(x) = z_3 + \psi_2(x) \\
&= \Phi_3(x) + \psi_2(x) && (43)\\
&\vdots \\
\hat{z}_n &= z_n + L^{n-1}_{\hat{f}}h(x) - z_n = z_n + L^{n-1}_{\hat{f}}h(x) - L^{n-1}_f h(x) = z_n + \psi_{n-1}(x) \\
&= \Phi_n(x) + \psi_{n-1}(x) && (44)
\end{aligned}$$

which implies (35). Substituting the linearizing feedback law (19) based on the parameters of the original model into the differential equation of $\hat{z}_n$ results in

$$\dot{\hat{z}}_n = L^n_{\hat{f}} h + L_g L^{n-1}_{\hat{f}} h \frac{w - L^n_f h}{L_g L^{n-1}_f h}. \tag{45}$$

Write $L_g L^{n-1}_{\hat{f}} h$ in the form

$$L_g L^{n-1}_{\hat{f}} h = (1+\rho) L_g L^{n-1}_f h, \tag{46}$$

which can be done in every point x where the system output has maximal relative degree, since this implies $L_g L^{n-1}_f h(x) \neq 0$, then (45) becomes

$$\begin{aligned}
\dot{\hat{z}}_n &= L^n_{\hat{f}} h + (1+\rho) L_g L^{n-1}_f h \frac{w - L^n_f h}{L_g L^{(n-1)}_f h} & (47)\\
&= L^n_{\hat{f}} h + (1+\rho)(w - L^n_f h) = L^n_{\hat{f}} h + w - L^n_f h + \rho(w - L^n_f h) & (48)\\
&= w + L^n_{\hat{f}} h - L^n_f h + \rho(w - L^n_f h) = w + \psi_n(x) + \rho(w - L^n_f h), & (49)
\end{aligned}$$

which implies (33). □

Example 5. Apply the feedback linearization in Example 2 to the perturbed model in Example 1, and suppose moreover that $\hat{b} \neq 0$, i.e., the relative degree of the output of the perturbed system is the same as the relative degree of the output of the original system, as it was shown in Example 4. We apply Theorem 2 to create the linear perturbed model resulting after the linearization. First, we construct $\rho(x)$. If the output of the original and the perturbed model have maximal relative degree, then $b \neq 0$ and $\hat{b} \neq 0$, and we can write $\hat{b} = b(1+\Delta b)$ with $\Delta b \neq -1$, thus

$$\rho(x) = \frac{L_{\hat{g}} L_{\hat{f}} \hat{h} - L_g L_f h}{L_g L_f h} = \frac{-\hat{b}x_1 + bx_1}{-bx_1} = \frac{-\hat{b} + b}{-b} = \frac{-b(1+\Delta b) + b}{-b} = \Delta b. \tag{50}$$

Using the results from Example 3, the differences ψ_1 and ψ_2 are

$$\begin{aligned}
\psi_1 &= L_{\hat{f}} \hat{h} - L_f h = (\hat{a} - a) L_{f_1} h + (\hat{b} - b) L_{f_2} h \\
&= (\hat{a} - a) x_1 + (\hat{b} - b)(-x_1 x_2) & (51)\\
\psi_2 &= L^2_{\hat{f}} \hat{h} - L^2_f h = (\hat{a}^2 - a^2) L_{f_1} L_{f_1} h + (\hat{a}\hat{b} - ab) \left(L_{f_2} L_{f_1} h + L_{f_1} L_{f_2} h\right) \\
&\quad + (\hat{a}\hat{c} - ac) \left(L_{f_3} L_{f_1} h + L_{f_1} L_{f_3} h\right) + (\hat{b}^2 - b^2) L_{f_2} L_{f_2} h \\
&\quad + (\hat{b}\hat{c} - bc) \left(L_{f_3} L_{f_2} h - L_{f_2} L_{f_3} h\right) + (\hat{c}^2 - c^2) L_{f_3} L_{f_3} h \\
&= (\hat{a}^2 - a^2) x_1 - 2(\hat{a}\hat{b} - ab) x_1 x_2 + (\hat{b}^2 - b^2) x_1 x_2^2 + (\hat{b}\hat{c} - bc) x_1 x_2. & (52)
\end{aligned}$$

Thus, the perturbed linear model has the states

$$\begin{aligned}
\hat{z}_1 &= z_1 = y & (53)\\
\hat{z}_2 &= z_2 + \Psi_1 = z_2 + (\hat{a} - a) x_1 + (\hat{b} - b)(-x_1 x_2) & (54)
\end{aligned}$$

and the differential equation of $\hat{z}_2$ is the perturbed linear model

$$\dot{\hat{z}}_2 = w + \varphi(x,w) \tag{55}$$

with $\varphi(x,w)$ being the model perturbation

$$\begin{aligned}
\varphi(x,w) &= \psi_2(x) + \rho(x)(w - L_f^2 h) \\
&= (\hat{a}^2 - a^2)x_1 - 2(\hat{a}\hat{b} - ab)x_1x_2 + (\hat{b}^2 - b^2)x_1x_2^2 \\
&\quad + (\hat{b}\hat{c} - bc)x_1x_2 + \Delta b(w - L_f^2 h) \\
&= (\hat{a}^2 - a^2(1+\Delta b))x_1 - 2(\hat{a}\hat{b} - ab(1+\Delta b))x_1x_2 \\
&\quad + (\hat{b}^2 - (b^2 + \Delta b))x_1x_2^2 + (\hat{b}\hat{c} - bc(1+\Delta b))x_1x_2 + \Delta bw
\end{aligned} \tag{56}$$

caused by the parametric perturbations.

Corollary 1. *The difference ψ_i for $i = 1,2,\ldots,n$ can be written as the sum*

$$\psi_i = L_{\hat{f}}^i \hat{h} - L_f^i h = \sum_{p_1=1}^{P}\sum_{p_2=1}^{P}\cdots\sum_{p_i=1}^{P}(\hat{\kappa}_{p_1}\hat{\kappa}_{p_2}\cdots\hat{\kappa}_{p_i} - \kappa_{p_1}\kappa_{p_2}\cdots\kappa_{p_i})L_{f_{p_i}}\ldots L_{f_{p_2}}L_{f_{p_1}}h, \tag{57}$$

while $\psi_n - \rho L_f^n h$ is the sum

$$\psi_n - \rho L_f^n h = \sum_{p_1=1}^{P}\sum_{p_2=1}^{P}\cdots\sum_{p_n=1}^{P}(\hat{\kappa}_{p_1}\hat{\kappa}_{p_2}\cdots\hat{\kappa}_{p_n} - (1+\rho)\kappa_{p_1}\kappa_{p_2}\cdots\kappa_{p_n})L_{f_{p_n}}\ldots L_{f_{p_2}}L_{f_{p_1}}h, \tag{58}$$

thus the model perturbation $\varphi(x,w)$ can be rewritten in the form

$$\begin{aligned}
\varphi(x,w) &= \\
&\sum_{p_1=1}^{P}\sum_{p_2=1}^{P}\cdots\sum_{p_n=1}^{P}(\hat{\kappa}_{p_1}\hat{\kappa}_{p_2}\cdots\hat{\kappa}_{p_n} - (1+\rho(x))\kappa_{p_1}\kappa_{p_2}\cdots\kappa_{p_n})L_{f_{p_n}}\ldots L_{f_{p_2}}L_{f_{p_1}}h(x) \\
&+ \rho(x)w.
\end{aligned} \tag{59}$$

2.4 Stabilization and Parametric Perturbation

In order to ensure stability of the linearized dynamics, we consider the feedback $w = -Kz + b_n u_{sl}$ with $K = (k_n, k_{n-1}, \ldots, k_1)$. Application of the feedback on the integrator series $y^{(n)} = w$ results in the system

$$\dot{z} = Az + Bu_{sl} \tag{60}$$

with

$$A = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \vdots & 1 & 0 \\ -k_n & -k_{n-1} & -k_{n-2} & \dots & -k_2 & -k_1 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_n \end{pmatrix}, \tag{61}$$

which is a stable linear system if the characteristic polynomial

$$\lambda(s) = s^n + k_1 s^{n-1} + k_2 s^{n-2} + \ldots + k_{n-1} s + k_n \tag{62}$$

of the system matrix is Hurwitz. The transfer function of the system is

$$G(s) = \frac{b_n}{s^n + k_1 s^{n-1} + k_2 s^{n-2} + \ldots + k_{n-1} s + k_n}, \tag{63}$$

thus, the static gain of the closed-loop system is b_n/k_n. Stabilization may be a crucial step, e.g., for robust controller design. Provided that all poles are real or critically damped, the H_∞-norm of the stabilized system is $b_n/k_n < \infty$, while the H_∞-norm of the integrator series (15)-(18) was infinite.

Theorem 3. *Suppose that the original and perturbed systems have maximal relative degree in a point x, the linearizing feedback law $w = -Kz + b_n u_{sl}$ designed for the nominal model is applied to the perturbed model, with z being the states of the linearized nominal model. This results in the perturbed linear model with dynamics*

$$\dot{\hat{z}} = A\hat{z} + Bu_{sl} + e_n \varphi(x, u_{sl}) \tag{64}$$

in each point $x \in U$, with e_n being the nth unit vector, the matrices A and B are as defined in (61), and

$$\begin{aligned} \varphi(x, u_{sl}) \quad = \quad & \sum_{i=2}^{n} k_{n+1-i} \psi_{i-1}(x) + \psi_n(x) \\ & -\rho(x) \left(L_f^n h(x) + \sum_{i=1}^{n} k_{n+1-i} \Phi_i(x) \right) + \rho(x) b_n u_{sl} \end{aligned} \tag{65}$$

with ρ and ψ_i, $i = 1, 2, \ldots, n$ as defined in Theorem 2.

Proof. From Theorem 2 we know, that the states of the perturbed linear system in point $x \in U$ are $\hat{z}_1 = z_1 = h(x)$, $\hat{z}_i = \Phi_i(x) - \psi_{i-1}(x)$ for $i = 2, 3, \ldots, n$, and the differential equation of $\hat{z}_n$ is

$$\dot{\hat{z}}_n = w + \psi_n(x) + \rho(x)(w - L_f^n h(x)), \tag{66}$$

while substituting the feedback law $w = -Kz + b_n u_{sl}$ results in

$$\dot{\hat{z}}_n = -Kz + b_n u_{sl} + \psi_n(x) + \rho(x)(-Kz + b_n u_{sl} - L_f^n h(x)). \tag{67}$$

The term Kz in the feedback can be written as

$$\begin{aligned} Kz &= \sum_{i=1}^{n} k_{n+1-i} z_i = \sum_{i=1}^{n} k_{n+1-i} \Phi_i(x) = \sum_{i=2}^{n} k_{n+1-i} \left(\hat{z}_i - \psi_{i-1}(x)\right) + k_n \hat{z}_1 \\ &= K\hat{z} - \sum_{i=2}^{n} k_{n+1-i} \psi_{i-1}(x). \end{aligned} \tag{68}$$

Using these expressions, (67) becomes

$$\begin{aligned} \dot{\hat{z}}_n &= -K\hat{z} + b_n u_{sl} + \sum_{i=2}^{n} k_{n+1-i} \psi_{i-1}(x) + \psi_n(x) \\ &\quad + \rho(x) \left(-\sum_{i=1}^{n} k_{n+1-i} \Phi_i(x) + b_n u_{sl} - L_f^n h(x) \right). \end{aligned} \tag{69}$$

Rearranging the terms results in (64) and (65). □

Example 6. Consider the perturbed linearized system from Example 5, and apply the control law $w = -Kz + b_2 u_{sl}$ with $b_2 > 0$. The differential equation of $\hat{z}_2$ becomes

$$\dot{\hat{z}}_2 = -K\hat{z} + b_2 u_{sl} + \varphi(x, u_{sl}) \tag{70}$$

and $\varphi(x, u_{sl})$ is

$$\begin{aligned} \varphi(x, u_{sl}) &= \psi_2(x) - \rho(x) L_f^2 h(x) + k_1 \psi_1(x) \\ &\quad - \rho(x) \left(k_2 \Phi_1(x) + k_1 \Phi_2(x)\right) + \rho(x) u_{sl} \\ &= L_{\hat{f}}^2 \hat{h}(x) - (1 + \Delta b) L_f^2 h(x) + k_1 \left(L_{\hat{f}} \hat{h}(x) - L_f h(x) \right) \\ &\quad - \Delta b \left(k_2 h(x) + k_1 L_f h(x) \right) + \Delta b b_2 u_{sl} \\ &= (\hat{a}^2 - a^2(1 + \Delta b)) x_1 - 2(\hat{a}\hat{b} - ab(1 + \Delta b)) x_1 x_2 \\ &\quad + (\hat{b}^2 - (b^2 + \Delta b)) x_1 x_2^2 + (\hat{b}\hat{c} - bc(1 + \Delta b)) x_1 x_2 \\ &\quad + k_1 \left((\hat{a} - a(1 + \Delta b)) x_1 + \left(\hat{b} - b(1 + \Delta b)\right)(-x_1 x_2) \right) \\ &\quad - k_2 \Delta b x_1 + \Delta b b_2 u_{sl}. \end{aligned} \tag{71}$$

3 Positive Control and Exact Linearization in the Presence of Parametric Uncertainty

3.1 Positive Input Dynamics

Certain classes of real-world systems can only accept nonnegative control inputs [21, 22, 24, 25]. Conventional controller structures typically ignore this constraint

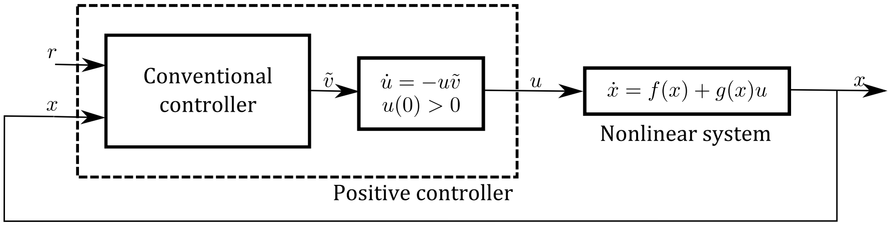

Figure 1
Controller architecture with positive input dynamics extension [8]

and may produce negative input values. In order to address this limitation, the system can be extended with a dynamic element that ensures positivity of the physical input [40, 42]:

$$\dot{u} = -u\tilde{v} \tag{72}$$

where $\tilde{v} \in \mathscr{L}^\infty(\mathbb{R},\mathbb{R})$ acts as a virtual control input. The solution to (72) remains positive for all $t \geq 0$ whenever $u(0) > 0$, regardless of the sign of $\tilde{v}$. Consequently, u never reaches zero in finite time but may become arbitrarily small.

For controller synthesis, the plant is augmented with the dynamics (72) to form the extended state vector

$$\tilde{x} = \begin{pmatrix} x \\ u \end{pmatrix} \tag{73}$$

whose evolution is described by

$$\underbrace{\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix}}_{\dot{\tilde{x}}} = \underbrace{\begin{pmatrix} f(x) + g(x)u \\ 0 \end{pmatrix}}_{\tilde{f}(\tilde{x})} + \underbrace{\begin{pmatrix} 0 \\ -u \end{pmatrix}}_{\tilde{g}(\tilde{x})} \tilde{v} \tag{74}$$

Here $\tilde{f}$ and $\tilde{g}$ denote the drift and control vector fields of the extended system, respectively, and the output mapping $\tilde{h}(x,u) = h(x)$ ensures that the external measured variable remains unchanged.

In practice, the controller is designed for this extended representation. The virtual input $\tilde{v}$ can take any real value, while the auxiliary dynamics guarantees that the actual actuator input u remains positive [39, 42]. The resulting structure (Figure 1) allows conventional design techniques to handle systems with inherent positivity constraints.

From (74) it follows that $\tilde{g}$ depends only on u, which is known and measurable. Therefore, if uncertainties are present in the original control vector field g, they are transferred into the drift term $\tilde{f}$ only. This property allows the analytical results of the previous sections to be applied directly to uncertain systems with constrained positive inputs.

Example 7. Consider again the minimal tumor growth model from Example 1 [19, 39, 42]. After including the positive input dynamics (72), the extended system becomes

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{u} \end{pmatrix}}_{\dot{\tilde{x}}} = \underbrace{\begin{pmatrix} ax_1 - bx_1x_2 \\ -cx_2 + u \\ 0 \end{pmatrix}}_{\tilde{f}(\tilde{x})} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ -u \end{pmatrix}}_{\tilde{g}(\tilde{x})} \tilde{v} \tag{75}$$

with output $\tilde{h}(x,u) = h(x) = x_1$. For any initial condition satisfying $x_1(0), x_2(0), u(0) > 0$, the trajectories remain positive for all $t \geq 0$ and any bounded virtual input $\tilde{v}(t)$.

3.2 Positive Input Dynamics with Feedback Linearization

The coordinate transformation corresponding to the extended system is

$$\tilde{z} = \begin{pmatrix} \tilde{h} \\ L_{\tilde{f}}\tilde{h} \\ L_{\tilde{f}}^2\tilde{h} \\ \vdots \\ L_{\tilde{f}}^n\tilde{h} \end{pmatrix} := \tilde{\Phi}(\tilde{x}), \tag{76}$$

while the feedback law for linearization is given by

$$\tilde{v} = \frac{\tilde{w} - L_{\tilde{f}}^{n+1}\tilde{h}}{L_{\tilde{g}}L_{\tilde{f}}^n\tilde{h}}, \tag{77}$$

where $\tilde{w}$ is the input of the resulting integrator series. Since the extended system has order $n+1$, the Lie derivatives in the coordinate transformation are required until the nth order, while the Lie derivatives in the feedback law are required until the $(n+1)$th order. Throughout the paper, we will denote the order of the original system by n, and the order of the extended system by $n+1$.

Lemma 2. *Suppose that the original system output has maximal relative degree (i.e., $r = n$) in a point x with u being the input. Then the output of the system after extension with positive input dynamics with the new input $\tilde{v}$ also has maximal relative degree (i.e., $r = n+1$) in the point x if and only if $u \neq 0$.*

Proof. See [8]. □

Corollary 2. *The coordinate transformation (76) and feedback law (77) for the extended system can be written in terms of the vector fields and variables of the* *original system as*

$$\tilde{z} = \begin{pmatrix} h \\ L_f h \\ L_f^2 h \\ \vdots \\ L_f^{n-1} h \\ L_f^n h + L_g L_f^{n-1} hu \end{pmatrix} \tag{78}$$

$$\tilde{v} = \frac{\tilde{w} - \left(L_f^{n+1} h + L_g L_f^n hu + L_f L_g L_f^{n-1} hu + L_g^2 L_f^{n-1} hu^2 \right)}{-L_g L_f^{n-1} hu}. \tag{79}$$

Example 8. The linearizing feedback of the extended system from Example 7 can be written as

$$\tilde{v} = \frac{\tilde{w} - \left(L_f^3 h + L_g L_f^2 hu + L_f L_g L_f hu + L_g^2 L_f hu^2 \right)}{-L_g L_f hu}, \tag{80}$$

where $-L_g L_f h(x)u = bx_1 u$, thus the extended system can be linearized using state feedback, i.e., the output has maximal relative degree if $x_1 \neq 0$ and $u \neq 0$, and

$$L_f^3 h = \sum_{p_1=1}^{3} \sum_{p_2=1}^{3} \sum_{p_3=1}^{3} \kappa_{p_1} \kappa_{p_2} \kappa_{p_3} L_{f_{p_1}} L_{f_{p_2}} L_{f_{p_3}} h \tag{81}$$

$$L_g L_f^2 h = \sum_{p_1=1}^{3} \sum_{p_2=1}^{3} \kappa_{p_1} \kappa_{p_2} L_g L_{f_{p_1}} L_{f_{p_2}} h \tag{82}$$

$$L_f L_g L_f h = \sum_{p_1=1}^{3} \sum_{p_2=1}^{3} \kappa_{p_1} \kappa_{p_2} L_{f_{p_1}} L_g L_{f_{p_2}} h \tag{83}$$

$$L_g^2 L_f h = \sum_{p_1=1}^{3} \kappa_{p_1} L_g^2 L_{f_{p_1}} h, \tag{84}$$

i.e., the Lie derivatives in the feedback law (79) can be calculated using the Lie derivatives of the vector fields of the original system. The states of the integrator series resulting from the linearization of the extended system are

$$\begin{aligned} \tilde{z} &= \begin{pmatrix} h \\ L_f h \\ L_f^2 h + L_g L_f hu \end{pmatrix} = \begin{pmatrix} \tilde{\phi}_1(\tilde{x}) \\ \tilde{\phi}_2(\tilde{x}) \\ \tilde{\phi}_3(\tilde{x}) \end{pmatrix} = \\ &= \begin{pmatrix} y \\ \sum_{p_1=1}^{3} \kappa_{p_1} L_{f_{p_1}} h \\ \sum_{p_1=1}^{3} \sum_{p_2=1}^{3} \kappa_{p_1} \kappa_{p_2} L_{f_{p_1}} L_{f_{p_2}} h + \sum_{p_1=1}^{3} \kappa_{p_1} L_g L_{f_{p_1}} hu \end{pmatrix}. \end{aligned} \tag{85}$$

Due to space limitations, we do not give the explicit form of the functions in the Lie derivatives for this example.

Lemma 2 states that the output of the extended system has maximal relative degree if $u \neq 0$ everywhere where the original system output has maximal relative degree, i.e., in every point except $x_1 = 0$. This is equivalent with the result that the extended system output has maximal relative degree if $-L_g L_f h(x) u = b x_1 u \neq 0$ in (80).

3.3 Stabilizaton and Parametric Perturbation

The positive input dynamics guarantees that the control signal of the original system remains nonnegative. However, this extension introduces nonlinearity, even when the nominal system itself is linear. Therefore, in Subsection 3.2, state feedback was used to linearize the extended model, resulting in a chain of $n+1$ integrators governed by

$$\tilde{z}^{(n+1)} = \tilde{w}. \tag{86}$$

Next, internal loop-shaping is applied (analogous to the approach in Section 2.4) to shift the zero poles of the system to new poles $s_1, s_2, \ldots, s_{n+1}$, each having a negative real part.

Let $\tilde{K} = \left(\tilde{k}_{n+1}, \tilde{k}_n, \ldots, \tilde{k}_2, \tilde{k}_1\right)$, and denote the input of the new system by $\tilde{u}$. Application of the control law

$$\tilde{w} = -\tilde{K}\tilde{z} + \tilde{b}_{n+1}\tilde{u} \tag{87}$$

results in the closed-loop system

$$\dot{\tilde{z}} = \tilde{A}\tilde{z} + \tilde{B}\tilde{u}, \tag{88}$$

with matrices

$$\tilde{A} = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ -\tilde{k}_{n+1} & -\tilde{k}_n & -\tilde{k}_{n-1} & \ldots & -\tilde{k}_2 & -\tilde{k}_1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \tilde{b}_{n+1} \end{pmatrix}. \tag{89}$$

The system matrix $\tilde{A}$ has a companion form, and its characteristic polynomial is

$$s^{n+1} + \tilde{k}_1 s^n + \tilde{k}_2 s^{n-1} + \ldots + \tilde{k}_{n+1}, \tag{90}$$

where the coefficients are selected so that (90) is Hurwitz, with its roots $p_1, p_2, \ldots, p_{n+1}$ representing the closed-loop poles. The static gain of the system equals $\tilde{b}_{n+1}/\tilde{k}_{n+1}$.

The transfer function of the loop-shaped linearized model is

$$G(s) = \frac{\tilde{b}_{n+1}}{s^{n+1} + \tilde{k}_1 s^n + \tilde{k}_2 s^{n-1} + \ldots + \tilde{k}_n s + \tilde{k}_{n+1}}, \tag{91}$$

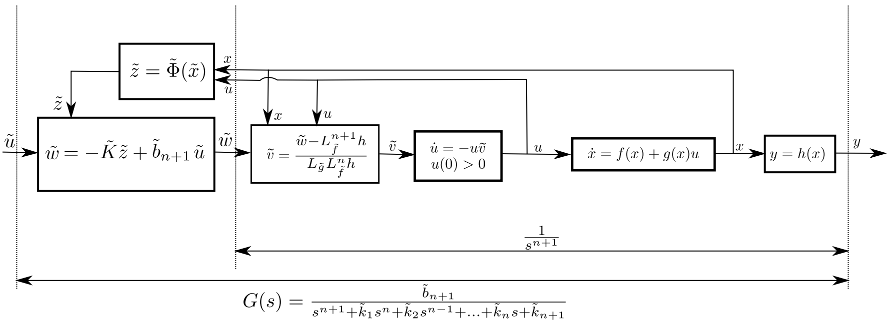

Figure 2
The positive input dynamics extension ensures positivity of the original input; the extended system is linearized: the result is the integrator series with transfer function $1/s^{n+1}$; finally the integrator series is transformed to $G(s)$ using internal loop-shaping

where the poles $p_1, p_2, \ldots, p_{n+1}$ are determined by the chosen coefficients $\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_{n+1}$. Provided that the poles are critically damped or real, the H_∞ norm of $G(s)$ is $\tilde{b}_{n+1}/\tilde{k}_{n+1} < \infty$, thus the model is appropriate as a nominal plant for H_2 or H_∞ controller synthesis.

In summary, the nonlinear plant is first augmented with positive input dynamics to guarantee input positivity, then converted into a chain of integrators, and finally reshaped into a stable linear system with finite H_∞ norm through internal loop-shaping, as illustrated in Figure 2.

Theorem 4. *Suppose that the original and perturbed models have maximal relative degree in a point x, design the dynamical extension (72), feedback linearization (77) and internal loop-shaping (87) for the nominal model and apply them for the perturbed model as shown in Figure 2. The linear perturbed model of the closed-loop system in Figure 2 in the presence of parametric uncertainties (which are as described in Subsection 2.1) is given by*

$$\dot{\hat{\tilde{z}}} = \tilde{A}\hat{\tilde{z}} + \tilde{B}\tilde{u} + e_{n+1}\varphi(\tilde{x}, \tilde{u}) \tag{92}$$

with $\tilde{A}$ and $\tilde{B}$ are as given in (88), e_{n+1} being the $(n+1)$th unit vector and

$$\begin{aligned} \varphi(\tilde{x}, \tilde{u}) \quad = \quad & \tilde{\psi}_{n+1}(\tilde{x}) + \sum_{i=2}^{n+1} \left(\tilde{\psi}_{i-1}(\tilde{x}) \tilde{k}_{n+2-i} \right) \\ & + \tilde{\rho}(\tilde{x}) \left(-\sum_{i=1}^{n+1} \tilde{\phi}_i(\tilde{x}) \tilde{k}_{n+2-i} + \tilde{b}_{n+1}\tilde{u} - L_{\tilde{f}}^{n+1}\tilde{h}(\tilde{x}) \right) \end{aligned} \tag{93}$$

where for all $i = 1, 2, \ldots, n+1$

$$\tilde{\psi}_i \quad = \quad L_{\hat{\tilde{f}}}^{i} \hat{\tilde{h}} - L_{\tilde{f}}^{i} \tilde{h} \tag{94}$$

$$\tilde{\phi}_i \quad = \quad L_{\tilde{f}}^{i-1} \tilde{h} \tag{95}$$

and

$$\tilde{\rho} = \frac{L_{\hat{\tilde{g}}} L_{\hat{\tilde{f}}}^{n} \hat{\tilde{h}} - L_{\tilde{g}} L_{\tilde{f}}^{n} \tilde{h}}{L_{\tilde{g}} L_{\tilde{f}}^{n} \tilde{h}}. \tag{96}$$

Proof. See [8]. □

Example 9. Consider the extended linearized model from Example 8, and apply the state feedback $w = -\tilde{K}\tilde{z} + \tilde{b}_3\tilde{u}$, where $\tilde{K} = (\tilde{k}_3, \tilde{k}_2, \tilde{k}_1)$. If the system parameters are perturbed, then according to Theorem 4, the resulting closed-loop dynamics can be expressed as a linear perturbed system

$$\dot{\hat{\tilde{z}}} = \tilde{A}\tilde{z} + \tilde{B}\tilde{u} + e_3\varphi(\tilde{x}, \tilde{u}), \tag{97}$$

where

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\tilde{k}_3 & -\tilde{k}_2 & -\tilde{k}_1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 \\ 0 \\ \tilde{b}_3 \end{pmatrix}. \tag{98}$$

The perturbation term is given by

$$\begin{aligned} \varphi(\tilde{x}, \tilde{u}) &= \tilde{\psi}_3(\tilde{x}) + \sum_{i=2}^{3} \left(\tilde{\psi}_{i-1}(\tilde{x})\tilde{k}_{4-i}\right) + \tilde{\rho}(\tilde{x}) \left(-\sum_{i=1}^{3} \tilde{\phi}_i(\tilde{x})\tilde{k}_{4-i} + \tilde{b}_3\tilde{u} - L_{\tilde{f}}^3\tilde{h}\right) \\ &= \tilde{\psi}_3(\tilde{x}) + \tilde{\psi}_2(\tilde{x})\tilde{k}_1 + \tilde{\psi}_1(\tilde{x})\tilde{k}_2 \\ &\quad + \tilde{\rho}(\tilde{x})(-\tilde{\phi}_1(\tilde{x})\tilde{k}_3 - \tilde{\phi}_2(\tilde{x})\tilde{k}_2 - \tilde{\phi}_3(\tilde{x})\tilde{k}_1 + \tilde{b}_3\tilde{u} - L_{\tilde{f}}^3\tilde{h}) \end{aligned} \tag{99}$$

After substituting the definitions of $\tilde{\psi}_i$ and $\tilde{\phi}_i$, this can be rewritten as

$$\begin{aligned} \varphi(\tilde{x}, \tilde{u}) &= L_{\hat{\tilde{f}}}^3\hat{\tilde{h}}(\tilde{x}) - L_{\tilde{f}}^3\tilde{h}(\tilde{x}) + \tilde{k}_1\left(L_{\hat{\tilde{f}}}^2\hat{\tilde{h}}(\tilde{x}) - L_{\tilde{f}}^2\tilde{h}(\tilde{x})\right) + \tilde{k}_2\left(L_{\hat{\tilde{f}}}\hat{\tilde{h}}(\tilde{x}) - L_{\tilde{f}}\tilde{h}(\tilde{x})\right) \\ &\quad - \tilde{\rho}(\tilde{x})\tilde{k}_3\tilde{h}(\tilde{x}) - \tilde{\rho}(\tilde{x})\tilde{k}_2 L_{\tilde{f}}\tilde{h}(\tilde{x}) - \tilde{\rho}(\tilde{x})\tilde{k}_1 L_{\tilde{f}}^2\tilde{h}(\tilde{x}) + \tilde{\rho}(\tilde{x})\tilde{u} - \tilde{\rho}(\tilde{x})L_{\tilde{f}}^3\tilde{h} \\ &= L_{\hat{\tilde{f}}}^3\hat{\tilde{h}}(\tilde{x}) - (1 + \tilde{\rho}(\tilde{x}))L_{\tilde{f}}^3\tilde{h}(\tilde{x}) + \tilde{k}_1\left(L_{\hat{\tilde{f}}}^2\hat{\tilde{h}}(\tilde{x}) - (1 + \tilde{\rho}(\tilde{x}))L_{\tilde{f}}^2\tilde{h}(\tilde{x})\right) \\ &\quad + \tilde{k}_2\left(L_{\hat{\tilde{f}}}\hat{\tilde{h}}(\tilde{x}) - (1 + \tilde{\rho}(\tilde{x}))L_{\tilde{f}}\tilde{h}(\tilde{x})\right) - \tilde{k}_3\tilde{\rho}(\tilde{x})\tilde{h}(\tilde{x}) + \tilde{\rho}(\tilde{x})\tilde{u}. \end{aligned}$$

The function $\tilde{\rho}(\tilde{x})$ is

$$\tilde{\rho}(\tilde{x}) = \frac{L_{\hat{\tilde{g}}} L_{\hat{\tilde{f}}}^2 \hat{\tilde{h}} - L_{\tilde{g}} L_{\tilde{f}}^2 \tilde{h}}{L_{\tilde{g}} L_{\tilde{f}}^n \tilde{2}} = \frac{\hat{b}x_1u - bx_1u}{bx_1u} = \frac{\hat{b} - b}{b}. \tag{100}$$

Assuming $\hat{b} = b(1 + \Delta b)$ with $\Delta b \neq -1$, we have $\tilde{\rho}(\tilde{x}) = \Delta b$.

Finally, the perturbation term can be expressed in terms of the Lie derivatives of the nominal vector fields as

$$
\begin{aligned}
\varphi(\tilde{x},\tilde{u}) \quad = \quad & \sum_{p_1=1}^{3}\sum_{p_2=1}^{3}\sum_{p_3=1}^{3}\left(\hat{\kappa}_{p_1}\hat{\kappa}_{p_2}\hat{\kappa}_{p_3}-(1+\Delta b)\,\kappa_{p_1}\kappa_{p_2}\kappa_{p_3}\right)L_{f_{p_1}}L_{f_{p_2}}L_{f_{p_3}}h(x) \\
& +\sum_{p_1=1}^{3}\sum_{p_2=1}^{3}\left(\hat{\kappa}_{p_1}\hat{\kappa}_{p_2}-(1+\Delta b)\,\kappa_{p_1}\kappa_{p_2}\right)L_gL_{f_{p_1}}L_{f_{p_2}}h(x)u \\
& +\sum_{p_1=1}^{3}\sum_{p_2=1}^{3}\left(\hat{\kappa}_{p_1}\hat{\kappa}_{p_2}-(1+\Delta b)\,\kappa_{p_1}\kappa_{p_2}\right)L_{f_{p_1}}L_gL_{f_{p_2}}h(x)u \\
& +\sum_{p_1=1}^{3}\left(\hat{\kappa}_{p_1}-(1+\Delta b)\,\kappa_{p_1}\right)L_g^2L_{f_{p_1}}h(x)u^2+ \\
& \tilde{k}_1\left(\sum_{p_1=1}^{3}\sum_{p_2=1}^{3}\left(\hat{\kappa}_{p_1}\hat{\kappa}_{p_2}-(1+\Delta b)\kappa_{p_1}\kappa_{p_2}\right)L_{f_{p_1}}L_{f_{p_2}}h(x)\right. \\
& \left.+\sum_{p_1=1}^{3}\left(\hat{\kappa}_{p_1}-(1+\Delta b)\kappa_{p_1}\right)L_gL_{f_{p_1}}h(x)u\right) \\
& +\tilde{k}_2\sum_{p_1=1}^{3}\left(\hat{\kappa}_{p_1}-(1+\Delta b)\kappa_{p_1}\right)L_{f_{p_1}}h(x) \\
& -\tilde{k}_3\Delta b x_1+\Delta b\tilde{u}:=\nu(x)+\Delta b\tilde{u}.
\end{aligned}
\tag{101}
$$

The controller gains $\tilde{k}_1,\tilde{k}_2,\tilde{k}_3$ can also be selected not only to ensure that the characteristic polynomial (90) is Hurwitz, but to minimize the influence of model uncertainties represented by $|\nu(x)|$. This provides an additional degree of freedom in the design, allowing the robustness of the closed-loop system to be increased by reducing the impact of parameter perturbations.

Conclusions

An analytical framework was developed for the exact linearization of nonlinear systems with parametric uncertainties and positive input dynamics. Equivalent linear perturbed models were derived to describe how parameter variations affect the system after feedback linearization and internal stabilization. Conditions were given for the invariance of the relative degree, and explicit formulas were obtained for the perturbed linear dynamics. The analysis also demonstrated how positive input dynamics can be incorporated into the controller structure to guarantee input nonnegativity while preserving the possibility of exact linearization.

The results provide a unified approach to handling uncertainty and positivity in nonlinear control design. The derived perturbed models form a basis for robust and adaptive controller synthesis, sensitivity evaluation, and fault detection. Moreover, recognizing that stabilization gains can reduce the effect of parametric perturbations introduces an additional tuning option to improve robustness. Future research may extend these findings to multi-input multi-output configurations and develop optimization-based methods for joint performance–robustness tuning.

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