On additions of interactive fuzzy numbers

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Abstract: In this paper we will summarize some properties of the extended addition operator on fuzzy numbers, where the interactivity relation between fuzzy numbers is given by their joint possibility distribution.

1 Introduction

A fuzzy number A is a fuzzy set of the real line $\mathbb{R}$ with a normal, fuzzy convex and continuous membership function of bounded support. Any fuzzy number can be described with the following membership function,

$$A(t) = \begin{cases} L\left(\dfrac{a-t}{\alpha}\right) & \text{if } t \in [a-\alpha, a] \\ 1 & \text{if } t \in [a,b], a \leq b, \\ R\left(\dfrac{t-b}{\beta}\right) & \text{if } t \in [b, b+\beta] \\ 0 & \text{otherwise} \end{cases}$$

where $[a,b]$ is the peak of A; a and b are the lower and upper modal values; L and R are shape functions: $[0,1] \to [0,1]$, with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$ which are non-increasing, continuous mappings. We shall call these fuzzy numbers of LR-type and use the notation $A = (a,b,\alpha,\beta)_{LR}$. If $R(x) = L(x) = 1-x$, we denote $A = (a,b,\alpha,\beta)$. The family of fuzzy numbers will be denoted by $\mathcal{F}$. A γ-level set of a fuzzy number A is defined by $[A]^\gamma = \{t \in \mathbb{R} | A(t) \geq \gamma\}$, if $\gamma > 0$ and $[A]^\gamma = \text{cl}\{t \in \mathbb{R} | A(t) > 0\}$ (the closure of the support of A) if $\gamma = 0$.

A triangular fuzzy number A denoted by (a, α, β) is defined as

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & \text{if } a-\alpha \leq t \leq a \\ 1 & \text{if } a \leq t \leq b \\ 1 - \dfrac{t-b}{\beta} & \text{if } a \leq t \leq b+\beta \\ 0 & \text{otherwise} \end{cases}$$

where $a \in \mathbb{R}$ is the centre and $\alpha > 0$ is the left spread, $\beta > 0$ is the right spread of A.

If $\alpha = \beta$, then the triangular fuzzy number is called symmetric triangular fuzzy number and denoted by (a, α).

An n-dimensional possibility distribution C is a fuzzy set in $\mathbb{R}^n$ with a normalized membership function of bounded support. The family of n-dimensional possibility distribution will be denoted by $\mathcal{F}_n$.

Let us recall the concept and some basic properties of joint possibility distribution introduced in [30]. If $A_1, \ldots, A_n \in \mathcal{F}$ are fuzzy numbers, then $C \in \mathcal{F}_n$ is said to be their joint possibility distribution if $A_i(x_i) = \max\{C(x_1, \ldots, x_n) \mid x_j \in \mathbb{R}, j \neq i\}$, holds for all $x_i \in \mathbb{R}$, $i = 1, \ldots, n$. Furthermore, A_i is called the i-th marginal possibility distribution of C. For example, if C denotes the joint possibility distribution of A_1, $A_2 \in \mathcal{F}$, then C satisfies the relationships

$$\max_y C(x_1, y) = A_1(x_1), \quad \max_y C(y, x_2) = A_2(x_2),$$

for all $x_1, x_2 \in \mathbb{R}$. Fuzzy numbers $A_1, \ldots, A_n$ are said to be non-interactive if their joint possibility distribution C satisfies the relationship

$$C(x_1, \ldots, x_n) = \min\{A_1(x_1), \ldots, A_n(x_n)\},$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

A function $T : [0,1] \times [0,1] \to [0,1]$ is said to be a triangular norm (t-norm for short) iff T is symmetric, associative, non-decreasing in each argument, and $T(x, 1) = x$ for all $x \in [0,1]$. Recall that a t-norm T is Archimedean iff T is continuous and $T(x,x) < x$ for all $x \in]0,1[$. Every Archimedean t-norm T is representable by a continuous and decreasing function $f\colon [0,1] \to [0,\infty]$ with $f(1) = 0$ and

$$T(x, y) = f^{[-1]}(f(x) + f(y))$$

where $f^{[-1]}$ is the pseudo-inverse of f, defined by

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \text{if } y \in [0, f(0)] \\ 0 & \text{otherwise} \end{cases}$$

The function f is the additive generator of T. Let T_1, T_2 be t-norms. We say that T_1 is weaker than T_2 (and write $T_1 \leq T_2$) if $T_1(x, y) \leq T_2(x, y)$ for each $x, y \in [0,1]$.

The basic t-norms are (i) the minimum: $\min(a, b) = \min\{a, b\}$; (ii) Łukasiewicz: $T_L(a, b) = \max\{a + b - 1, 0\}$; (iii) the product: $T_P(a, b) = ab$; (iv) the weak:

$$T_W(a, b) = \begin{cases} \min\{a, b\} & \text{if } \max\{a, b\} = 1 \\ 0 & \text{otherwise} \end{cases}$$

(v) Hamacher [10]:

$$H_\gamma(a, b) = \frac{ab}{\gamma + (1-\gamma)(a + b - ab)}, \ \gamma \geq 0$$

and (vi) Yager

$$T_p^Y(a, b) = 1 - \min\{1, \sqrt[p]{[(1-a)^p + (1-b)^p]}\}, \ p > 0.$$

Using the concept of joint possibility distribution we introduced the following extension principle in [3].

Definition 1.1. *[3] Let C be the joint possibility distribution of (marginal possibility distributions) $A_1, \dots, A_n \in \mathcal{F}$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then*

$$f_C(A_1, \dots, A_n) \in \mathcal{F},$$

will be defined by

$$f_C(A_1, \dots, A_n)(y) = \sup_{y = f(x_1, \dots, x_n)} C(x_1, \dots, x_n). \quad (1)$$

We have the following lemma, which can be interpreted as a generalization of Nguyen's theorem [28].

Lemma 1. *[3] Let $A_1, A_2 \in \mathcal{F}$ be fuzzy numbers, let C be their joint possibility distribution, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then,*

$$[f_C(A_1, \dots, A_n)]^\gamma = f([C]^\gamma),$$

for all $\gamma \in [0, 1]$. Furthermore, $f_C(A_1, \dots, A_n)$ is always a fuzzy number.

Let C be the joint possibility distribution of (marginal possibility distributions) $A_1, A_2 \in \mathcal{F}$, and let $f(x_1, x_2) = x_1 + x_2$ be the addition operator. Then $A_1 + A_2$ is defined by

$$(A_1 + A_2)(y) = \sup_{y = x_1 + x_2} C(x_1, x_2). \quad (2)$$

If A_1 and A_2 are non-interactive, that is, their joint possibility distribution is defined by

$$C(x_1, x_2) = \min\{A_1(x_1), A_2(x_2)\},$$

then (2) turns into the extended addition operator introduced by Zadeh in 1965 [29],

$$(A_1 + A_2)(y) = \sup_{y=x_1+x_2} \min\{A_1(x_1), A_2(x_2)\}.$$

Furthermore, if $C(x_1, x_2) = T(A_1(x_1), A_2(x_2))$, where T is a t-norm then we get the t-norm-based extension principle,

$$(A_1 + A_2)(y) = \sup_{y=x_1+x_2} T(A_1(x_1), A_2(x_2)). \tag{3}$$

For example, if A_1 and A_2 are fuzzy numbers, T is the product t-norm then the sup-product extended sum of A_1 and A_2 is defined by

$$(A_1 + A_2)(y) = \sup_{x_1+x_2=y} A_1(x_1)A_2(x_2), \tag{4}$$

and the $sup - H_\gamma$ extended addition of A_1 and A_2 is defined by

$$(A_1 + A_2)(y) = \sup_{x_1+x_2=y} \frac{A_1(x_1)A_2(x_2)}{\gamma + (1-\gamma)(A_1(x_1) + A_2(x_2) - A_1(x_1)A_2(x_2))}.$$

If T is an Archimedean t-norm and $\tilde{a}_1, \tilde{a}_2 \in \mathcal{F}$ then their T-sum

$$\tilde{A}_2 := \tilde{a}_1 + \tilde{a}_2$$

can be written in the form

$$\tilde{A}_2(z) = f^{[-1]}(f(\tilde{a}_1(x_1)) + f(\tilde{a}_2(x_2))), z \in \mathbb{R},$$

where f is the additive generator of T. By the associativity of T, the membership function of the T-sum $\tilde{A}_n := \tilde{a}_1 + \cdots + \tilde{a}_n$ can be written as

$$\tilde{A}_n(z) = \sup_{x_1+\cdots+x_n=z} f^{[-1]}\left(\sum_{i=1}^{n} f(\tilde{a}_i(x_i))\right), z \in \mathbb{R}.$$

Since f is continuous and decreasing, $f^{[-1]}$ is also continuous and non-increasing, we have

$$\tilde{A}_n(z) = f^{[-1]}\left(\inf_{x_1+\cdots+x_n=z} \sum_{i=1}^{n} f(\tilde{a}_i(x_i))\right), z \in \mathbb{R}.$$

2 Additions of interactive fuzzy numbers

Dubois and Prade published their seminal paper on additions of interactive fuzzy numbers in 1981 [5]. Since then the properties of additions of interactive fuzzy numbers, when their joint possibility distribution is defined by a t-norm have been extensively studied in the literature [1-3, 5-26]. In 1991 Fullér [6, 7] extended the results presented in [5] to product-sum and Hamacher-sum of triangular fuzzy numbers.

Theorem 2.1. *[6] Let* $\tilde{a}_i = (a_i, \alpha)$, $i \in \mathbf{N}$ *be symmetrical triangular fuzzy numbers and let their addition operator be defined by sup-product convolution (4). If*

$$A := \sum_{i=1}^{\infty} a_i$$

exists and it is finite, then with the notations

$$\tilde{A}_n := \tilde{a}_1 + \cdots + \tilde{a}_n,\ A_n := a_1 + \cdots + a_n,\ n \in \mathbf{N},$$

we have

$$\left(\lim_{n\to\infty} \tilde{A}_n\right)(z) = \exp(-|A - z|/\alpha),\ z \in \mathbb{R}.$$

Theorem 2.1 can be interpreted as a central limit theorem for mutually product-related identically distributed fuzzy variables of symmetric triangular form.

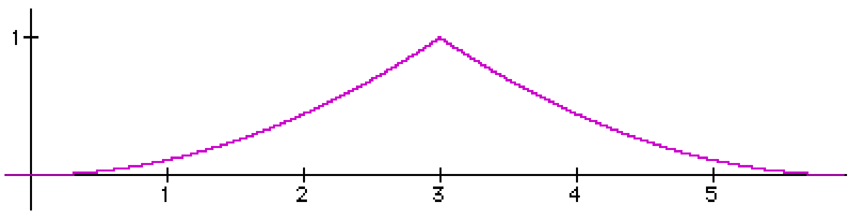

Figure 1: Product-sum of two triangular fuzzy numbers.

Theorem 2.2. *[7] Let* $\tilde{a}_i = (a_i, \alpha)$, $i \in N$ *and let their addition operator be defined by sup-*H_0 *convolution. Suppose that* $A := \sum_{i=1}^{\infty} a_i$ *exists and it is finite, then with the notation*

$$\tilde{A}_n = \tilde{a}_1 + \cdots + \tilde{a}_n, \quad A_n = a_1 + \cdots + a_n$$

we have

$$\left(\lim_{n\to\infty} \tilde{A}_n\right)(z) = \frac{1}{1 + |A - z|/\alpha},\ z \in \mathbb{R}.$$

Theorem 2.3. *[7] (Einstein-sum). Let* $\tilde{a}_i = (a_i, \alpha)$, $i \in N$ *and let their addition operator be defined by sup-*H_2 *convolution If* $A := \sum_{i=1}^{\infty} a_i$ *exists and it is finite, then with the notations of Theorem 2.2 we have*

$$\left(\lim_{n\to\infty} \tilde{A}_n\right)(z) = \frac{2}{1 + \exp(-2|A - z|/\alpha)},\ z \in \mathbb{R}.$$

In 1992 Fullér and Keresztfalvi [8] generalized and extended the results presented in [5, 6, 7]. Namely, they determined the exact membership function of the t-norm-based sum of fuzzy intervals, in the case of Archimedean t-norm having strictly convex additive generator function and fuzzy intervals with concave shape functions. They proved the following theorem,

Theorem 2.4. *[8] Let T be an Archimedean t-norm with additive generator f and let $\tilde{a}_i = (a_i, b_i, \alpha, \beta)_{LR}$, $i = 1, \dots, n$, be fuzzy numbers of LR-type. If L and R are twice differentiable, concave functions, and f is twice differentiable, strictly convex function then the membership function of the T-sum $\tilde{A}_n = \tilde{a}_1 + \cdots + \tilde{a}_n$ is*

$$\tilde{A}_n(z) = \begin{cases} 1 & \text{if } A_n \le z \le B_n \\ f^{[-1]}\left(n \times f\left(L\left(\dfrac{A_n - z}{n\alpha}\right)\right)\right) & \text{if } A_n - n\alpha \le z \le A_n \\ f^{[-1]}\left(n \times f\left(R\left(\dfrac{z - B_n}{n\beta}\right)\right)\right) & \text{if } B_n \le z \le B_n + n\beta \\ 0 & \text{otherwise} \end{cases}$$

where $A_n = a_1 + \cdots + a_n$ and $B_n = b_1 + \cdots + b_n$.

We shall illustrate Theorem 2.4 for Yager's, Dombi's and Hamacher's parametrized t-norm. For simplicity we shall restrict our consideration to the case of symmetric fuzzy numbers $\tilde{a}_i = (a_i, a_i, \alpha, \alpha)_{LL}$, $i = 1, \dots, n$. Denoting

$$\sigma_n := \frac{|A_n - z|}{n\alpha}$$

we get the following formulas for the membership function of t-norm-based sum $\tilde{A}_n = \tilde{a}_1 + \cdots + \tilde{a}_n$:

(i) Yager's t-norm with $p > 1$:

$$T_p^Y(x, y) = 1 - \min\left\{1, \sqrt[p]{(1-x)^p + (1-y)^p}\right\}.$$

This has additive generator

$$f(x) = (1-x)^p$$

and then

$$\tilde{A}_n(z) = \begin{cases} 1 - n^{1/p}(1 - L(\sigma_n)) & \text{if } \sigma_n < L^{-1}(1 - n^{-1/p}) \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Hamacher's t-norm with $p \le 2$:

$$H_p(x, y) = \frac{xy}{p + (1-p)(x + y - xy)}$$

having additive generator

$$f(x) = \ln \frac{p + (1-p)x}{x}$$

Then

$$\tilde{A}_n(z) = \begin{cases} \dfrac{p}{\left[(p + (1-p)L(\sigma_n))/L(\sigma_n)\right]^n - 1 + p} & \text{if } \sigma_n < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Dombi's t-norm with $p > 1$:

$$D_p(x, y) = \frac{1}{1 + \sqrt[p]{(1/x - 1)^p + (1/y - 1)^p}}$$

with additive generator

$$f(x) = \left(\frac{1}{x} - 1\right)^p.$$

Then

$$\tilde{A}_n(z) = \begin{cases} \left[1 + n^{1/p}(1/L(\sigma_n) - 1)\right]^{-1} & \text{if } \sigma_n < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(iv) Product t-norm (i.e. the Hamacher's t-norm with $p = 1$), that is $T_P(x, y) = xy$ having additive generator $f(x) = -\ln x$ Then

$$\tilde{A}_n(z) = L^n(\sigma_n), \; z \in \mathbb{R}.$$

The results of Theorem 2.4 have been extended to wider classes of fuzzy numbers and shape functions by many authors.

In 1994 Hong and Hwang [11] provided an upper bound for the membership function of T-sum of LR-fuzzy numbers with different spreads. They proved the following theorem,

Theorem 2.5. *[11] Let T be an Archimedean t-norm with additive generator f and let $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}$, $i = 1, 2$, be fuzzy numbers of LR-type. If L and R are concave functions, and f is a convex function then the membership function of the T-sum $\tilde{A}_2 = \tilde{a}_1 + \tilde{a}_2$ is less than or equal to*

$$A_2^*(z) =$$

$$
\begin{cases}
f^{[-1]}\left(2f\left(L\left(1/2+\dfrac{(A_2-z)-\alpha^*}{(2\alpha_*}\right)\right)\right) & \text{if } A_2-\alpha_1-\alpha_2\le z\le A_2-\alpha^* \\
f^{[-1]}\left(2f\left(L\left(\dfrac{A_2-z}{2\alpha^*}\right)\right)\right) & \text{if } A_2-\alpha^*\le z\le A_2 \\
f^{[-1]}\left(2f\left(R\left(\dfrac{z-A_2}{2\beta^*}\right)\right)\right) & \text{if } A_2\le z\le A_2+\beta^* \\
f^{[-1]}\left(2f\left(R\left(1/2+\dfrac{(z-A_2)-\beta^*}{2\beta_*}\right)\right)\right) & \text{if } A_2+\beta^*\le z\le A_2+\beta_1+\beta_2 \\
0 & \text{otherwise}
\end{cases}
$$

where $\beta^* = \max\{\beta_1, \beta_2\}$, $\beta_* = \min\{\beta_1, \beta_2\}$, $\alpha^* = \max\{\alpha_1, \alpha_2\}$, $\alpha_* = \min\{\alpha_1, \alpha_2\}$ *and* $A_2 = a_1 + a_2$.

The In 1995 Hong [12] proved that Theorem 2.4 remains valid for concave shape functions and convex additive t-norm generator. In 1996 Mesiar [25] showed that Theorem 2.4 remains valid if both $L \circ f$ and $R \circ f$ are convex functions.

In 1997 Mesiar [26] generaized Theorem 2.4 to the case of nilpotent t-norms (nilpotent t-norms are non-strict continuous Archimedean t-norms). In 1997 Hong and Hwang [14] gave upper and lower bounds of T-sums of LR-fuzzy numbers $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}$, $i = 1, \ldots, n$, with different spreads where T is an Archimedean t-norm. They proved the following two theorems,

Theorem 2.6. *[14] Let T be an Archimedean t-norm with additive generator f and let $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}$, $i = 1, \ldots, n$, be fuzzy numbers of LR-type. If $f \circ L$ and $f \circ R$ are concvex functions, then the membership function of their T-sum $\tilde{A}_n = \tilde{a}_1 + \cdots + \tilde{a}_n$ is less than or equal to*

$$
A_n^*(z) = \begin{cases}
f^{[-1]}\left(nf\left(L\left(\dfrac{1}{n}I_L\left(A_n-z\right)\right)\right)\right) & \text{if } A_n-\sum_{i=1}^n\alpha_i\le z\le A_n \\
f^{[-1]}\left(nf\left(R\left(\dfrac{1}{n}I_R\left(z-A_n\right)\right)\right)\right) & \text{if } A_n\le z\le A_n+\sum_{i=1}^n\beta_i \\
0 & \text{otherwise},
\end{cases}
$$

where

$$
I_L(z) = \inf\left\{\frac{x_1}{\alpha_1}+\cdots+\frac{x_n}{\alpha_n}\,\middle|\, x_1+\cdots+x_n=z,\ 0\le x_i\le\alpha_i,\ i=1,\ldots,n\right\},
$$

and

$$
I_R(z) = \inf\left\{\frac{x_1}{\beta_1}+\cdots+\frac{x_n}{\beta_n}\,\middle|\, x_1+\cdots+x_n=z,\ 0\le x_i\le\beta_i,\ i=1,\ldots,n\right\}.
$$

Theorem 2.7. *[14] Let T be an Archimedean t-norm with additive generator f and let $\tilde{a}_i = (a_i, \alpha_i, \beta_i)_{LR}$, $i = 1, \ldots, n$, be fuzzy numbers of LR-type. Then*

$$\tilde{A}_n(z) \geq A_n^{**}(z) =$$

$$\begin{cases} f^{[-1]}\left(nf\left(L\left(\dfrac{A_n - z}{\alpha_1 + \cdots + \alpha_n}\right)\right)\right) & \text{if } A_n - (\alpha_1 + \cdots + \alpha_n) \leq z \leq A_n \\ f^{[-1]}\left(nf\left(R\left(\dfrac{A_n - z}{\beta_1 + \cdots + \beta_n}\right)\right)\right) & \text{if } A_n \leq z \leq A_n + (\beta_1 + \cdots + \beta_n) \\ 0 & \text{otherwise}, \end{cases}$$

In 1997, generalizing Theorem 2.4, Hwang and Hong [18] studied the membership function of the t-norm-based sum of fuzzy numbers on Banach spaces and they presented the membership function of finite (or infinite) sum (defined by the sup-t-norm convolution) of fuzzy numbers on Banach spaces, in the case of Archimedean t-norm having convex additive generator function and fuzzy numbers with concave shape function. In 1998 Hwang, Hwang and An [19] approximated the strict triangular norm-based addition of fuzzy intervals of L-R type with any left and right spreadss. In 2001 Hong [15] showed a simple method of computing T-sum of fuzzy intervals having the same results as the sum of fuzzy intervals based on the weakest t-norm T_W.

2.1 Shape preserving arithmetic operations

Shape preserving arithmetic operations of LR-fuzzy intervals allow one to control the resulting spread. In practical computation, it is natural to require the preservation of the shape of fuzzy intervals during addition and multiplication. Hong [16] showed that T_W , the weakest t-norm, is the only t-norm T that induces a shape-preserving multiplication of LR-fuzzy intervals. In 1995 Kolesarova [22, 23] proved the following theorem,

Theorem 2.8. *(a) Let T be an arbitrary t-norm weaker than or equal to the Łukasiewicz t-norm T_L; $T(x, y) \leq T_L(x, y) = \max(0, x + y - 1)$, $x, y \in [0, 1]$. Then the addition $\oplus$ based on T coincides on linear fuzzy intervals with the addition $\oplus$ based on the weakest t-norm T_W; i.e.,*

$$\begin{aligned}(a_1, b_1, \alpha_1, \beta_1) \oplus (a_2, b_2, \alpha_2, \beta_2) = \\ (a_1 + a_2, b_1 + b_2, \max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2)).\end{aligned}$$

(b) Let T be a continuous Archimedean t-norm with convex additive generator f . Then the addition $\oplus$ based on T preserves the linearity of fuzzy intervals if and only if the t-norm T is a member of Yager's family of nilpotent t-norms with parameter $p \in [1, \infty)$, $T = T_p^Y$, and $f(x) = (1 - x)^p$. Then $T_1^Y = T_L$ and for $p \in (0, \infty)$,

$$\begin{aligned}(a_1, b_1, \alpha_1, \beta_1) \oplus (a_2, b_2, \alpha_2, \beta_2) = \\ (a_1 + a_2, b_1 + b_2, (\alpha_1^q + \alpha_2^q)^{1/q}, (\beta_1^q + \beta_2^q)^{1/q}),\end{aligned}$$

where $1/p + 1/q = 1$*, i.e.* $q = p/(p-1)$.

In 1997 Mesiar [27] studied the triangular norm-based additions preserving the LR-shape of LR-fuzzy intervals and conjectured that the only t-norm-based additions preserving the linearity of fuzzy intervals are those described in Theorem 2.8. He proved the following theorem,

Theorem 2.9. *[27] Let a continuous t-norm T be not weaker than or equal to* T_L *(i.e., there are some* $x, y \in [0,1]$ *so that* $T(x,y) > x + y - 1 > 0$*). Let the addition based on T preserve the linearity of fuzzy intervals. Then either T is the strongest t-norm,* $T = T_M$*, or* T *is a nilpotent t-norm.*

In 2002 Hong [17] proved Mesiar's conjecture.

Theorem 2.10. *[17] Let a continuous t-norm T be not weaker than or equal to* T_L*. Then the addition* $\oplus$ *based on T preserves the linearity of fuzzy intervals if and only if the t-norm T is either* T_M *or a member of Yager's family of nilpotent t-norms with parameter* $p \in (1, \infty)$*,* $T = T_p^Y$*, and* $f(x) = (1-x)^p$*.*

2.2 Additions of completely correlated fuzzy numbers

Until now we have summarized some properties of the addition operator on interactive fuzzy numbers, when their joint possibility distribution is defined by a t-norm. It is clear that in (3) the joint possibility distribution is defined *directly* and *pointwise* from the membership values of its marginal possibility distributions by an aggregation operator. However, the interactivity relation between fuzzy numbers may be given by a more general joint possibility distribution, which can not be directly defined from the membership values of its marginal possibility distributions by any aggregation operator.

Drawing heavily on [3] we will now consider some properties of the addition operator on completely correlated fuzzy numbers, where the interactivity relation is given by their joint possibility distribution.

Let C be a joint possibility distribution with marginal possibility distributions A and B, and let

$$f(x_1, x_2) = x_1 + x_2,$$

the addition operator in $\mathbb{R}^2$. In [3] we introduced the notation,

$$A +_C B = f_C(A, B).$$

Definition 2.1. *[9] Fuzzy numbers* A *and* B *are said to be completely correlated, if there exist* $q, r \in \mathbb{R}$*,* $q \neq 0$ *such that their joint possibility distribution is defined by*

$$C(x_1, x_2) = A(x_1) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2) = B(x_2) \cdot \chi_{\{qx_1 + r = x_2\}}(x_1, x_2), \quad (5)$$

where $\chi_{\{qx_1+r=x_2\}}$*, stands for the characteristic function of the line*

$$\{(x_1, x_2) \in \mathbb{R}^2 | qx_1 + r = x_2\}.$$

In this case we have,

$$[C]^\gamma = \left\{(x, qx + r) \in \mathbb{R}^2 \middle| x = (1-t)a_1(\gamma) + ta_2(\gamma), t \in [0,1]\right\}$$

where $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$*; and* $[B]^\gamma = q[A]^\gamma + r$*, for any* $\gamma \in [0,1]$.

We should note here that the interactivity relation between two fuzzy numbers is defined by their joint possibility distribution. Fuzzy numbers A and B with $A(x) = B(x)$ for all $x \in \mathbb{R}$ can be non-interactive, positively or negatively correlated depending on the definition of their joint possibility distribution.

Definition 2.2. *[9] Fuzzy numbers* A *and* B *are said to be completely positively (negatively) correlated, if* q *is positive (negative) in (5).*

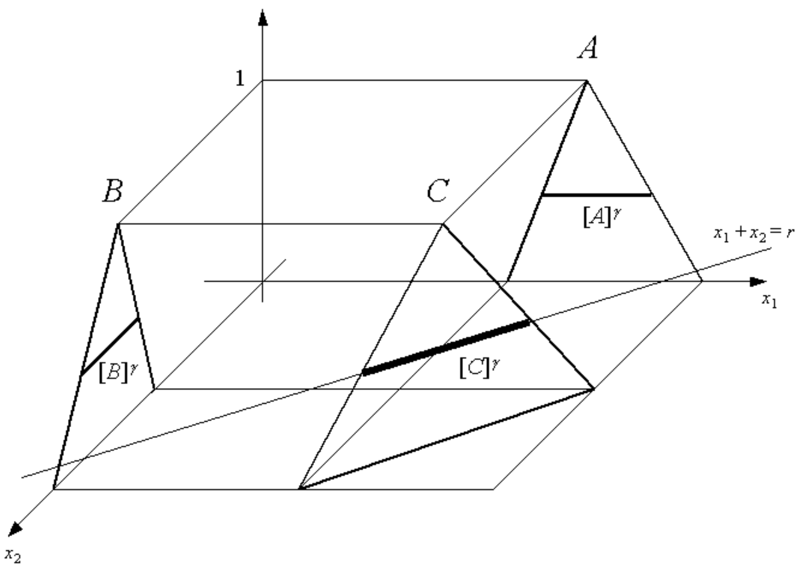

Figure 2: Completely negatively correlated fuzzy numbers with $q = -1$.

We note that if $A, B \in \mathcal{F}$ are completely positively correlated then their correlation coefficient is equal to one, furthermore, if they are completely negatively correlated then their correlation coefficient is equal to minus one [4, 9]. In the case of complete positive correlation, if $A(u) \geq \gamma$ for some $u \in \mathbb{R}$ then there exists a *unique* $v \in \mathbb{R}$ that B can take, furthermore, if u is moved to the left (right) then the corresponding value (that B can take) will also move to the left (right). In case of complete negative correlation, if $A(u) \geq \gamma$ for some $u \in \mathbb{R}$ then there exists a *unique* $v \in \mathbb{R}$ that B can take, furthermore, if u is moved to the left (right) then the corresponding value (that B can take) will move to the right (left). It is also clear that in these two cases, given q

and r, the first marginal possibility distribution completely determines the second one, and vica versa. Finally, if A and B are not completely correlated then if $A(u) \geq \gamma$ for some $u \in \mathbb{R}$ then there may exist *several* $v \in \mathbb{R}$ that B can take (see [9]).

Now let us consider the extended addition of two completely correlated fuzzy numbers A and B,

$$(A +_C B)(y) = \sup_{y=x_1+x_2} C(x_1, x_2).$$

That is,

$$(A +_C B)(y) = \sup_{y=x_1+x_2} A(x_1) \cdot \chi_{\{qx_1+r=x_2\}}(x_1, x_2).$$

Then from (2) and (5) we find,

$$[A +_C B]^\gamma = (q + 1)[A]^\gamma + r, \tag{6}$$

for all $\gamma \in [0, 1]$. If A and B are completely negatively correlated with $q = -1$, that is, $[B]^\gamma = -[A]^\gamma + r$, for all $\gamma \in [0, 1]$, then $A +_C B$ will be a crisp number. Really, from (6) we get $[A +_C B]^\gamma = 0 \times [A]^\gamma + r = r$, for all $\gamma \in [0, 1]$.

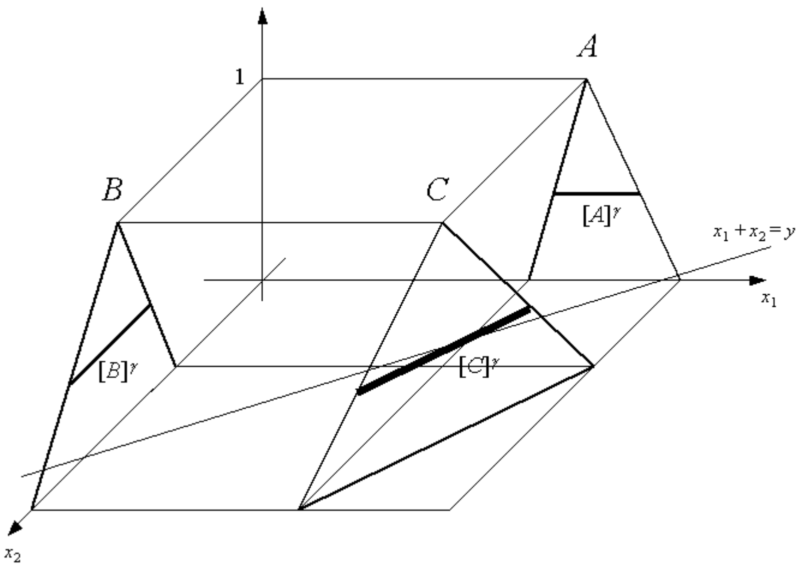

Figure 3: Completely negatively correlated fuzzy numbers with $q \neq -1$.

That is, the interactive sum, $A +_C B$, of two completely negatively correlated fuzzy numbers A and B with $q = -1$ and $r = 0$, i.e.

$$A(x) = B(-x), \forall x \in \mathbb{R},$$

will be (crisp) zero. On the other hand, a γ-level set of their non-interactive sum, $A + B$, can be computed as,

$$[A + B]^\gamma = [a_1(\gamma) - a_2(\gamma), a_2(\gamma) - a_1(\gamma)],$$

which is a fuzzy number.

In this case (i.e. when $q = -1$) any γ-level set of C are included by a certain level set of the addition operator, namely, the relationship,

$$[C]^\gamma \subset \{(x_1, x_2) \in \mathbb{R} | x_1 + x_2 = r\},$$

holds for any $\gamma \in [0, 1]$ (see Fig. 2). On the other hand, if $q \neq -1$ then the fuzziness of $A +_C B$ is preserved, since

$$[A +_C B]^\gamma = (q + 1)[A]^\gamma + r \neq \text{constant},$$

for all $\gamma \in [0, 1]$ and $y \in \mathbb{R}$. (see Fig. 3).

Really, in this case the set $\{(x_1, x_2) \in [C]^\gamma | x_1 + x_2 = y\}$ consists of a single point at most for any $\gamma \in [0, 1]$ and $y \in \mathbb{R}$.

Note 2.1. *The interactive sum of two completely negatively correlated fuzzy numbers A and B with $A(x) = B(-x)$ for all $x \in \mathbb{R}$ will be (crisp) zero.*

3 Summary

In this paper we have summarized some properties of the addition operator on interactive fuzzy numbers, when their joint possibility distribution is defined by a t-norm or by a more general type of joint possibility distribution.

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