Number of Minimal Paths in a Honeycomb Grid

Mousumi Dutt[1], Arindam Biswas[2] and Benedek Nagy[3,4]

[1] Department of Computer Science and Engineering, St. Thomas' College of Engineering and Technology, 4, Diamond Harbour Road, Kidderpore, Kolkata-700023, West Bengal, India, mousumi.dutt@stcet.ac.in

[2] Department of Information Technology, Indian Institute of Engineering Science and Technology, Shibpur, Howrah 711103, West Bengal, India, abiswas@it.iiests.ac.in

[3] Department of Mathematics, Eastern Mediterranean University, Shakespeare Street, 99628 Famagusta, North Cyprus, Mersin-10, Türkiye, benedek.nagy@emu.edu.tr

[4] Institute of Mathematics and Informatics, Eszterházy Károly Catholic University, 3300 Eger, Leányka utca 4, Hungary

Abstract: The computation to find the number of minimal-length paths from a point to other in honeycomb grid is presented in this paper. There are three kinds of neighborhood used in the honeycomb grid. We solve the shortest-path counting problem for two of the neighborhoods: in case of 1- and 2-neighborhoods, given the coordinate triplets of the two points, closed formulae are proven. The case of 3-neighborhood is also considered, as well as a brief discussion is also stated for the counting the minimal paths in other grids. The number of minimal paths in isometric grid is also proposed here. The problem has theoretical aspects and can be applied in networking and in digital image processing.

Keywords: hexagonal grids; combinatorics; path counting; digital distances; shortest paths

1 Introduction

Digital geometry is related to some mathematical disciplines like discrete mathematics and geometry, but it is mainly connected to other disciplines including crystallography, image processing, computer graphics and network theory. In the discrete space points are addressed by integer coordinates. The two usual neighborhood criteria are defined in digital geometry for the rectangular grid [1]—cityblock and chessboard. The cityblock step allows movements only on the gridlines: horizontally and vertically (4-neighborhood); at the chessboard step diagonal directions are also allowed (8-neighborhood). These two types of motions allow us to define two kinds of distances. A brief summary of studies on the rectangular grid is presented in [2, 3]. In this grid, the coordinates of a point are independent of each other. Generally, in n dimensions, to describe the rectangular grids, n coordinates are used that are independent of each other. In the n dimensional rectangular grid, the structure of the n-dimensional cubes is isomorphic to the structure of the nodes. These grids are the subjects of the 'Geometry of Numbers' [4, 5, 6, 7]. The concepts of 'array' and 'lattice' are also used which are synonymous with the concept 'grid', that we use here.

Digital geometry also involves the honeycomb (hexagonal) and isometric (triangular) grids in the studies. A relation among the cubic, isometric and honeycomb grids is established [8, 9, 10, 11, 12], and thus, three integer coordinates can be used on these grids to address their points (see Fig. 2 (right)). The hexagonal grid has various advantages for representing digital images over rectangular grid and used in image processing and computer vision [13, 14]. Here, we analyze the stated path counting problem on the honeycomb grid, which has 3 types of neighborhoods. Analogously, in [15], three types of neighborhood relation on the triangular grid are discussed along with the thinning algorithms on the three regular 2-dimensional grids. The honeycomb grid has 3 coordinates which are not independent [10, 16, 17, 18]. The length of any shortest paths connecting two points using a kind of neighborhood gives the digital distances of the two points. In these paths, in each step one moves to the next point that is the given type of neighbor of the previous point. Distances when the neighborhood can vary on a path based on so-called neighborhood sequences are studied: metrical properties [19], optimality [20, 21] and other properties are discussed [22, 23]. The weighted distance is given in [24, 25], where the distance function is defined using the three weights assigned to the types of neighbors. Binary tomography is developed in [26, 27, 28]. Rotations and other transformations are studied in [29, 30, 31]. Various advantages of non-traditional grids are highlighted in [32]. Shortest paths play important roles in various communication scenarios in networks. The honeycomb network was analyzed in [18, 33, 34, 35] from various points of view including its topology and algorithmic approaches.

The shortest paths have applications in various fields, e.g., networking (social networking, routing, etc.), robotics (finding shortest paths), VLSI floor planning (usage of minimum wires), geographic information systems, digital geometry, image analysis, etc. In discrete space and in networks, the shortest path is usually not defined uniquely. Similar problems are considered on various types of graphs since they are interesting combinatorial problems and also important for several applications in computer networks. This problem also has applications in inter-processor networks where alternative routes for communication are important. For instance, in [36], the path planning algorithm for robotics is proposed on a hexagonal grid map. Also, in digital grids (which can be seen as infinite graphs), their number between any two points could be more than one. Their number, e.g., inside a shape can reflect some properties of the shape on the figure. Path counting in digital images was already introduced in [1] as a technique to analyse images.

As a related combinatorial puzzle, see Fig. 1, and consider the question how many ways it is possible to read out the word 'RECTANGLE' by stepping only to the right and down. The recursive formulation to count the number of minimal paths (in the term of number of steps) in case of cityblock, chessboard, and octagonal paths, from a point to another in the 2D rectangular grid was computed in [37]. In [38], Das computed the number of shortest paths between each pair of points of digital images with various neighborhood criteria. He considered the images as matrices, therefore, his algorithm is based on matrix operations. A shortest grid path (isothetic, cityblock) from any point to another inside a digital object is presented in [39, 40]. The stated algorithm uses combinatorial technique to obtain the resulting shortest isothetic path. There could be more than one minimal isothetic path between any two given points. Thus, counting the shortest isothetic paths between any two given points becomes essential. An algorithm is shown in [41] to find out the total number of shortest isothetic paths between any two given points inside a digital image for the given grid size. Now, in this paper, we contribute to the field, namely we count the number of minimal paths in the honeycomb grid with all three neighborhoods. Our previous conference paper [42], in which the counting on the triangular grid is addressed, is extended and reformulated here, e.g., by considering 3-neighborhood and to shift the notation to the honeycomb grid. For the sake of completeness and comparisons, we also include similar results on the other regular grids, the isometric and rectangular the grids. For the latter, new direct proofs are presented here (comparing them to the proofs given in [37, 38]).

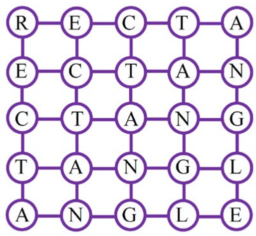


Figure 1
A way to read the word RECTANGLE is a shortest path from the corner 'R' to corner 'E'

This paper has the following structure. Section 2 presents the preliminaries of the honeycomb grids which are required to understand the problem. The formulation for the number of minimal paths for the three neighborhoods along with the proof of correctness are given in Sec. 3. This section also includes the combined formula for 1- and 2-neighborhoods. The discussion on the number of minimal paths for various grids (rectangular and isometric) is presented in Sec. 4. The number of minimal paths between two points in isometric grid is also given. Minimal paths with chessboard neighborhood in 2D rectangular grid are counted by Das [37, 38] using recurrence relations. Here, we present a direct proof that is shorter based on combinatorics. Concluding remarks are given in Sec. 5.

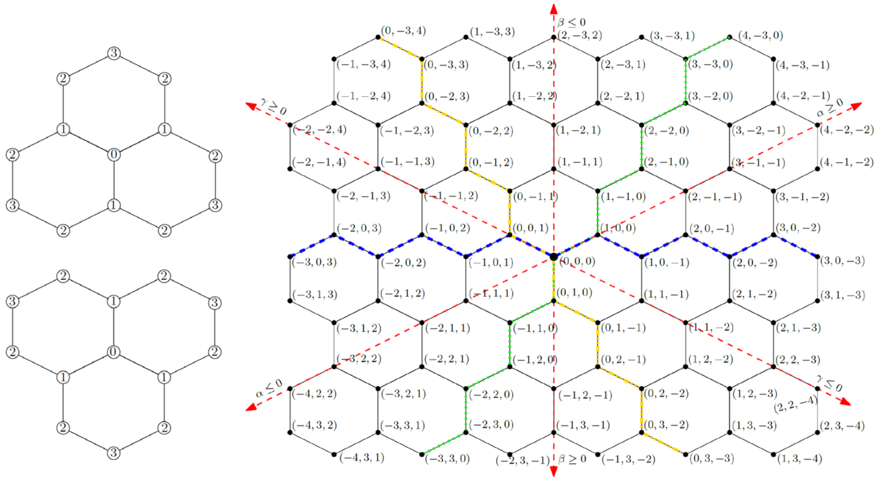


Figure 2

Types of neighbors in honeycomb grid and coordinate system with lanes in honeycomb grid

2 Basic Notions and Definitions

Some basic notations and definitions are recalled to be able to read the paper. There are 3 kinds of neighbors in the honeycomb and in the triangular grids [15, 16] as we show in Fig. 2 (left) for the latter. Every node (we also refer them as points) of the honeycomb grid has three edges. There are two orientations of the edges at a node: shape Y and upside-down Y. There are three 1-neighbors (connected directly by the edges), nine 2-neighbors (six 2-neighbors in the shorter diagonal positions additionally to the previous 1-neighbors), and there are twelve 3-neighbors (three of them in the longer diagonal positions additionally to the nine 2-neighbors) w.r.t. each hexagon.

As we have mentioned, we use three coordinate values to address the points. One point in the honeycomb grid is the origin with coordinate triplet (0,0,0). There are 3 lines through the origin, in the directions of the edges connected to it: they are the coordinate axes γ, β, and α (as they are shown Fig. 2 (right)). The coordinate triplet assigned to a node is computed from the triplet of another node based on the step, which is parallel to an axis. In case the movement is in the same direction as the axis, then the coordinate value is incremented by 1, however, if the step and the axis are in the opposite direction, then the coordinate is decremented by 1, respectively. Starting from the origin, by this method, all nodes get their unique coordinate triple [16, 18, 29]. The honeycomb grid is 2-dimensional, therefore, the 3 coordinates depend on each other. A point is called odd (of shape upside-down Y) and even (of shape Y) based on the sum of their coordinates: it is either 1 or 0, respectively.

Let E and F be 2 grid points of the honeycomb grid. They are h-neighbors ($h \in \{1, 2, 3\}$), if both the next conditions are fulfilled:

1. $|E(i) - F(i)| \leq 1$ for $i \in \{1, 2, 3\}$, and (1)
2. $|E(1) - F(1)| + |E(2) - F(2)| + |E(3) - F(3)| \leq h$. (2)

In case for a value of h, there is an equality in Eqn. (2), then, the points are in strict h-neighborhood. If one of the points (E or F) is not the origin, then translation or reflection can be used (isometric transformations are described in [29]).

The set points that are sharing one of their coordinate values, i.e., z, y, or x-coordinate, is a *lane*. Every lane is perpendicular to a coordinate axis, as the corresponding coordinate is fixed for these points. When stepping to neighbors in a lane, the other two coordinates are changing by ±1. When two points do not have any common coordinate value, then they can be connected using 2 lanes with angle $\frac{2}{3}\pi$ between.

By the symmetry of the grid, we cut it basically to 6 parts. These sextants are called direct and indirect sextants: if two of the coordinates of a point are positive, then it in an indirect sextant. Actually, by the properties of the coordinate system, the third value is necessarily negative in these cases. On the other hand, if a point has two negative and a positive coordinate, then it is in a direct sextant. Thus, the honeycomb grid built up by three lanes that go through the origin and six pairwise disjoint sextants. Two of these lanes border two of the sextants (in fact, a sextant and half of these lanes contain neighbor points).

In case none of the points for those we are doing the computation is the origin, we may need to do some transformations of the honeycomb grid. Assume that E is not the origin of the grid. In case E is an even point, e.g., (x,y,z) with $x+y+z=0$, a translation of the grid via vector $(-x,-y,-z)$ helps: each node $(u\ v,w)$ is translated to $(u - x,\ v - y,\ w - z)$ [29]. On the other hand, it could be that E is odd point. Assume that the odd point E' has coordinate triplet (x,y,z) with $x + y + z = 1$. It is transformed to the origin by a reflection [29], and any point F' having, e.g., coordinate triplet (u,v,w) will be transformed to F with the same transformation, where the coordinates of F are $(x - u,\ y - v,\ z - w)$. Both these transformations are isometric: the structure, the distance and the number of shortest paths are kept. Therefore, in the followings, wlog., we consider only counting the minimal paths between the origin $E(0,0,0)$ with other points.

3 Formulation of the Number of Minimal Paths

Let us count the number of minimal paths between two points (let them be E and F) in the honeycomb grid. The result depends on the used neighborhood. As we have described above, without loss of generality, let E be the origin addressed by (0,0,0) and F be another point, let its coordinates be (u,v,w). As we have mentioned, there are 3 lanes through E (see in Fig. 2). Let $f_h(u,v,w)$ be the number of minimal paths from E to F by using h-neighborhood.

From the symmetry of the honeycomb grid, it can be said that $f_h(u,v,w) = f_h(u,w,v) = f_h(v,u,w) = f_h(v,w,u) = f_h(w,u,v) = f_h(w,v,u)$. So only the values of a coordinate triplet are important. In each sextant two of the coordinates have the same sign—either negative or positive, and the third value has the opposite sign, this latter coordinate is referred as the *prime coordinate*. The former two (with the same sign), are referred as *secondary coordinates*. In the top-most sextant of Fig. 2(right) $u \geq 0$ and $w \geq 0$ with $v \leq 0$. Therefore, v is the prime coordinate (corresponding to axis β.). The coordinates u and w are secondary ones. A sextant is referred by its prime coordinate, thus the previously described sextant is the v-indirect sextant. Further, the value of the prime coordinate cannot be smaller than the values of a secondary one (by taking their absolute values). It is implied that the secondary coordinates play an important role in counting the number of minimal length paths based on 1-neighborhood.

The computation of the number of minimal paths by 1-neighborhood, 2-neighborhood, and 3-neighborhood are shown below along with the proof of correctness. We determine the number of shortest paths in 2-neighborhood based on the formulation for 1-neighborhood. A combined formula for 1- and 2-neighborhoods are also provided.

3.1 The Case of 1-Neighborhood

The number of minimal paths by 1-neighborhood for points with various coordinates are presented in Fig. 3. It is to be noted that if F lies in one of the 3 lanes that pass through E, then there is a unique minimal path between F and E in 1-neighborhood.

Theorem 1. Let $f_1(u,v,w)$ be the number of minimal length paths from $E(0,0,0)$ to $F(u,v,w)$ by 1-neighborhood in the honeycomb grid. Then,

$$f_1(u,v,w) = \binom{|e|+|d|}{|e|}, \quad \begin{array}{l} \text{where} \quad e,d \geq 0 \text{ or } e,d \leq 0, \\ e \in \{w,u,v\}, d \in \{w,u,v\} \backslash \{e\} \end{array} \tag{3}$$

Proof. We prove the formula by induction on the length of the paths for both the odd and even points. In both cases for the points of the mentioned lanes and for the points of the (indirect and direct) sextants. Based on that we have 5 cases.

Case 1: We start with points on the lanes. It is easy to see that any minimal path connecting the origin E to a point with triplet $(u,v,0)$ or $(0,v,w)$ with $v > 0$ has exactly $|u|+v$ or $v+|w|$ steps, respectively. Actually, in these cases, the minimal path is unique, going neighbor to neighbor point of the considered lane. Applying Equation 3, the result is $\binom{|u|}{0} = 1$ (or $\binom{|u|}{|u|} = 1$, or $\binom{v}{0}$, or equivalently, $\binom{v}{v} = 1$) and

$\binom{|w|}{0} = 1$ (or $\binom{|w|}{|w|}$, or $\binom{v}{0}$, or equivalently, $\binom{v}{v} = 1$), respectively. Thus, by using the symmetry of the grid, equation (3) has been proven for all points on the 3 lanes going through on (0,0,0).

Case 2: Let us consider now points of the direct sextants. The proof is shown for the v-direct sextant, and based on symmetry, it must hold for the all direct sextants. The proof goes by induction based on the sum of absolute values of secondary coordinates of the sextant. Considering first, an even point (u,v,w) of the v-direct sextant, the conditions $v > 0$ and $u,w < 0$ must hold. In a minimal length path starting at E, we can reach (u,v,w) from $(u,v,w+1)$ or from $(u+1,v,w)$. There are no other possibilities. Therefore, $f_1(u,v,w) = f_1(u,v,w+1)+f_1(u+1,v,w) = \binom{|u|+|w|-1}{|u|} + \binom{|u|+|w|-1}{|u|-1} = \binom{|u|+|w|}{|u|}$. Notice that both points $(u,v,w+1)$, $(u+1,v,w)$ are odd.

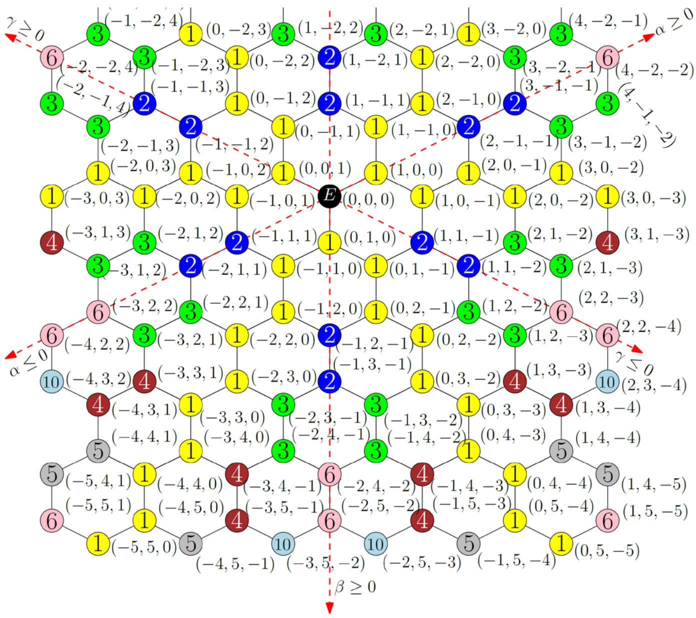


Figure 3

Number of minimal paths (inside the circles) by 1-neighborhood

Case 3: Let us see what is the case if the point is odd in this sextant, i.e., $v>0$ and $u,w<0$. Now, in a minimal path from E, the last step of the path can be only from the even point $(u,v-1,w)$ to the point (u,v,w), therefore, $f_1(u,v,w) = f_1(u,v-1,w)$.

Notice that we can use the already proven cases for the lanes, where one of the coordinates is 0. One can see that the binomial coefficients appear in the form of a Pascal's triangle. However, every value appears twice assigned to and shared by an even point and one of its odd point neighbors, right below.

Case 4: Consider now the v-indirect sextant. Here, $v<0$ and $u,w>0$. As for the basis of the induction, consider points with a 0 coordinate value and see that the proof for points on the lanes that are bordering this sextant is already shown. Further, consider the odd points inside the sextant. In a minimal path from E, to reach the point (u,v,w) (with $u+v+w=1$) the last step can be from $(u,v,w-1)$ or from $(u-1,v,w)$. There are no other possibilities. Therefore, $f_1(u,v,w) = f_1(u,v,w-1)+f_1(u-1,v,w)$. Notice that both these points are even.

Case 5: Considering even points inside the v-indirect sextant, in a minimal path starting at E, the path goes through and reaches the odd point $(u,v-1,w)$ just one step before reaching (u,v,w). Therefore, $f_1(u,v,w) = f_1(u,v-1,w)$. Thus, the proof based induction shows also the binomial coefficients in this case in a very similar manner as we have seen in the j-direct sextant. ■

3.2 The Case of 2-Neighborhood

The number of minimal paths with 2-neighborhood will be shown as a function of the 1-neighborhood result; while some specific values are shown in Fig. 4.

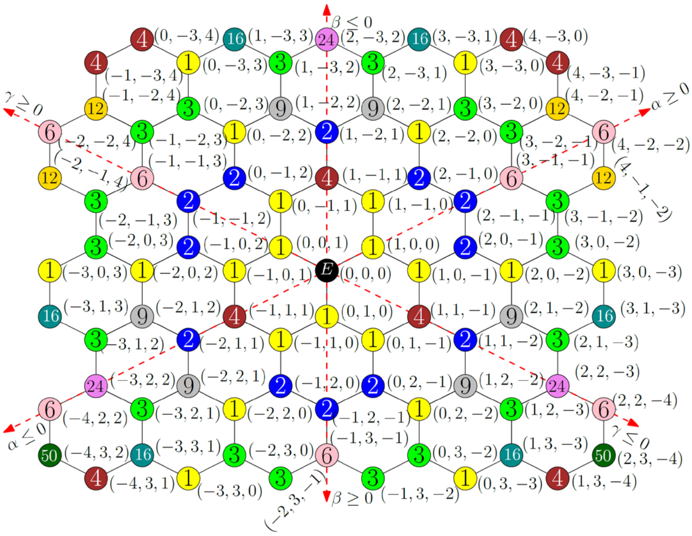
Theorem 2. Let $f_2(u,v,w)$ be the number of minimal length paths between two points $F(u,v,w)$ and $E(0,0,0)$ by 2-neighborhood and computed as follows.

$$f_2(u,v,w) = \alpha \times f_1(u,v,w), \quad \begin{array}{ll} \alpha = 1, & \text{when } w+v+u=0; \\ \alpha = \max\{|w|,|v|,|u|\}, & \text{when } w \times v \times u \geq 0; \\ \alpha = \max\{|w|,|v|,|u|\}+1, & \text{when } w \times v \times u < 0. \end{array} \tag{4}$$

i.e., for even points $f_2(u,v,w)$ is the same as $f_1(u,v,w)$ and for odd points in the direct sextants and on the lanes passing through the origin it is multiplied by the largest absolute value of the coordinates. Finally, for points in the indirect sextants that are odd, $f_1(u,v,w)$ is multiplied by the sum of the absolute values of the secondary coordinates.

Proof. The proof in the case of 2-neighborhood goes by cases for odd and even points, and by their locations: we prove for the lanes through (0,0,0), and for the v-indirect and v-direct sextants. Thus, altogether there are 2×3 cases based on the parity and location. Since the grid is rotational invariant, the proof for other locations follows immediately.

Generally, in a minimal path from E with 2-neighborhood to an even point all steps change exactly two coordinates (i.e., the path built up by strict 2-steps). On the other hand, to an odd point starting at E, there is exactly one 1-step in a minimal path, the other steps must change 2 coordinates (see, e.g., [10, 17]).



Number of minimal paths by 2-neighborhood

Case 1: Consider, first, an even point on any of the lanes passing through E (0,0,0). Clearly, any minimal path starting at E to a point with coordinates $(u,v,0) = (-v,v,0)$ or $(0,v,w) = (0,v,-v)$ with $v > 0$ built up by v steps, respectively. In fact, there is a unique minimal path in these cases in which steps to strict 2-neighbors are used in the lane. From Equation (4), in these cases, the result is also $\binom{|u|}{0} = 1$ and $\binom{|w|}{0} = 1$ that is exactly the same value that we have in the case of 1-neighborhood.

Case 2: Next, we do the computation for odd points in the lanes passing through E. Let the point be $(u,v,0)$ or $(0,v,w)$ with $v>0$ that is $(1-v,v,0)$ or $(0,v,1-v)$, respectively. A minimal path starting at E and ending at $(1-v,v,0)$, one arrives to there either from $(1-v,v-1,0)$ or from $(2-v,v-1,0)$. In a similar manner, to $(0,v,1-v)$, the last step is either from $(0,v-1,1-v)$ or from $(0,v-1,2-v)$ in a minimal path; it cannot be otherwise. Therefore, the number of minimal paths between E and these odd points is computed as $f_2(0,v,w) = f_2(0,v-1,w)+f_2(0,v-1,w+1)$ and $f_2(u,v,0) = f_2(u,v-1,0)+f_2(u+1,j-v,0)$. In both of these formulae, the first parts count minimal paths to even points and the other parts count shortest paths to odd points which can be expanded similarly. Using induction of Equation (4), $f_2(0,v,w) = f_1(0,v-1,w) + f_1(0,v-1,w+1)\times(v-1)$ and $f_2(u,v,0)=f_1(u,v-1,0)+f_1(u-1,v-1,0)\times(v-1)$, respectively. Thus, $f_2(0,v,w) = \binom{0+|w|}{|w|} + \binom{0+|w|-1}{|w|-1} \times (v-1)$ or $f_2(u,v,0) = \binom{|u|-1+0}{|u|-1} + \binom{|u|-1+0}{|u|-1} \times (v-1)$, i.e., $f_2(0,v,w)=1+1\times(v-1)$ or $f_2(u,v,0)= 1+1\times(v-1)$. Hence, $f_2(0,v,w) = v$ and also $f_2(u,v,0) = v$. Therefore, applying symmetry of the grid, Equation (4) has been proven for all points on the three lanes passing through the origin.

Case 3: Now, let an even point $F(u,v,w)$ be given in v-direct sextant ($v>0$) and compute the number of minimal paths starting at E. Any of these paths built up by strict 2-steps, therefore, only even points are contained in them. There could be 2 points from which $F(u,v,w)$ is reached in the last step: from $(u,v-1,w+1)$ or from $(u+1,v-1,w)$. Therefore, to count these paths we have $f_2(u,v,w)= f_2(u,v-1,w+1)+f_2(u+1,v-1,w)$. Both terms are formulae for even points. Therefore, $f_2(u,v,w)=f_1(u,v-1,w+1)+f_1(u+1,v-1,w)$, i.e., $f_2(u,v,w)=\binom{|u|+|w|-1}{|u|}+\binom{|u|-1+|w|}{|w|-1}=\binom{|u|+|w|}{|u|}=f_1(u,v,w)$.

Case 4: In a minimal path starting at E, we may reach the odd point (u,v,w) of the v-direct sextant in the last step from $(u,v-1,w+1)$ or $(u,v-1,w)$ or $(u+1,v-1,w)$. Therefore, $f_2(u,v,w)=f_2(u,v-1,w+1)+f_2(u+1,v-1,w)+f_2(u,v-1,w)$. Two terms represent values for odd points but the last term represents a value corresponding to an even point. Applying Equation (4), $f_2(u,v,w)=f_1(u,v-1,w+1)\times(v-1)+f_1(u+1,v-1,w)\times(v-1)+ f_1(u,v-1,w)$. Therefore, $f_2(u,v,w)=$

$(v-1)\times\left(\binom{|u|+|w|-1}{|u|}+\binom{|u|-1+|w|}{|u|-1}\right)+f_1(u,v-1,w)=(v-1)\times\binom{|u|+|w|}{|u|}+\binom{|u|+|w|}{|u|}=v\times\binom{|u|+|w|}{|u|}=v\times f_1(u,v,w)$. Observe that the formula is the one we intended to prove for this case.

Case 5: Next, consider minimal paths starting at E and finishing at even point (u,v,w) in the v-indirect sextant ($v<0$). The number of them is counted as $f_2(u,v,w)=f_2(u-1,v+1,w)+f_2(u,v+1,w-1)$ because in the last step (u,v,w) is reached from $(u-1,v+1,w)$ or $(u,v+1,w-1)$. Both parts are values for even points. Consequently, $f_2(u,v,w)=f_1(u-1,v+1,w)+f_1(u,v+1,w-1)$, i.e., $f_2(u,v,w)=\left(\binom{u-1+w}{u-1}+\binom{u+w-1}{w}\right)=\binom{u+w}{u}=f_1(u,v,w)$.

Case 6: Finally, consider the only remaining case. Let (u,v,w) be an odd point inside the j-indirect sextant. It can be reached by a minimal path only from the following points in the last step: $(u-1,v,w)$, $(u,v,w-1)$, $(u-1,v+1,w)$ and $(u,v+1,w-1)$. Consequently,
$f_2(u,v,w)=f_2(u-1,v,w)+f_2(u,v,w-1)+f_2(u-1,v+1,w)+f_2(u,v+1,w-1)$. Two parts are counting paths for even points and the other two parts are counting paths for odd points. Applying Equation (4),
$f_2(u,v,w)=f_1(u-1,v,w)+f_1(u,v,w-1)+f_1(u-1,v+1,w)\times(u-1+w)+$
$f_1(u,v+1,w-1)\times(u+w-1)$. Therefore, $f_2(u,v,w)=\binom{u-1+w}{u-1}+\binom{u+w-1}{u}+$

$\binom{u-1+w}{u-1} \times (u-1+w) + \binom{u+w-1}{u} \times (u+w-1)$. Thus, $f_2(u,v,w) = \binom{u+w}{u} + (u+w-1) \times \left(\binom{u-1+w}{u-1} + \binom{u+w-1}{u}\right) = \binom{u+w}{u} + (u+w-1) \times \binom{u+w}{u} = \binom{u+w}{u} \times (u+w-1+1) = f_1(u,v,w) \times (u+w)$. In the *v*-indirect sextant, for any odd point $u + v + w = 1$ and $|v| + 1 = u + w$, since $v<0$. Hence, $f_2(u,v,w) = f_1(u,v,w) \times (|v|+1)$.

In similar way, the proof can be extended to other sextants including *u*-indirect, *w*-indirect, *u*-direct and *w*-direct sextants. ■

In the next corollary we will make a connection of the results shown in Theorems 1 and 2.

Corollary 1. Let $f(u,v,w)$ be the number of minimal paths between points $F(u,v,w)$ and $E(0,0,0)$ in β-neighborhood where $\beta \in \{1,2\}$ and is defined as follows,

$$f(u,v,w) = \binom{|c|+|d|}{|c|} \times \alpha \times \beta, \quad \begin{array}{ll} \text{where} & c,d \geq 0 \text{ or } c,d \leq 0; \\ c \in \{u,v,w\}, & d \in \{u,v,w\} \backslash \{c\} \\ \alpha = 1, & \text{when } \beta = 1. \\ \text{Further,} & \text{when } \beta = 2: \\ \alpha = \frac{1}{2}, & \text{when } u+v+w = 0; \\ \alpha = \frac{max\{|u|,|w|,|v|\}}{2}, & \text{when } u \times v \times w \geq 0; \\ \alpha = \frac{max\{|u|,|w|,|v|\}+1}{2}, & \text{when } u \times v \times w < 0. \end{array} \quad (5)$$

Proof. When the value of β is 1, $f(u,v,w)$ is equal to the number of minimal paths between two points in 1-neighborhood as given in Eqn. (3), i.e., $f(u,v,w)=f_1(u,v,w)$ if $\beta = 1$. The proof of this formula is already shown for Theorem 1. The number of shortest paths between two points, $f(u,v,w)$ is equal to the number of minimal paths between two points in 2-neighborhood as given in Eqn. (4) when the value of β is 2, i.e., $f(u,v,w) = f_2(u,v,w)$, if $\beta = 2$. The resulting formula with its proof is already proposed in Theorem 2. ■

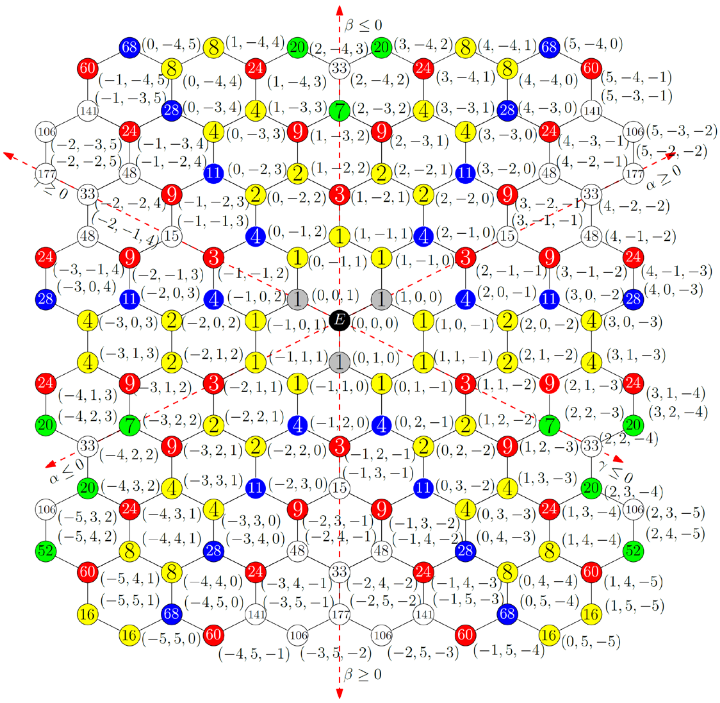
3.3 The Case of 3-Neighborhood

Path counting in 3-neighborhood is formulated here in two parts— for boundary points and other points. The formulations are given as follows.

Theorem 3. Let $f_3(u,v,w)$ be the number of minimal paths between two points $F(u,v,w)$ and $E(0,0,0)$ in 3-neighborhood. The formulation for the boundary points, i.e., points close to the lanes separating the sextants, (see colored nodes in Fig. 5) is given below.

$$f_3(u,v,w) = \begin{cases} 2^a, & \text{where } a,b \in \{w,v,u\}, a=b=0, \{w,v,u\}\backslash\{b,a\}=\{1\} \\ 2^{|a|-1}, & \text{where } a,b \in \{w,v,u\}, |a|=|b| \neq 0, \{w,v,u\}\backslash\{b,a\} \in \{0,1\} \\ \lceil 3 \times a \times 2^{a-3} \rceil, & \begin{array}{l} \text{where } a \in \{u,v,q\}, min\{|w|,|v|,|u|\}=1, \\ w+v+u=0 \text{ and } max\{|w|,|v|,|u|\}=a; \end{array} \\ \lfloor 3 \times a \times 2^{a-3} + 2^{a-2} \rfloor, & \begin{array}{l} \text{where } a \in \{w,v,u\}, \ |v| \neq |u|, \ |u| \neq |w|, \\ |v| \neq |w|, \ min\{|w|,|v|,|u|\}=0, \\ w+v+u=1 \text{ and } max\{|w|,|v|,|u|\}=a; \end{array} \\ 3 \times a \times 2^{|a|-3} - 2^{|a|-2}, & \begin{array}{l} \text{where } a \in \{u,v,w\}, min\{|w|,|v|,|u|\}=2, \\ w+v+u=1 \text{ and } min\{w,v,u\}=a \leq -3. \end{array} \end{cases} \quad (6)$$

Proof. For 3-neighborhood, the proof can be done for even and odd points using induction. The cases are by the locations of the points shown in various colors in Fig. 5.



Number of paths with minimal length by 3-neighborhood

Case 1: Let us first consider the gray points (see Fig. 5). Three such points are there which can be reached from origin by one and only one path only. These three points are in 1-neighborhood of *E*. Two of the coordinates are zero. The value as per Eqn. (6) is always $2^0 = 1$.

Case 2: Now, let us consider the yellow points in Fig. 5. Nine yellow points are in 3-neighborhood of E, i.e., those points can be reached by an appropriate single step. The values at other yellow points (whether even or odd) which can be reached by two steps are dependent on the values of the two of their neighbors. This yields the value 2. The yellow points which can be reached by appropriate three steps are actually the sum of values of the two points which are reachable by appropriate two steps. Thus, the value is 4. This leads to exponential growth in the values — powers of two. All the yellow points are at the boundary of each sextants.

Case 3: Let us consider the red points (Fig. 5), which are even points. In v-indirect sextant, a red point is reached from three of its neighboring points. Let us consider the red point, F, in v-indirect sextant which can be reached by an appropriate two steps from the origin. If the coordinate triplet of F is (u,v,w), then it can be reached by an appropriate single step from $F_1 = (u-1,v+1,w)$, $F_2 = (u,v+1,w)$, and $F_3(u,v+1w-1)$, which are yellow points. As F is in the v-indirect sextant and red point, $\max\{|u|,|v|,|w|\} = |v|$ and F_1, F_2, and F_3 are exactly at single step distant from origin. So, F can be reached by exactly two steps from origin by $f_3(u-1,v+1,w) + f_3(u,v+1,w) + f_3(u,v+1,w-1) = 1+1+1 = 3$ ways, i.e., $3 \times 2^{|a|-1} = 3 \times 2^{|v|-1-1} = 3 \times 2^{|v|-2}$ as $v < 0$ as per Eqn. (6). Let $F' = (u+1,v-1,w)$ be the point which can be reached by exactly two steps from origin and by an appropriate single step from F, $F'_2(u+1,v,w)$, $F'_3(u+1,v,w-1)$, and $F'_4(u,v,w+1)$ (from one red point and three yellow points). Here, $\max\{|u+1|,|v|+1,|w|\} = |v|+1$ as $v < 0$. $f_3(u+1,v,w) = f_3(u+1,v,w-1) = f_3(u,v,w+1) = 2^{|v|-1}$ (Eqn. 6). $f_3(u+1,v-1,w) = 3 \times 2^{|v|-2} + 3 \times 2^{|v|-1} = 3 \times 2^{|v-1|-3} + 3 \times 2^{|v-1|-2} = 3 \times |v-1| \times 2^{|v-1|-3}$. Let $F'' = (u'',v'',w'')$ be a red point in v-indirect sextant which can be reached exactly by w steps in 3-neighborhood. Then, F'' can be reached from four points which are exactly $w-1$ steps from origin. Let $F''_1 = (u''-1,v''+1,w'')$, $F''_2 = (u'',v''+1,w'')$, $F''_3 = (u'',v''+1,w''-1)$, and $F''_4 = (u''-1,v''+1,w''+1)$ be such points (one red point, one green point, and two yellow points). So, F'' can be reached by an appropriate two steps from origin by $f_3(u''-1,v''+1,w'') + f_3(u'',v''+1,w'') + f_3(u'',v''+1,w''-1) + f_3(u''-1,v''+1,w''+1) = 3 \times |v''| \times 2^{|v|-3}$ ways. Similarly, it can be proven for other sextants also.

Case 4: Now consider the blue points (odd points) which has one zero value in coordinate triplet. Let $F = (u,v,0)$ be an odd point in j-indirect sextant and $u > 0 > v$, $u - |v| = 1$. F can be reached from origin by an appropriate u steps and $F_1 = (u-1,v,1)$, $F_2 = (u-1,v,0)$, $F_3 = (u-1,v+1,0)$, and $F_4 = (u-1,v+1,-1)$ be such points which can be reached by an appropriate $u-1$ steps from origin. Thus, the value for F will be $f_3(u-1,v,1) + f_3(u-1,v,0) + f_3(u-1,v+1,0) + f_3(u-1,v+1,-1)$. The red point at $(u,v,-1)$ can be reached from F_2, F_3, and F_4 by an appropriate single step. The value for the red point will be $f_3(u-1,v,0) + f_3(u-1,v+1,0) + f_3(u-1,v+1,-1) = 3 \times u \times 2^{u-3}$ as $\max\{u,|v|,|-1|\} = u$ by Eqn. (6). F_1 is the yellow point which has the value $2^{|v|-1} = 2^{u-2}$ as $|v| = u-1$. Hence, the value for F will be $3 \times u \times 2^{u-3} + 2^{u-2}$.

Case 5: Now consider the green points (odd points) which has one value '2' in their coordinate triplet. Let $F = (u,v,w)$ be an odd point in j-indirect sextant and $u = 2$ or $w = 2$, $\max\{|u|,|v|,|w|\} = |v|$. F can be reached from origin exactly by $|v|$ steps and $F_1 = (u-1,v+1,w)$, $F_2 = (u-1,v+1,w-1)$, and $F_3 = (u,v+1,w-1)$ be such points which can be reached by an appropriate $|v| - 1$ steps from origin (on yellow, one red, one green). Thus, the value for F will be $f_3(u-1,v+1,w) + f_3(u-1,v+1,w-1) + f_3(u,v+1,w-1)$. The red point at $(u,v,w-1)$ or $(u-1,v,w)$ can be reached from F_1, F_2, F_3, and one more yellow point by an appropriate single step. The value for the green point will be $3 \times u \times 2^{u-3} - 2^{u-2}$. ■

In Fig. 5, the formulation for the boundary points are shown in five different colors for each condition in Eqn. (6). The first, second, third, fourth, and fifth conditions in Eqn. (6) are shown in gray, yellow, red, blue, and green color respectively.

For other points (white points in Fig. 5), the total number of minimal paths in 3-neighborhood can be generated as sum of the number of shortest paths for previous neighboring points. In indirect sextant, for an odd point, the number of minimal paths is sum of the paths for its three neighbors, whereas for an even point, it is sum of the paths for its five neighbors. For direct sextant it is just the reverse of the indirect sextant. The generating function for the v-indirect and v-direct sextants are given below.

Theorem 4. Let $f_3(u,v,w)$ be the number of shortest paths between two points $F(u,v,w)$ and $E(0,0,0)$ in 3-neighborhood. The formulation for the points other than boundary points for the v-direct and v-indirect sextants (see the white balls in Fig. 5) is given below.

$$f_3(u,v,w) = \begin{cases} f_3(u-1,v+1,w+1) + f_3(u-1,v+1,w) + f_3(u,v+1,w) \\ \quad +f_3(u,v+1,w-1) + f_3(u+1,v+1,w-1), \text{where} \\ \quad u+v+w=0 \text{ and } w,u \geq 0, v \leq 0. \\ f_3(u-1,v+1,w) + f_3(u-1,v+1,w-1) + f_3(u,v+1,w-1) \\ \quad \text{where } u+v+w=1 \text{ and } w,u \geq 0, v \leq 0. \\ f_3(u+1,v-1,w-1) + f_3(u+1,v-1,w) + f_3(u,v-1,w) \\ \quad +f_3(u,v-1,w+1) + f_3(u-1,v-1,w+1), \text{where} \\ \quad u+v+w=0 \text{ and } w,u \leq 0, v \geq 0. \\ f_3(u+1,v-1,w) + f_3(u+1,v-1,w+1) + f_3(u,v-1,w+1) \\ \quad \text{where } u+v+w=1 \text{ and } w,u \leq 0, v \geq 0. \end{cases} \tag{7}$$

Proof. For white points the number of minimal paths is generated recursively by using the Eqn. (7). Let us consider an even white point $F(u,v,w)$ which can be reached from E by appropriate $|v|$ steps. F can be reached by exactly single step from $F_1(u-1,v+1,w+1)$, $F_2(u-1,v+1,w)$, $F_3(u,v+1,w)$, $F_4(u,v+1,w-1)$, and $F_5(u+1,v+1,w-1)$ which can be reached exactly by $|v| - 1$ steps from origin. Thus, $f_3(u,v,w) = f_3(u-1,v+1,w+1)+f_3(u-1,v+1,w) + f_3(u,v+1,w)+f_3(u,v+1,w-1) + f_3(u+1,v+1,w-1)$. Let us consider an odd white point $F(u,v,w)$ which can be reached from E exactly by $|v|$ steps. F can be reached by an appropriate single step from $F_1(u-1,v+1,w)$, $F_2(u-1,v+1,w-1)$, and $F_3(u,v+1,w-1)$ which can be reached by exactly $|v|-1$ steps from origin. Thus, $f_3(u,v,w) = f_3(u-1,v+1,w) + f_3(u-1,v+1,w-1) + f_3(u,v+1,w-1)$. ■

Example 1. Let us consider the even point (2, −4, 2) in *v*-indirect sextant. $f_3(2,-4,2) = f_3(1,-3,3)+f_3(1,-3,2)+f_3(2,-3,2)+f_3(2,-3,1)+f_3(3,-3,1)$. All the points are at boundary, values can be put according to the Eqn. (6). $f_3(2,-4,2) = 4+9+7+9+4 = 33$. Now, consider the odd point (3, −5, 3) in *v*-indirect sextant. $f_3(3,-5,3) = f_3(2,-4,3) + f_3(2,-4,2) + f_3(3,-4,2)$. The first and third points are at boundary, so we can directly calculate their values. The middle point is not at boundary, it can be expanded recursively until we will get boundary values. Thus, $f_3(3,-5,3) = 20+f_3(1,-3,3)+f_3(1,-3,2)+f_3(2,-3,2)+f_3(2,-3,1)+f_3(3,-3,1)+20 = 20+4+9+7+9+4+20 = 73$. The above formulation for other sextants can be done similarly.

4 Counting Minimal Paths in Other Grids

The number of minimal paths between two points in the other two regular grids are discussed here for the sake of completeness and comparisons. There are differences also in the number of neighborhoods. First, we consider the rectangular grid and then, the isometric grid. The former has two neighborhoods—city block (4-neighborhood) and chessboard (8-neighborhood). The explanation for the number of minimal paths in rectangular grids for the two neighborhoods are given in [37, 38]. A portion of enumeration of the number of minimal paths between two points in 4-neighborhood is shown in Fig. 6 and that of 8-neighborhood is shown in Fig. 7.

Let $f_{r-4}(i,j)$ be the number of shortest paths from (i,j) to (0, 0) considering 4-neighborhoods, i.e., grid paths in this case, in rectangular grid. The number of shortest paths between $p(i,j)$ to $q(0,0)$ is given below.

$$f_{r-4}(i,j) = \binom{|i|+|j|}{|i|} \tag{8}$$

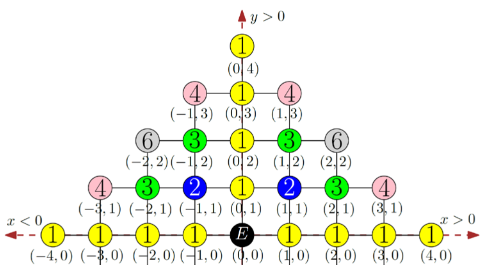


Figure 6

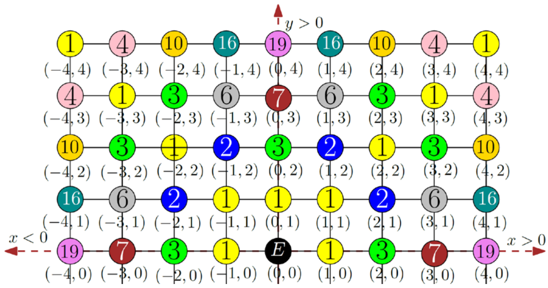


Figure 7

The result for the number of shortest paths in 8-neighborhood in rectangular grid has already been proven by Das [37, 38] with recurrence relations. Here, we show a shorter direct proof with combinatorial tools for the formulation of 8-neighborhood in rectangular grid as follows.

Theorem 5. Let $f_{r\text{–}8}(i,j)$ be the number of shortest paths from the origin $q(0,0)$ to any other points $p(i,j)$ in 2D rectangular grid in 8-neighborhood and

$$f_{8-r}(i,j) = \sum_{l=0,\, r-l=|i|,\, r\geq l}^{r+l\leq |j|} \frac{|j|!}{l!\, r!\, (|j|-r-l)!}, \quad \text{where } |j| \geq |i| \tag{9}$$

Proof. Consider the two points $p(i,j)$ and $q(0,0)$ for which the number of shortest paths in 8-neighborhood in 2D has to be proved. Let $|j| \geq |i|$. The distance between p and q is $\max\{|i|,|j|\}$. Here, the distance between p and q is $|j|$. When the value of i is zero, all possible paths between two points is shown in Fig. 8(bottom, purple background). There are $\frac{|j|}{2}$ left movements and $\frac{|j|}{2}$ right movements at maximum. The total number of left and right movements is less than or equal to $|j|$. When $i \neq 0$, the total number of left and right movements is less than or equal to $|j|$ (see the top of Fig. 8, orange background). It is also evident from the figure that if the number of right movements is $|i|$, then there is no left movement or if there is a left movement, then there will be two right movements (in case $i = 1$). From these the conditions of summation in the Eqn. (9) is evident. There will be in total $|j|$ steps to reach p from q. These $|j|$ steps can be taken in $|j|!$ ways. The letters l, r, and $|j| - r - l$ represent the total number of left moves, right moves, and forward moves (when only the coordinate with the larger difference changes). Correspondingly, $l!$, $r!$, and $(|j| - r - l)!$ are used in the denominator considering all possible left moves, right moves, and forward moves. For a particular value of left moves, the corresponding number of paths is $\frac{|j|!}{l!r!(|j|-r-l)!}$. The sum of all possible number of paths for each left movements is given in Eqn. (9). ■

It is to be noted here that formulation for $|i| \geq |j|$, is just the reverse (exchanging i with j) of the above equation (Eqn 9). $f_{r\text{–}4}(i,j)$ has a similarity in values with $f_1(i,j,k)$.

$f_3(i,j,k)$ has some boundary conditions as we have for $f_{r-4}(i,j)$. Thus, it can be stated that there are some similarities in honeycomb grid and rectangular grid for the number of minimal paths between two points.

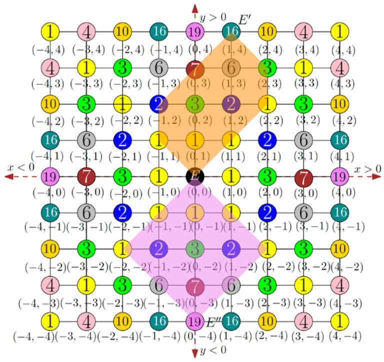


Figure 8

The isometric grid has only one neighborhood whereas the honeycomb grid, which is its dual, has three neighborhoods. In the isometric grid, the origin has six neighbors. Let $f_{iso}(i,j,k)$ be the number of shortest paths from (i,j,k) to $(0,0,0)$ considering only one neighborhood in isometric grid. The number of shortest paths between $p(i,j,k)$ to $q(0,0,0)$ is given below.

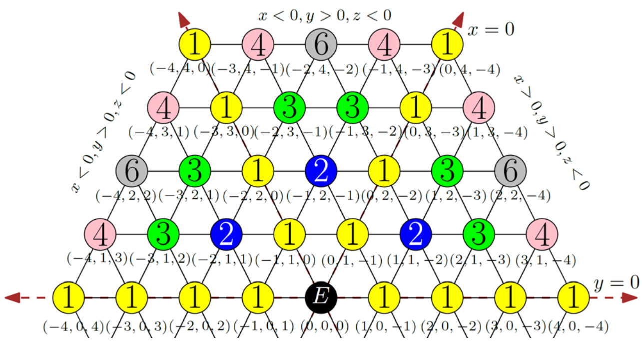


Figure 9

Counting minimal paths in the isometric grid

$$f_{iso}(i,j,k) = \binom{|a|+|b|}{|a|}, \begin{array}{l} \text{where } a,b \in \{i,j,k\} \text{ and} \\ |a+b| = |\{i,j,k\} \setminus \{a,b\}| \end{array} \tag{10}$$

are zero and one only (even and odd nodes). In 3-neighborhood of honeycomb grid, there are twelve neighbors among them six are even points and six are odd points w.r.t. q. The coordinate sets of that six even points are present in the six neighbors of q in the isometric grid.

Conclusions

In digital geometry step-based, digital distances are defined on various grids. They are applied in various research areas, especially in digital image processing and in network modeling. Non-traditional grids have various advantages comparing them to traditional rectangular grids based on, e.g., their symmetric properties [32]. For instance, there are three kinds of usual neighborhood in the honeycomb grid, while there are only 2 in the 2D rectangular grid, and only 1 in the isometric grid. This gives more flexibility in applications to the honeycomb grid. The minimal paths are counted in this paper based on all the three neighborhoods of the honeycomb grid. The result is of theoretical nature: we have formulated the problem and proved the correctness of the result. One can also expand the problem to obtain one combined formula which is suitable to find shortest paths between any two points in the honeycomb grid for the three neighborhoods. This could be a future task. On the other hand, the numbers of minimal paths in the cubic grid for the different neighborhoods, and the number of minimal paths in the square grid with weighted distances, as some of the related results have been explored in [43] and in [44], respectively. Possible applications are in networking, when to deliver some data, various shortest paths can be used simultaneously. Path counting was also applied in two-dimensional images [1]. Papers [42, 45] show similar path counting results on the triangular grid.

Acknowledgements

Some of the results of the paper were formulated on the triangular grid and were presented in 17th International Workshop on Combinatorial Image Analysis (IWCIA 2015) in Kolkata, India [42].

References

[1] A. Rosenfeld, and J. Pfaltz. Distance functions on digital pictures. Pattern Recognition, 1(1), 33-61, 1968, DOI: 10.1016/0031-3203(68)90013-7

[2] R. A. Melter. A survey of digital metrics. Contemporary Mathematics, 119:95-106, 1991

[3] A. Rosenfeld and R. A. Melter. Digital geometry. The mathematical intelligencer, 11(3):69-72, 1989

[4] P. Crawley and R. P. Dilworth (1973) Algebraic theory of lattices. Prentice-Hall Inc., Englewood Cliffs, NY, USA

[5] P. M. Gruber. Geometry of numbers. In Handbook of convex geometry, Vol. A, B, pp. 739-763, Elsevier, North-Holland, Amsterdam, 1993

[6] P. M. Gruber and C. G. Lekkerkerker. Geometry of numbers. Second edition. North-Holland Mathematical Library, 37, North-Holland Publishing Co., Amsterdam, 1987

[7] C. G. Lekkerkerker. Geometry of numbers. Bibliotheca Mathematica, Vol. VIII. Wolters-Noordhoff Publishing, Groningen, North-Holland Publishing Co., Amsterdam-London, 1969

[8] I. Her. Geometric transformations on the hexagonal grid. IEEE Transaction on Image Processing, 4(9):1213-1221, 1995

[9] B. Nagy. A family of triangular grids in digital geometry. In Proceedings of the 3rd International Symposium on Image and Signal Processing and Analysis, ISPA 2003, Vol. 1, pp. 101-106, Rome, Italy, 2003

[10] B. Nagy. Calculating distance with neighborhood sequences in the hexagonal grid. In R. Klette and J. Zunic, editors, Proceedings of the 10th International Workshop on Combinatorial Image Analysis, IWCIA 2004, Vol. 3322 of LNCS, pp. 98-109, Auckland, New Zealand, 2004. Springer-Verlag

[11] B. Nagy and R. Strand. A connection between $\mathbf{Z}^n$ and generalized triangular grids. In Proceedings of the 4th International Symposium on Advances in Visual Computing (Part II), ISVC'08, pp. 1157-1166, Las Vegas, NV, USA, 2008, Springer

[12] C. A. Wüthrich and P. Stucki. An algorithmic comparison between square- and hexagonal-based grids. CVGIP: Graphical Models and Image Processing, 53:324-339, 1991

[13] X. He and W. Jia. Hexagonal structure for intelligent vision. In International Conference on Information and Communication Technologies, 52-64, 2005, IEEE

[14] L. Middleton and J. Sivaswamy. Hexagonal image processing: a practical approach, 2005, Springer Verlag, Berlin

[15] E. S. Deutsch. Thinning algorithms on rectangular, hexagonal and triangular arrays. Communications of the ACM, 15(3):827-837, 1972

[16] B. Nagy. A symmetric coordinate frame for hexagonal networks. In: IS-TCS'04, Theoretical Computer Science – Information Society (ACM Slovenia conference), Ljubljana, Slovenia, pp. 193-196 (2004)

[17] B. Nagy. Digital geometry of various grids based on neighbourhood structures. In Proceedings of the 6th Conference of Hungarian Association for Image Processing and Pattern Recognition, KEPAF 2007, pp. 46-53, Debrecen, Hungary, 2007

[18] I. Stojmenovic. Honeycomb Networks: Topological Properties and Communication Algorithms. IEEE Trans. Parallel Distributed Syst. 8(10): 1036-1042, 1997

[19] B. Nagy. Nonmetrical Distances on the Hexagonal Grid Using Neighborhood Sequences, Pattern Recognition and Image Analysis 17: 183-190, 2007

[20] B. Nagy. Optimal Neighborhood Sequences on the Hexagonal Grid, ISPA 2007, 5th International Symposium on Image and Signal Processing and Analysis, Istanbul, Turkey, 310-315, 2007, IEEE

[21] B. Nagy and R. Strand. Approximating Euclidean circles by neighbourhood sequences in a hexagonal grid Theoretical Computer Science 412: 1364-1377, 2011

[22] B. Nagy. Geometry of neighborhood sequences in hexagonal grid, DGCI 2006, Discrete Geometry for Computer Imagery, Szeged, Hungary, Lecture Notes in Computer Science, LNCS 4245: 53-64, 2006, Springer

[23] B. Nagy. Theory of Neighborhood Sequences on Hexagonal Grids, ISPA 2007, 5th International Symposium on Image and Signal Processing and Analysis, Istanbul, Turkey, 391-396, 2007, IEEE

[24] B. Nagy. Weighted distances on a triangular grid. In Proceedings of the 16th International Workshop on Combinatorial Image Analysis, IWCIA'14, Vol. 8466 of LNCS, pp. 37-50, Brno, Czech Republic, 2014, Springer-Verlag

[25] B. Nagy. Weighted distances and distance transforms on the triangular tiling. Trans. GIS 27, 2042-2098, 2023

[26] T. Lukic and B. Nagy: Regularized binary tomography on the hexagonal grid, Physica Scripta 94, 025201 (9pp), 2019

[27] S. Matej, G. T. Herman and A. Vardi. Binary tomography on the hexagonal grid using Gibbs priors. Int. J. Imaging Syst. Technol. 9, 126-131, 1998

[28] B. Nagy and T. Lukic. Binary Tomography on Triangular Grid Involving Hexagonal Grid Approach. IWCIA 2018: pp. 68-81, 2018, Springer

[29] B. Nagy. Isometric transformations of the dual of the hexagonal lattice. In Proceedings of the 6th International Symposium on Image and Signal Processing and Analysis ISPA'09, pp. 432-437, Salzburg, Austria, 2009, IEEE

[30] K. Pluta, P. Romon, Y. Kenmochi, N. Passat. Honeycomb Geometry: Rigid Motions on the Hexagonal Grid. In: Discrete Geometry for Computer Imagery, 20th IAPR International Conference, DGCI 2017, Vienna, Austria, Proceedings, LNCS 10502: 33-45, 2017, Springer

[31] M. Saadetoglu, B. Nagy, A. Avkan. Digital continuity of rotations in the 2D regular grids. Ann. Math. Artif. Intell. 92(1): 115-137, 2024

[32] B. Nagy. Non-traditional 2D Grids in Combinatorial Imaging – Advances and Challenges. In Proceedings of the 21st International Workshop on

Combinatorial Image Analysis, IWCIA 2022, Proceedings. Vol. 13348 of LNCS, pp. 3-27, 2022, Springer-Verlag

[33] J. F. Sibeyn. Routing or Triangles, Tori and Honeycombs. Int. J. Found. Comput. Sci. 8(3): 269-287, 1997

[34] M. Imran, A. Q. Baig, H. Ali, S. U. Rehman. On topological properties of poly honeycomb networks. Period. Math. Hung. 73: 100-119, 2016

[35] B. Nagy. Application of neighborhood sequences in communication of hexagonal networks. Discret. Appl. Math. 216, 424-440, 2017

[36] L. Shan, L. Chang, X. Li, Y. Qu, and J. Chen. Improved JPS+ Path Planning Algorithm Based on Hexagonal Grid Map. In Proceedings of 2022 International Conference on Autonomous Unmanned Systems (ICAUS 2022), Lecture Notes in Electrical Engineering, Vol. 1010, pp. 2885-2896, 2022 Springer, Singapore

[37] P. P. Das. Counting minimal paths in digital geometry. Pattern Recognition Letters, 12(10), 595-603, 1991, DOI: 10.1016/0167-8655(91)90013-c

[38] P. P. Das. An algorithm for computing the number of the minimal paths in digital images. Pattern Recognition Letters, 9(2), 107-116, 1989, DOI: 10.1016/0167-8655(89)90043-3

[39] M. Dutt, A. Biswas, P. Bhowmick, and B. B. Bhattacharya. On finding shortest isothetic path inside a digital object. In IWCIA'12, Vol. 7655 of LNCS, pp. 1-15, Austin, Texas, November 2012, Springer-Verlag

[40] M. Dutt, A. Biswas, P. Bhowmick, and B. B. Bhattacharya. On finding a shortest isothetic path and its monotonicity inside a digital object. Annals of Mathematics and Artificial Intelligence, 75(1-2):27-51, 2015

[41] M. Dutt, A. Biswas, and B. B. Bhattacharya. Enumeration of shortest isothetic paths inside a digital object. In 6th International Conference on Pattern Recognition and Machine Intelligence, Vol. 9124 of LNCS, pp. 105-115, Warsaw, Poland, July 2015. Springer-Verlag

[42] M. Dutt, A. Biswas, and B. Nagy. Number of shortest paths in triangular grid for 1- and 2-neighborhoods. In R. P. Barneva, B. B. Bhattacharya, and V. E. Brimkov, editors, Proceedings of the 17th International Workshop on Combinatorial Image Analysis, IWCIA'15, Vol. 8448 of LNCS, pp. 115-124, Kolkata, India, 2015, Springer-Verlag

[43] M. Dutt, A. Biswas, and B. Nagy. Counting of Shortest Paths in a Cubic Grid. Acta Polytechnica Hungarica, 21(6), 169-186, 2024

[44] L. Alzboon, B. Khassawneh and B. Nagy. Counting the number of shortest chamfer paths in the square grid, Acta Polytechnica Hungarica 17/4, 67-87, 2020

[45] B. Nagy, A. A. Akkeles. Trajectories and Traces on Non-traditional Regular Tessellations of the Plane. In: Combinatorial Image Analysis - 18th International Workshop, IWCIA 2017, Plovdiv, Bulgaria, Proceedings, LNCS 10256: 16-29, 2017, Springer