A Note on Aggregation of t-norm-based, Fuzzy Vector Subspaces

Dragan Jočić[1] and Ivana Štajner-Papuga[2]

[1]Mathematical Institute of the Serbian Academy of Sciences and Arts, Kneza Mihaila 36, 11000 Belgrade, Serbia

[2]University of Novi Sad, Department of Mathematics and Informatics, Faculty of Sciences, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia

e-mails: dragan.jocic@mi.sanu.ac.rs, ivana.stajner-papuga@dmi.uns.ac.rs

Abstract: Aggregation operators, as mathematical tools, play an important role in fusing data, within many knowledge-based systems. The question that arises while aggregating data of a specific structure, is under which conditions the structure can be preserved. The focus of this paper, is on an overview of the aggregation of special algebraic structures, known as T -fuzzy vector subspaces, (i.e., fuzzy vector subspaces with the main axiom based on an arbitrary triangular norm T) and to also explore the possibilities for directions for future research.

Keywords: aggregation operators; t-norms; fuzzy sets; fuzzy vector subspaces

1 Introduction

One of the contemporary research directions in the field of mathematics is based on modelling the process of fusion of data, commonly referred to as aggregation of data. From a mathematical point of view, this process is carried out by mathematical operators, called aggregation operators. Although aggregation operators are usually given on the unit interval, they are also studied in many other frameworks such as finite chains, interval-valued frameworks, lattices and bounded partially ordered sets, fuzzy sets, etc. One of current topics, highly investigated by researchers, is preservation of the structure of aggregated data. That is, if the input data possess a specific structure, it is of particular interest to know under which conditions this structure is preserved under fusion. Currently, the focus is on aggregation of fuzzy subgroups, fuzzy vector subspaces, fuzzy quasi-metrics, indistinguishability operators, etc. (see [1-4, 18, 21, 22]).

As the theory of fuzzy sets, and consequently fuzzy systems, continues to find applications across various areas of science and real life, with undiminished intensity (see e.g. [5, 9, 14, 19]), the focus of this paper is on preserving the structure of the T-fuzzy vector spaces, during the aggregation process. While T_M-fuzzy vector spaces are also T-fuzzy vector spaces for any t-norm T the reverse does not generally hold. Therefore, the question of structure preservation in T-fuzzy vector spaces for arbitrary t-norms T needs to be addressed. For that reason, a review of the results from [4] is provided, however from the perspective of a more general approach based on t-norms in general, along with new result that are analysis of the impact of using combinations of multiple aggregation operators.

This paper is organized as follows. The second section contains preliminary notions concerning the aggregation operators, fuzzy sets and t-norm based fuzzy vector subspaces. Aggregation of T-fuzzy vector subspaces done by a single aggregation operator is considered in the third section, while Section 4 investigates aggregation based on a family of aggregation operators. Finally, Section 5 contains some concluding remarks and discussions on possible future work.

2 Preliminaries

The most general definition of an aggregation operator that provides fundamental properties of this type of operators is given by the following definition (see [11, 16]).

Definition 1: [16] *An aggregation operator is a function*

$$Aggo: \bigcup_{n\in N}[0,1]^n \rightarrow [0,1]$$

such that, $Aggo(x) = x$ for all $x \in [0,1]$, is nondecreasing in each variable, and it fulfills the boundary conditions:

$$Aggo(0,0,0,\ldots,0) = 0 \text{ and } Aggo(1,1,1,\ldots,1) = 1$$

Of course, further on, binary and consequently, n-ary aggregation operators ([11]) are being considered.

The previous definition can be easily expanded to aggregation operator on arbitrary nonempty real interval, as well as, on other structures such as fuzzy sets. Also, depending on the type of the problem that is being investigated or modelled by aggregation, more properties such as commutativity, associativity, continuity, conjunctivity, disjunctivity, strict monotonicity, can and are required. For example, an n-ary aggregation operator $Aggo: [0,1]^n \rightarrow [0,1]$ such that:

$$Aggo(x_1,\ldots,x_n) \le \min\{x_1,\ldots,x_n\}$$

where $x_i \in [0,1]$ for all $i \in \{1,2,\ldots,n\}$ is conjunctive aggregation operator. On the other hand, an aggregation operator $Aggo: [0,1]^n \rightarrow [0,1]$ is disjunctive if:

$$Aggo(x_1, \ldots, x_n) \geq \max\{x_1, \ldots, x_n\},$$

for $x_i \in [0,1]$, $i \in \{1, 2, \ldots, n\}$

As previously noted, the present study rely on two additional, well-established concepts - those of fuzzy sets and triangular norms:

- For X being a universe of discourse, a fuzzy set A in X is defined as a set of ordered pairs $A = \{(x, \mu(x)) | x \in X\}$ where $\mu : X \to [0, 1]$ is the membership function that assigns to each element $x \in X$ a real number from the unit interval that provides an information of the degree to which it belongs to the fuzzy set A ([23]). Further on, in order to obtain a simpler and more concise notation, a fuzzy set is identified with its membership function. That is, instead of writing ”μ is the membership function of a fuzzy set A”, we simply write ”μ is a fuzzy set”.
- A t-norm T: $[0, 1]^2 \to [0, 1]$ is a commutative, associative, and monotone operation on $[0, 1]^2$, such that 1 is its neutral element ([13]).

Now, having all previous notions in mind, and letting $(X, +, \cdot)$ be a vector space on a field F, by definitions from [4] and [7], a fuzzy set μ: $X \to [0, 1]$ is called t- norm based fuzzy vector subspace of X, or, for short, T-fuzzy vector subspace if the following holds:

(TF1) $\mu(x + y) \geq T(\mu(x), \mu(y))$ *for all* $x, y \in X$

(TF2) $\mu(ax) \geq \mu(x)$ *for all* $x \in X$ *and* $a \in F$

where T is an arbitrary t-norm.

While the authors in [7], as above, are considering an arbitrary t-norm in definition of fuzzy vector subspaces, the results from [4] are obtained specifically for $T = T_M$ in condition (TF1), where T_M is the strongest t-norm, i.e., min (see [13]). Conversely, [4] explores aggregation over the broadest possible class of aggregation operators, whereas [7] focuses solely on T-sum of fuzzy sets ([10, 15, 17]) for aggregation. This paper considers the approach from [4] while accepting this broader definition of a fuzzy vector subspace based on an arbitrary t-norm.

The subtle difference between two approaches can also be observed through the following equivalences. For $T = T_M$ the following statements are equivalent (see [4]):

(i) μ is a T-fuzzy vector subspace of X

(ii) For all $x, y \in X$ and $a, b \in F$ holds $\mu(ax + by) \geq T(\mu(x), \mu(y))$

(iii) Each non-empty α-cut of μ, i.e., each

$$[\mu]_\alpha = \{x \in X \mid \mu(x) \geq \alpha\} \neq \emptyset$$

$\alpha \in [0, 1]$, is a vector subspace of X

If T is an arbitrary t-norm, following from general properties on t-norms, statements (i) and (ii) are equivalent for fuzzy sets that maps null element from X to value 1 (see [7]). Of course, $(i) \Rightarrow (ii)$ holds for all. For the statement (iii) situation is even more restrictive, and the following proposition holds.

Proposition 2: Let $(X, +, \cdot)$ be a vector space on a field F, let $\mu: X \rightarrow [0, 1]$ be a fuzzy set, and let T be an arbitrary t-norm.

a) $(i) \Rightarrow (ii)$

b) $(iii) \Rightarrow (ii)$

c) Let fuzzy set μ maps null element from X to value 1. Then $(ii) \Rightarrow (i)$

d) Let fuzzy set μ maps null element from X to value 1. Then $(iii) \Rightarrow (i)$

Proof. Proofs for a), c), and d) are straightforward and based on basic properties of triangular norms. Proof for b) is a generalization of one from [4] based on properties of triangular norms. If $\alpha = T(\mu(x), \mu(y))$, for some $x, y \in X$, from $T(\mu(x), \mu(y)) \leq \mu(x)$ and $T(\mu(x), \mu(y)) \leq \mu(y)$ follows that both x and y belongs to the α-cut $[\mu]_\alpha$, therefore $[\mu]_\alpha$ is non-empty. Now, since by assumption of (iii), each α-cut $[\mu]_\alpha$ is a vector subspace of X, and $ax + by$ is also from $[\mu]_\alpha$ for some $a, b \in F$, i.e., $\mu(ax + by) \geq \alpha = T(\mu(x), \mu(y))$. □

Remark 3: It should be stressed that, due to the fact that T_M is the strongest t-norm, i.e., for all t-norms T holds $T \leq T_M$, for an arbitrary T_M-fuzzy vector subspace μ holds:

$$\mu(x + y) \geq T_M\big(\mu(x), \mu(y)\big) \geq T(\mu(x), \mu(y))$$

for all t-norms T**.** Therefore, μ is also a T-fuzzy vector subspace for an arbitrary T and results obtained for μ as a T_M-fuzzy vector subspace are valid for this corresponding T-fuzzy vector subspace. However, there are T-fuzzy vector subspaces where $T \neq T_M$ that are not T_M-fuzzy vector subspaces at the same time. Therefore, there are some localized results valid for a specific T-fuzzy vector subspace that cannot be extended to T_M-fuzzy vector subspace.

Example 4: Let us consider the real vector space R^2, and the following fuzzy vector subspace $\mu: R^2 \rightarrow [0, 1]$ given by:

$$\mu(x, y) = \begin{cases} 1, & (x, y) = (0,0) \\ 0.3, & x - y = 0 \text{ } and \text{ } (x, y) \neq (0,0) \\ 0.5, & otherwise \end{cases}$$

It can be easily checked that μ is a T_P-fuzzy vector subspace where T_P is the product t-norm. Let us consider two arbitrary elements (x_1, y_1) and (x_2, y_2) from R^2. It should be shown that

$$\mu\big((x_1, y_1) + (x_2, y_2)\big) \geq \mu\big((x_1, y_1)\big) \cdot \mu\big((x_2, y_2)\big) \qquad (1)$$

holds:

- If $\mu\big((x_1, y_1) + (x_2, y_2)\big)$ assumes value 1, then (1) holds trivial
- If $\mu\big((x_1, y_1) + (x_2, y_2)\big)$ assumes value 0.5, the only option for (1) to fail is when both $\mu\big((x_1, y_1)\big)$ and $\mu\big((x_2, y_2)\big)$ are equal to 1. However, from $\mu\big((x_1, y_1)\big) = \mu\big((x_2, y_2)\big) = 1$ follows $(x_1, y_1) = (x_2, y_2) = (0,0)$, which provides $(x_1, y_1) + (x_2, y_2) = (0,0)$, and contradiction $\mu\big((x_1, y_1) + (x_2, y_2)\big) = 1$ is obtained. Therefore (1) holds
- If $\mu\big((x_1, y_1) + (x_2, y_2)\big)$ assumes value 0.3, the only option for (1) to fail is when $\mu\big((x_1, y_1)\big) = 1$ and $\mu\big((x_2, y_2)\big) = 0.5$ (or vice versa). Since from $\mu\big((x_1, y_1)\big) = 1$ follows $(x_1, y_1) = (0,0)$ that provides $(x_1, y_1) + (x_2, y_2) = (x_2, y_2)$, and $\mu\big((x_1, y_1) + (x_2, y_2)\big) = \mu\big((x_2, y_2)\big) = 0.5$ is in contradiction to $\mu\big((x_1, y_1) + (x_2, y_2)\big) = 0.3$. Therefore, (1) holds

As previously seen, the given μ is a T_P-fuzzy vector subspace. However, since:

$$\mu((1,2) + (2,1)) = \mu((3,3)) = 0.3$$

and

$$T_M(\mu((1,2)), \mu((2,1))) = min\{0.5, 0.5\} = 0.5$$

the required inequality (TF1) does not hold, i.e.,

$$\mu((1,2) + (2,1)) < T_M(\mu((1,2)), \mu((2,1)))$$

3 Aggregation of T-fuzzy Vector Spaces: Single Aggregation Operator

For an arbitrary set X, an aggregation operator $Aggo$ and two fuzzy subsets $\mu, \nu: X \to [0,1]$, the mapping $Aggo(\mu, \nu): X \to [0,1]$ given by:

$$Aggo(\mu, \nu)(x) = Aggo(\mu(x), \nu(x))$$

is a new fuzzy set (membership function). In order to use a bit simpler notation, further on it will be denoted with $A_{\mu,\nu}$.

3.1 Self-Aggregation

The first case concerning the aggregation of T-fuzzy vector spaces is self-aggregation. That is, the question is: if μ is a T-fuzzy vector subspace of X, is $A_{\mu,\mu}$ also a T-fuzzy vector subspace of X, when T is an arbitrary t-norm? As shown in [4], the answer is straightforward for $T = T_M$: yes, $A_{\mu,\mu}$ is again T_M-fuzzy vector subspace of X. Let us observe that in this case no restrictions are imposed on aggregation operator that is being used.

In general case, i.e., for a T-fuzzy vector space where T is an arbitrary t-norm the following holds.

Proposition 5: Let $Aggo: [0,1]^2 \to [0,1]$ be an aggregation operator, $\mu: X \to [0,1]$ be a T-fuzzy vector subspace of a vector space X, and let T be a t-norm that is distributive over $Aggo$. Then, $A_{\mu,\mu}: X \to [0,1]$ is a T-fuzzy vector subspace of X.

Proof. Let $x, y \in X$. Since μ is a T-fuzzy vector subspace, there holds

$$\mu(x + y) \geq T(\mu(x), \mu(y))$$

Now, from monotonicity of $Aggo$, and assumed distributivity (which insures idempotency of $Aggo$, see [12]) follows:

$$\begin{aligned} A_{\mu,\mu}(x + y) &= Aggo(\mu(x + y), \mu(x + y)) \\ &\geq Aggo(T(\mu(x), \mu(y)), T(\mu(x), \mu(y))) \\ &= T(\mu(x), Aggo(\mu(y), \mu(y))) \\ &= T(Aggo(\mu(x), \mu(x)), Aggo(\mu(y), \mu(y))) \\ &= T(A_{\mu,\mu}(x), A_{\mu,\mu}(y)). \end{aligned}$$

Also, based on monotonicity of $Aggo$ and starting assumption that μ is a T-fuzzy vector subspace, i.e., that $\mu(ax) \geq \mu(x)$ for $a \in F$, holds

$$A_{\mu,\mu}(ax) = Aggo\big(\mu(ax), \mu(ax)\big) \geq Aggo\big(\mu(x), \mu(x)\big) = A_{\mu,\mu}(x). \quad \square$$

As seen in [4], extension to n-ary aggregation is possible, i.e., if $Agoo: [0,1]^n \to [0,1]$ is an n-ary aggregation operator and $\mu: \to [0,1]$ a T_M-fuzzy vector subspace of a vector space X, then $A_{\mu,\dots,\mu}: X \to [0,1]$ is again a T_M-fuzzy vector subspace of X. However, generalization to T-fuzzy vector subspace of X, for any t-norm T, requires some additional restriction on aggregation operators.

Corollary 6: Let $Aggo: [0,1]^n \to [0,1]$ be an associative aggregation operator, $\mu : X \to [0,1]$ be a T-fuzzy vector subspace of a vector space X, and let T be a t-norm that is distributive over $Aggo$. Then $A_{\mu,\dots,\mu}: X \to [0,1]$, where

$$A_{\mu,\dots,\mu}(\mathrm{x}) = Aggo(\mu, \dots, \mu)(x) = Aggo\big(\mu(x), \dots, \mu(x)\big)$$

is a T-fuzzy vector subspace of X

The required distributivity of a t-norm, over an aggregation operator, is somewhat restrictive, since it leads to idempotent aggregation operators, such as, for example, max, or to a projection in the degenerate and non-symmetric case. If conditional distributivity is required instead, the variety of aggregation operators is extended, however conjunctivity for the aggregation operator $Agoo$ is needed. By conditional distributivity (see [12]) the following is considered:

$$T\big(x, Agoo(y,z)\big) = Agoo\big(T(x,y), T(x,z)\big), \qquad Agoo(y,z) < 1.$$

This raises the question of distributive pairs of operators, as well as the necessity of modifying the operator T within the condition (TF1) and potentially introducing new types of fuzzy vector subspaces. At this point, it should be strongly emphasized that, although the additional conditions required for the aggregation operator $Aggo$ and t-norm T may not seem particularly demanding at first glance, the conditional distributivity under the given conditions is indeed challenging and calls for further analysis, imposing new directions for research.

Remark 7: Both papers [2] and [19] consider T-fuzzy subgroups, i.e., mappings $\mu: X \to [0,1]$ such that:

$$\mu(e) = 1, \quad \mu(x^{-1}) = \mu(x) \quad and \quad T\big(\mu(x), \mu(y)\big) \leq \mu(x * y)$$

where $(X, *, \ {}^{-1}, e)$ is a group with binary operation $*$, unary operation $\ {}^{-1}$ and the neutral element e, and T is a t-norm:

- In the case when group X has two or three elements, for any t-norm T and any aggregation operator $Aggo$, if μ, ν are T-fuzzy subgroup, then $A_{\mu,\nu}$ is a T-fuzzy subgroup (see Proposition 3.4 from [2]).
- In the case when groups X has at least four elements it has been proved (see Theorem 3.7 from [2]) that $A_{\mu,\mu}$is a T-fuzzy subgroup for every aggregation operator $Aggo$ and every T-fuzzy subgroup μ if and only if $T = T_M$. Subsequently, the paper [2] focuses on the aggregation of two T_M-fuzzy subgroups (which are not necessarily equal).
- Results presented in [19] are also focused on aggregation of T-fuzzy subgroups. It is shown that strictly monotone binary aggregation operator applied to two T-fuzzy subgroups leads to a T_D-fuzzy subgroup, where T_D is the weakest t-norm. On the other hand, aggregation of two of T-fuzzy subgroups is again a T-fuzzy subgroup of the same type if $Aggo$ dominates T, i.e., if for $x, y, z, w \in [0,1]$

$$Agoo\big(T(x,y), T(z,w)\big) \geq T\big(Agoo(x,z), Agoo(y,w)\big)$$

- Also, construction of n-ary aggregation operators is considered in [19]. It has been shown that aggregation of n T-fuzzy subgroups can lead to a new T-fuzzy subgroup (with the same t-norm in the core of T-fuzzy subgroups) under certain additional conditions. Specifically, pairs of input T-fuzzy subgroups aggregated by a binary operator used in construction of the n-ary aggregation operator have to provide again a T-fuzzy subgroup.

Since the structure observed in this paper is a T-fuzzy vector space, and a T-fuzzy vector space can be interpreted as a special T-fuzzy subgroup, results obtained here can be used to identify T-fuzzy subgroups that are capable of maintaining their form (core t-norm) under aggregation. Also, based on Theorem 1 from [19] the following corollary is obtained.

Corollary 8: Let $Aggo: [0,1]^2 \rightarrow [0,1]$ be an aggregation operator, and let $\mu : X \rightarrow [0,1]$ and $\nu : X \rightarrow [0,1]$ be two T-fuzzy vector subspace of a vector space X, and let T be a t-norm that is distributive over $Aggo$. If $Aggo$ dominates T, then $A_{\mu,\nu}$ is a T-fuzzy vector subspace of X.

3.2 Similar Fuzzy Vector Subspaces

As seen in the previous section self-aggregation of T_M-fuzzy vector subspaces elegantly leads to a new T_M-fuzzy vector subspace. However, structure preservation under self-aggregation for T-fuzzy vector subspaces in general is bit more complex and requires more research. The important question is when aggregation of two not necessarily equal T-fuzzy vector subspaces results in a new T-fuzzy vector subspace.

The next example shows that, in general, even the aggregation of two T_M-fuzzy vector subspaces does not need to be T_M-fuzzy vector subspace.

Example 9: Let us consider the real vector space R^3, and the following T_M-fuzzy vector subspaces $\mu, \nu: R^3 \rightarrow [0,1]$ given by

$$\mu(x,y,z) = \begin{cases} 1, & (x,y,z) = (0,0,0) \\ 0.8, & x+y+z = 0 \; and \; (x,y,z) \neq (0,0) \\ 0.6, & otherwise \end{cases}$$

and

$$\nu(x,y,z) = \begin{cases} 1, & (x,y,z) = (0,0,0) \\ 0.7, & y-z = 0 \; and \; (x,y,z) \neq (0,0) \\ 0.5, & otherwise \end{cases}$$

It can be easily checked that μ and ν are T_M-fuzzy vector subspaces of R^3, and, therefore, also T_P-fuzzy vector subspaces of R^3.

Let us suppose that aggregation operator is the operator of maximum, i.e., $Aggo = max$. Then,

$$A_{\mu,\nu}(1,-2,1) = Aggo(\mu,\nu)(1,-2,1) = max\{0.8, 0.5\} = 0.8$$

$$A_{\mu,\nu}(3,0,0) = Aggo(\mu,\nu)(3,0,0) = max\{0.6, 0.7\} = 0.7$$

Therefore,

$$A_{\mu,\nu}(4,-2,1) = Aggo(\mu,\nu)(4,-2,1) = max\{0.6, 0.5\} = 0.6$$

Now,

$$A_{\mu,\nu}\big((1,-2,1) + (3,0,0)\big) = 0.6 < 0.7 = T_M(A_{\mu,\nu}(1,-2,1), A_{\mu,\nu}(3,0,0))$$

i.e., $max(\mu,\nu)$ does not satisfy axiom (TF1) and is not a T_M-fuzzy vector subspace of R^3. However, this newly obtained structure $max(\mu,\nu)$ is T_P-fuzzy vector subspace of R^3.

In [4] [6] this problem was solved for T_M-fuzzy vector subspaces by the introduction of similar T_M-fuzzy vector subspaces.

Definition 10: [6] Let $(X,+,\cdot)$ be a vector space on a field F, and $\mu,\nu: X \to [0,1]$ two fuzzy vector subspaces of X. Vector subspace μ is similar to vector subspace ν, denoted by $\mu \sim \nu$, if their respective α-cuts coincide.

Of course, relation ∼ is an equivalence relation, and in [4] was proved more operational method for determining classes of similar T_M-fuzzy vector subspaces of X:

$$\mu \sim \nu \Leftrightarrow \mu(x) < \mu(y) \Leftrightarrow \nu(x) < \nu(y)$$

Now, the following result resolves this issue for T_M-fuzzy vector subspaces.

Theorem 11: [4] Let $(X,+,\cdot)$ be a vector space on a field F, $Aggo: [0,1]^2 \to [0,1]$ an aggregation operator, and $\mu,\nu: X \to [0,1]$ two T_M-fuzzy vector subspaces of X. If $\mu \sim \nu$, then $A_{\mu,\nu}: X \to [0,1]$ is a T_M-fuzzy vector subspace of X.

As in case of self-aggregation, the previous result can be easily extended to n-ary aggregation operators.

Corollary 12: [4] Let $(X,+,\cdot)$ be a vector space on a field F, $Aggo: [0,1]^n \to [0,1]$ an aggregation operator, and $\mu_1,\mu_2,\ldots,\mu_n: X \to [0,1]$ T_M-fuzzy vector subspaces of X. If $\mu_i \sim \mu_j$ for all $i,j \in \{1,2,\ldots,n\}$, then:

$$A_{\mu_1,\mu_2,\ldots,\mu_n}: X \to [0,1]$$

is also a T_M-fuzzy vector subspace of X.

It should be emphasized that the aggregation of two similar T_M-fuzzy vector subspaces does not have to be from the same class of equivalence. The problem of remaining within the same class of equivalence was solved in [4] by refereeing to jointly strictly monotone aggregation operators ([11]). That is for any two T_M-fuzzy vector subspaces μ and ν of X satisfying $\mu \sim \nu$ holds $A_{\mu,\nu} \sim \mu$ if and only if $Aggo$ is a jointly strictly monotone aggregation operator, i.e., if from $x_i < y_j$ for all $i \in \{1,2,\ldots,n\}$ follows

$$Aggo(x_1,\ldots,x_n) < Aggo(y_1,\ldots,y_n)$$

where $x_i,\ y_j \in [0,1]$ for all $i \in \{1,2,\ldots,n\}$ (see [11]).

4 Aggregation of T-fuzzy Vector Spaces: Family of Aggregation Operator

Now, based on previous, there can be developed a method for n-ary aggregation applicable to similar T_M-fuzzy vector subspaces, based on n-ary aggregation operator generated by a family of binary jointly strictly monotone aggregation operators. A similar construction as given in Theorem 13 is applied in [19] to the aggregation of fuzzy subgroups.

Theorem 13: Let $(X, +, \cdot)$ be a vector space on a field F, and family of and $\mu_1, \mu_2, \ldots, \mu_n : X \to [0,1]$ a family of T_M-fuzzy vector subspaces of X such that $\mu_i \sim \mu_j$ for all $i, j \in \{1, 2, \ldots, n\}$. Let $Aggo_i : [0,1]^2 \to [0,1]$, for $i \in \{1, 2, \ldots, n-1\}$, be a family of jointly strictly monotone aggregation operators, and let:

$$Aggo(x_1, \ldots, x_n) \\ = Aggo_{n-1}(Aggo_{n-2}(\ldots Aggo_2(Aggo_1(x_1, x_2), x_3) \ldots x_{n-1}), x_n)$$

be an n-ary aggregation operator $Aggo: [0,1]^n \to [0,1]$ defined by the family of binary aggregation operators. Then, $A_{\mu_1,\mu_2,\ldots,\mu_n} : X \to [0,1]$ is again a T_M-fuzzy vector subspace such that

$$A_{\mu_1,\mu_2,\ldots,\mu_n} \sim \mu_i$$

for all $i \in \{1, 2, \ldots, n\}$

Proof. It is easy to prove by induction that $Aggo(x_1, \ldots, x_n)$ is an n-ary aggregation operator that is jointly strictly monotone. Also, by induction there can be proved that $A_{\mu_1,\mu_2,\ldots,\mu_n} = Aggo(\mu_1, \mu_2, \ldots, \mu_n)$ is a T_M-fuzzy vector subspace of X. It remains to show that $A_{\mu_1,\mu_2,\ldots,\mu_n} \sim \mu_i$ for all $i \in \{1, 2, \ldots, n\}$. Again, the proof is by induction.

For n = 2 it follows from results from [4].

Suppose that for $\mu_1, \mu_2, \ldots, \mu_{n-1}$

$\psi = Aggo_{n-2}(\ldots Aggo_2(Aggo_1(\mu_1, \mu_2), \mu_3) \ldots \mu_{n-1})$is a T_M-fuzzy vector subspace, denote by ψ, similar to μ_i for all $i \in \{1, 2, \ldots, n\}$. Now,

$A_{\mu_1,\mu_2,\ldots,\mu_n} = Aggo(\mu_1, \mu_2, \ldots, \mu_n) = Aggo_{n-1}(\psi, \mu_n)$ is a T_M-fuzzy vector subspace similar to μ_i for all $i \in \{1, 2, \ldots, n\}$

Conclusions

This paper presents an overview of the aggregation of fuzzy vector subspaces, defined using the t-norm T_M. It also discusses the effects of generalization to T-fuzzy vector subspaces where T is an arbitrary t-norm. We have seen in [4] that the self-aggregation of any given T_M-fuzzy vector subspace, results in another T_M-fuzzy vector subspace, without any limitations or constraints. Conversely, transition to T-fuzzy vector subspaces presents a more complex issue. In general, the aggregation of two arbitrary T_M-fuzzy vector subspaces does not necessarily yield another T_M-fuzzy vector subspace (see Example 8), and therefore, certain conditions must be imposed. Furthermore, Theorem 12, which presents a new result, addresses the problem of n-ary aggregation applicable to similar T_M-fuzzy vector subspaces, based on an n-ary aggregation operator, generated by a family of binary jointly, strictly monotone, aggregation operators. The general case of T-fuzzy vector subspaces remains an open problem in several areas.

Future work will focus on T-fuzzy vector subspaces, where T is an arbitrary t-norm.

Acknowledgement

This research was supported by the Science Fund of the Republic of Serbia, Grant no 6565, Advanced Techniques of Mathematical Aggregation and Approximative Equations Solving in Digital Operational Research- AT-MATADOR

References

[1] Ardanza-Trevijano, S., Chasco, M. J., Elorza, J., de Natividade, M., Talavera, F. J.: *Aggregation of T-subgroups.* Fuzzy Sets and Systems, 463, 2023

[2] Bejines, C., Ardanza-Trevijano, S., Chasco, M. J., Elorza, J.: *Aggregation of indistinguishability operators.* Fuzzy Sets and Systems, 446, 2022, 53-67

[3] Bejines, C., Chasco, M. J., Elorza, J.: *Aggregation of fuzzy subgroups.* Fuzzy Sets and Systems, 418, 2021, 170-184

[4] Bejines, C.: *Aggregation of fuzzy vector spaces.* Kybernetika, 59(5), 2023, 752-767

[5] Borisov, V., Chernovalova, M., Dulyasova, M., Morozov, D., Vasiliev, A.: *Fuzzy methods for comparing project situations and selecting precedent decisions.* Acta Polytechnica Hungarica, 19(10), 2022, 83-98

[6] Das, P.: *Fuzzy vector spaces under triangular norms.* Fuzzy Sets and Systems, 25, 1988, 73-85

[7] Das, P.: *Fuzzy groups and level subgroups.* Journal of Mathematical Analysis and Applications, 84, 1981, 264-269

[8] Fodor, J. C., Yager, R. R., Rybalov, A.: *Structure of uninorms.* International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 5, 1997, 411-427

[9] Forgacs, A., Lukacs, J., Horvath, R.: *The investigation of the applicability of fuzzy rule-based systems to predict economic decision-making.* Acta Polytechnica Hungarica, 18(11), 2021, 97-115. (DOI ako postoji)

[10] Fuller, R., Keresztfalvi, T.: *t-norm-based addition of fuzzy intervals.* Fuzzy Sets and Systems, 51(2), 1992, 155-159

[11] Grabisch, M., Marichal, J. L., Mesiar, R., Pap, E.: *Aggregation Functions.* Cambridge University Press, New York, 2009

[12] Klement, E. P., Mesiar, R., Pap, E.: *Triangular Norms.* Kluwer Academic Publishers, Dordrecht, 2000

[13] Katsaras, A., Liu, D.: *Fuzzy vector spaces and fuzzy vector topological spaces.* Journal of Mathematical Analysis and Applications, 58, 1977, 135-146

[14] Laufer, E. P.: *Similarity measure supported fuzzy failure mode and effect analysis.* Acta Polytechnica Hungarica, 21(2), 2024, 187-202

[15] Markova, A.: *T-sum of L-R fuzzy numbers.* Fuzzy Sets and Systems, 85(3), 1997, 379-384

[16] Mesiar, R., Komornikova, M.: *Aggregation operators.* In: Herceg, D., Surla, K. (Eds.), *Proceedings of the XI Conference on Applied Mathematics PRIM'96*, Institute of Mathematics, Novi Sad, 1997, pp. 193-211

[17] Mesiar, R.: *Shape preserving additions of fuzzy intervals.* Fuzzy Sets and Systems, 86(1), 1997, 75-80

[18] Pedraza, T., Rodriguez-Lopez, J., Valero, O.: *Aggregation of fuzzy quasi-metrics.* Information Sciences, 581, 2021, 362-389

[19] Štajner-Papuga, I., Tepavčević, A.: *A note on aggregation of T-fuzzy subgroups.* In: *Computational Intelligence and Mathematics for Tackling Complex Problems 6*, Studies in Computational Intelligence, Springer, accepted (2025)

[20] Stanković, A., Petrović, G.: *Priority decision rules with a fuzzy MCDM approach for solving flexible job shop problem: A real case study of optimizing manufacturing.* Acta Polytechnica Hungarica, 22(1), 2025, 143-162

[21] Talavera, F. J., Ardanza-Trevijano, S., Bragard, J., Elorza, J.: *Aggregation of T-subgroups of groups whose subgroup lattice is a chain.* Fuzzy Sets and Systems, 473, 2023

[22] Talavera, F. J., Ardanza-Trevijano, S., Bragard, J., Elorza, J.: *New type of domination to characterize the preservation of T-subgroups under aggregation.* Fuzzy Sets and Systems, 498, 2025

[23] Zadeh, L. A.: *Fuzzy sets.* Information and Control, 8, 1965, 338-353