Unifunctions and Nullfunctions: A New Generalization of Overlap and Grouping Functions for Bipolar Modeling

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Abstract: Uninorms and nullnorms are important aggregation functions that exhibit bipolar behavior. However, their lack of continuity, particularly for uninorms, presents challenges in practical applications. Since associativity is often not essential, especially when aggregating only two values, we propose two new classes of functions: unifunctions and nullfunctions. These are inspired by overlap and grouping functions and aim to retain the structural advantages of uninorms and nullnorms while ensuring continuity. This paper introduces these new classes, explores their fundamental properties, and demonstrates their role in solving generalized Frank's equations under relaxed axioms.

Keywords: overlap function; grouping function; unifunction; nullfunction; uninorm; bipolar aggregation

1 Introduction and Motivation

Uninorms [23] and nullnorms [4] are intensively studied classes of associative binary aggregation functions because they are applied in various fields of application such as neural networks [1, 5, 14], decision making [20, 21, 22], bipolar aggregation [9, 11, 15, 24] and many others. They both generalize the concepts of t-norms and t-conorms.

Since associativity is superfluous in many cases, Bustince et al. propose overlap [2] and grouping functions [3], focusing on continuity instead of associativity, which is often desired by the continuous nature of the model problem. We want to extend this approach to uninorms and nullnorms. Our aim is to define and study a new class of aggregation functions that generalize overlap and grouping functions in a similar way as uninorms and nullnorms generalize overlap and grouping functions.

Bipolar aggregation is based on the psychological evidence, which shows that humans judge their decision upon positive and negative aspects of the decision. Therefore, several approaches of bipolar aggregation from mathematical point of view were proposed to model such human preference. For a more exhaustive overview, see [9].

Both uninorms and nullnorms exhibit some kind of bipolar behavior, which is in fact somehow complementary. On the one hand nullnorms tend to aggregate inputs into the middle of the domain and on the other hand uninorms do it to the boundaries. The neutral element (annihilator) of a uninorm (nullnorm) divides the square of the unit interval into four subsets, where the behavior is different. In the case of uninorms, aggregation of two negative or positive incomes results in a more negative or positive outcome. Such structure supports the human's intuitive preference. If positive and negative outcome is aggregated, then uninorm is able to capture human preference due to its averaging behavior on the respective domain. Therefore, uninorms are interesting in the domain of bipolar aggregation [24], where negative and positive inputs can appear and where the reluctance to negative and the preference for positive experience appear.

However, we know from [7] that no uninorm is continuous on the whole domain. This led to a study of continuity of uninorms in [6, 12, 18]. In [19] it was shown that the points of discontinuity of a uninorm with continuous underlying functions U are covered by a characterizing set-valued function of U. From this paper, it also follows that every uninorm is discontinuous on the set $[0, e] \times [e, 1]$. In bipolar decision making, this represents the situation where one of the inputs is negative and the other is positive. In applications, the discontinuity of U does not allow a gradual change of the decision depending on the inputs. This is in contrary to situations where small changes in preference do not lead to sudden changes in the decision. Thus, such model is often limited due to these discontinuities of uninorms [23] and unintuitive behavior from the preference point of view [24]. Moreover, discontinuity brings instability of computation and other problematic issues.

Therefore, continuity may appear to be a more important property than associativity, especially if we aggregate only two elements. In order to be able to capture continuous change of preference depending on the values and preserve the most of properties of uninorms and nullnorms for which they are used in these situations as much as possible, we will propose new concepts. Note that overlap functions and grouping functions are distinct aggregation functions to t-norms and t-conorms, respectively, though with significant overlap between these classes and similar motivation. Therefore, we see these classes rather as a parallel approach to aggregation than a generalization, and hence we want to extend overlap and grouping functions in a similar manner as uninorms (nullnorms) did to t-norms and t-conorms. We expect that unifunctions (and nullfunctions), which we propose as a generalization of overlap and grouping functions, may be useful to overcome problematic issues of uninorms that cannot be continuous on the whole unit square. We have named these functions to emphasize the deep connection of the newly proposed unifunctions and nullfunctions to uninorms and nullnorms on the one hand, and to overlap and grouping functions on the other hand.

2 Preliminaries

T-norms and t-conorms are widely used in fuzzy logic theory as conjunctions and disjunctions, respectively. Since continuity is important in both computational stability and modeling gradual transition, the class of continuous t-norms and t-conorms is undoubtedly the most studied and used since the inception of the concept.

Definition 1 [13]

A binary function $T: [0,1]^2 \to [0,1]$ ($S: [0,1]^2 \to [0,1]$) *is called a t-norm (t-conorm) if it is associative, commutative, non-decreasing and it possesses the neutral element* 1 *(*0*).*

In some fields, the associativity of t-norms can appear as too restrictive and an unnecessary property. To address these phenomena, Bustince et al. [2, 3] proposed different classes of continuous binary aggregation functions, which partially cover the class of continuous t-norms and continuous t-conorms, respectively.

Definition 2 [2]

A mapping $O: [0,1]^2 \to [0,1]$ *is an overlap function if it satisfies the following conditions:*

1. O *is symmetric.*
2. $O(\mathrm{x}, \mathrm{y}) = 0$ *if and only if* $\mathrm{xy} = 0$.
3. $O(\mathrm{x}, \mathrm{y}) = 1$ *if and only if* $\mathrm{xy} = 1$.
4. O *is non-decreasing.*
5. O *is continuous.*

Definition 3 [3]

A mapping $G: [0,1]^2 \to [0,1]$ *is a grouping function if it satisfies the following conditions:*

1. G *is symmetric.*
2. $G(x, y) = 0$ *if and only if* $\mathrm{x} = \mathrm{y} = 0$.
3. $G(x, y) = 1$ *if and only if* $\mathrm{x} = 1$ *or* $\mathrm{y} = 1$.
4. G *is non-decreasing.*
5. G *is continuous.*

Another generalization of t-norms and t-conorms are uninorms, which were proposed by Yager and Rybalov [23].

Definition 4 [23]

A binary function $U: [0,1]^2 \to [0,1]$ *is called a uninorm if*

1. *U is associative.*
2. *U is commutative.*
3. *U is non-decreasing.*
4. *U possesses a neutral element* $e \in [0,1]$.

There exists a standard classification of aggregation functions into four families. A binary aggregation function $A: [0,1]^2 \to [0,1]$ is called (see [10]):

Conjunctive if $\boldsymbol{A(\mathrm{x},) \leq \min(x,y)}$,

Disjunctive if $\boldsymbol{A(x,y) \geq \max(x,y)}$,

Averaging if $\boldsymbol{\min(\mathrm{x},\mathrm{y}) \leq A(\mathrm{x},\mathrm{y}) \leq \max(\mathrm{x},\mathrm{y})}$,

Mixed if $\boldsymbol{A}$ is in none of the above four families.

The structure of a uninorm shows that it has conjunctive behavior on the square $[0,e]^2$, disjunctive on $[e,1]^2$ and averaging otherwise. For the sake of simplicity in the following, we define the set $A(x) = [0,x] \times [x,1] \cup [x,1] \times [0,x]$ for $x \in [0,1]$.

A uninorm U restricted to $[0,e]^2$ is a linear transformation of a t-norm T_U and restricted to $[e,1]^2$ is a linear transformation of a t-conorm S_U, which are called the underlying t-norm and t-conorm, respectively, or jointly the underlying functions of U. Note that no uninorm is continuous on the whole unit square [7].

Other important class of associative aggregation functions are t-operators [16] and nullnorms [4], which as was later shown, coincide [17].

Definition 5 [4]

A nullnorm $V: [0,1]^2 \to [0,1]$ *is a commutative, associative and non-decreasing binary operator with the annihilator* $a \in [0,1]$ *that satisfies*

1. $V(x,0) = x$, *for* $x \in [0,a]$.
2. $V(x,1) = x$ *for* $x \in [a,1]$.

Remark 6

For completeness-sake we also provide the definition of t-operators (see [16]): A function $V: [0,1]^2 \to [0,1]$ *is said to be a t-operator if it is commutative, associative, non-decreasing and if*

1. $V(0,0) = 0$ *and* $V(1,1) = 1$,
2. *functions* $f(x) = V(0,x)$ *and* $g(x) = V(1,x)$ *are continuous.*

The structure of nullnorms is similar to that of uninorms, i.e., it is based on underlying functions. A nullnorm restricted to $[0, e]^2$ is a linear transform of a t-conorm (i.e., disjunctive), restricted to $[e, 1]^2$ is a linear transform of a t-norm (i.e., conjunctive) and constant on $A(a)$.

In what follows we will need also the 1-Lipschitz property which in applications often occur.

Definition 7

Let $f: [0,1] \to [0,1]$ *be a unary function.* f *is called 1-Lipschitz if*

$|f(x) - f(y)| \le |x - y|$

holds for all $(x, y) \in [0,1]^2$.

We say that a binary function $F: [0,1]^2 \to [0,1]$ *is 1-Lipschitz if it is 1-Lipschitz in both of its coordinates.*

3 Unifunctions and Nullfunctions

In this section, we will propose and discuss a modification of uninorms inspired by the notions of overlap and grouping functions (Definitions 2 and 3), to overcome the discontinuity limitation of uninorms, which were mentioned in the previous section.

Definition 8

Let $e \in [0,1]$. *A binary function* $UF_e: [0,1]^2 \to [0,1]$ *is called a unifunction if the following conditions hold.*

1. UF_e *is non-decreasing.*
2. UF_e *is symmetric.*
3. *For* $(x, y) \in [0, e]^2$, $UF_e(x, y) = 0$ *if and only if* $x = 0$ *or* $y = 0$.
4. *For* $(x, y) \in [e, 1]^2$, $UF_e(x, y) = 1$ *if and only if* $x = 1$ *or* $y = 1$.
5. $UF_e(x, y) = e$ *then* $x < e < y$ *or* $y < e < x$ *or* $x = y = e$.
6. UF_e *is continuous.*

In [4], the authors studied whether there exists a solution to the well-known Frank's equation [8], if one of the functions in that equation is a uninorm, and this problem led to introducing nullnorms. However, they showed that there is no pair of a uninorm U_e and a nullnorm N_a solving the Frank's equation. Similarly to uninorms, nullnorms are a generalization of t-norms and t-conorms, which locate the annihilator to the interior domain rather than to the end-points of the interval. Following the steps of such an approach we introduce a modification of nullnorms in the next definition.

Definition 9

Let $a \in [0,1]$. A binary function $NF_a: [0,1]^2 \to [0,1]$ is called a nullfunction if the following conditions hold.

1. *NF_a is non-decreasing.*
2. *NF_a is symmetric.*
3. *a is the annihilator of NF_a.*
4. *$NF_a(x,y) \neq a$ if $(x,y) \notin A(a)$.*
5. *$NF_a(x,y) = 0$ if and only if $x = y = 0$.*
6. *$NF_a(x,y) = 1$ if and only if $x = y = 1$.*
7. *NF_a is continuous.*

Remark 10

Observe that if $e = 0$ (or $a = 1$) then the corresponding unifunction UF_0 (nullfunction NF_1) reduces to a grouping function, similarly if $e = 1$ (or $a = 0$) then the corresponding unifunction UF_1 (nullfunction NF_0) degenerates into an overlap function.

The basic structure of unifunctions and nullfunctions is similar to the structure of uninorms and nullnorms, respectively.

Proposition 11

Let $UF_e; [0,1]^2 \to [0,1]$ be a unifunction then the following statements hold.

1. *There exists an overlap function $O: [0,1]^2 \to [0,1]$ such that $UF_e(x,y) = eO(\frac{x}{e}, \frac{y}{e})$ for $x, y \in [0,e]^2$.*
2. *There exists a grouping function $G: [0,1]^2 \to [0,1]$ such that $UF_e(x,y) = e + (1-e)G(\frac{x-e}{1-e}, \frac{y-e}{1-e})$ for $x, y \in [e,1]^2$.*
3. *$UF_e(x,y) \in [min(UF_e(x,e), UF_e(y,e)), max(UF_e(x,e), UF_e(y,e))]$ for pairs $(x,y) \in A(e)$.*

PROOF:

1. Since $f(x) = \frac{x}{e}$ is a continuous increasing bijection from $[0,1]$ to $[0,e]$, there are only two items to prove, namely that $UF_e(x,y) = e$ for $(x,y) \in [0,e]^2$ if and only if $x = y = e$. This follows from item 5 of Definition 8. The other item is that $UF_e(x,y) = 0$ for $(x,y) \in [0,e]^2$ if and only if $x = 0$ or $y = 0$, but that follows from item 3 of Definition 8. Hence, UF_e restricted to $[0,e]^2$ is an overlap function.
2. The proof is analogous to that of the previous item.
3. If $x \leq e \leq y$ then $UF_e(x,e) \leq UF_e(x,y) \leq UF_e(y,e)$ and similarly vice versa. □

Proposition 12

Let $NF_a; [0,1]^2 \to [0,1]$ *be a nullfunction then the following statements hold.*

1. *There exists a grouping function* $G: [0,1]^2 \to [0,1]$ *such that* $NF_a(x,y) = aG(\frac{x}{a}, \frac{y}{a})$ *for* $x, y \in [0,a]^2$.
2. *There exists an overlap function* $O: [0,1]^2 \to [0,1]$ *such that* $NF_a(x,y) = a + (1-a)O(\frac{x-a}{1-a}, \frac{y-a}{1-a})$ *for* $x, y \in [a,1]^2$.
3. $NF_a(x,y) = a$ *for all* $(x,y) \in A(a)$.

We omit to prove this assertion since the proof is similar to that of Proposition 11.

We will refer to respective overlap and grouping function from Propositions 11 and 12 as underlying overlap and grouping function or jointly as underlying functions.

Similarly to uninorms, we propose some important subclasses of unifunctions and nullfunctions.

Definition 13

Let $UF_e: [0,1]^2 \to [0,1]$ *be a unifunction then*

1. UF_e *is called a strong unifunction, if* e *is its neutral element, i.e., if* $UF_e(e,x) = UF_e(x,e) = x$ *for all* $x \in [0,1]$.
2. UF_e *is called a conjunctive unifunction if* $UF_e(0,1) = 0$.
3. UF_e *is called a disjunctive unifunction if* $UF_e(0,1) = 1$.
4. UF_e *is called an idempotent unifunctions if* $UF_e(x,x) = x$ *for all* $x \in [0,1]$.

Definition 14

Let $NF_a: [0,1]^2 \to [0,1]$ *be a nullfunction then*

1. NF_a *is called strong, if* $NF_a(0,x) = x$ *for all* $x \in [0,a]$ *and* $NF_a(1,x) = x$ *for all* $x \in [a,1]$.
2. NF_a *is called idempotent, if* $NF_a(x,x) = x$ *for all* $x \in [0,1]$.

Now we will focus on properties of unifunctions. Note that the following statements hold.

Proposition 15

Let $UF_e: [0,1]^2 \to [0,1]$ *be a unifunction. Then it is strong if and only if the following properties hold.*

1. $min(x,y) \leq UF_e(x,y) \leq max(x,y)$ *for all* $(x,y) \in A(e)$.
2. UF_e *is conjunctive on* $[0,e]^2$, *i.e.,* $UF_e(x,y) \leq min(x,y)$.
3. UF_e *is disjunctive on* $[e,1]^2$, *i.e.,* $UF_e(x,y) \geq max(x,y)$.

PROOF: Let UF_e be a strong unifunction then

1. for $(x,y) \in [0,e]^2$, $UF_e(x,y) \leq UF_e(x,e) = x$ and $UF_e(x,y) \leq UF_e(e,y) = y$, i.e., $UF_e(x,y) \leq \min(x,y)$.
2. for $(x,y) \in [e,1]^2$, $UF_e(x,y) \geq UF_e(x,e) = x$ and $UF_e(x,y) \geq UF_e(e,y) = y$, i.e., $UF_e(x,y) \geq \max(x,y)$.
3. for $x \leq e \leq y$ we obtain $x = UF_e(x,e) \leq UF_e(x,y) \leq UF_e(e,y) = y$ and similarly we can show that $x \leq UF_e(y,x) \leq y$.

Let UF_e be a unifunction such that items 1.-3. hold. If for $x < e$, $UF_e(x,e) < x$ then this violates item 3 and $UF_e(x,e) > x$ violates item 1. Similarly, if $x > e$ then, $UF_e(x,e) < x$ violates item 2 and $UF_e(x,e) > x$ violates item 3. Hence, UF_e is a strong unifunction.

□

Proposition 16

Let $e \in]0,1[$ *then*

1. *no unifunction* UF_e *is a conjunctive aggregation function.*
2. *no unifunction* UF_e *is a disjunctive aggregation function.*
3. *unifunction* UF_e *is an averaging aggregation function if and only if both its underlying functions are idempotent.*

PROOF: Let UF_e be any unifunction.

1. Choose $x \in]e,1[$ then $UF_e(x,1) = 1 > x$, which implies that UF_e is not conjunctive.
2. Choose $x \in]0,e[$ then $UF_e(x,0) = 0 < x$, which implies that UF_e is not disjunctive.
3. Let $x,y \in [0,1]$ and $x \leq y$ then $x = UF_e(x,x) \leq UF_e(x,y) \leq UF_e(y,y) = y$, and since UF_e is symmetric we see that UF_e is averaging. Vice versa, $x \leq UF_e(x,x) \leq x$ implies that UF_e is an idempotent unifunction and hence both its underlying functions are idempotent.

□

Proposition 17

Let $a \in]0,1[$ *then*

1. *no nullfunction* NF_a *is a conjunctive aggregation function.*
2. *no nullfunction* NF_a *is a disjunctive aggregation function.*
3. *a nullfunction* NF_a *is an averaging aggregation function if and only if both of its underlying functions are idempotent.*

The proof of Proposition 17 is similar to that of Proposition 16. For this reason, it is omitted.

Proposition 18

Let $e \in]0,1[$. *Then, unifunction* UF_e *possesses an annihilator* a *if and only if* UF_e *is a conjunctive or a disjunctive unifunction.*

PROOF: Clearly for a conjunctive unifunction UF_e 0 is an annihilator since $0 = UF_e(0,0) \leq UF_e(0,x) = UF_e(x,0) \leq UF_e(1,0)$. Similarly we can show that 1 is an annihilator of a disjunctive unifunction UF_e.

Assume that a is the annihilator of UF_e. If $a \in [0,e]$ then $a = UF_e(0,a) = 0$, which implies $a = 0$ and since it is the annihilator of UF_e we see that $UF_e(0,1) = 0$, i.e., UF_e is a conjunctive unifunction. In the same way we can show that if $a \in [e,1]$ then UF_e is a disjunctive unifunction.

□

Theorem 19

Let $e \in]0,1[$. *Then, there is no associative unifunction* UF_e.

PROOF: Let $e \in]0,1[$. Assume UF_e is associative. Then, from Proposition 11 we obtain that UF_e restricted to $[0,e]^2$ is isomorphic to some associative overlap function. Since each associative overlap function is a continuous t-norm (see [2]), we get $UF_e(x,e) = UF(e,x) = x$ for $x \leq e$. Similarly, using the duality between overlap and grouping functions and t-norms and t-conorms, we can show that $UF_e(y,e) = UF_e(e,y) = y$ for $y \geq e$. However, this implies that e is the neutral element of UF_e, which is non-decreasing and associative, hence it is a uninorm. Since there exists no proper uninorm that is continuous, we get a contradiction.

□

Theorem 20

Let $a \in [0,1]$. *Then, a nullfunction* $NF_a: [0,1]^2 \to [0,1]$ *is associative if and only if* NF_a *is a continuous nullnorm with its annihilator* a, *such that its underlying t-norm has no* 0*-divisor and its underlying t-conorm has no* 1*-divisor.*

PROOF: Since NF_a is continuous, the sections $NF_a(0,\cdot)$, $NF_a(1,\cdot)$ are also continuous functions. From the associativity and the non-decreasing nature of NF_a we see that NF_a is a t-operator, i.e., a nullnorm. Then both underlying functions of the nullnorm NF_a are continuous since NF_a is a nullfunction. Moreover from $NF_a(x,y) \neq a$ for $(x,y) \in [0,a[^2$ follows that its underlying t-conorm has no 1-divisor and $NF_a(x,y) \neq a$ for $]a,1]^2$ implies that its underlying t-norm has no 0-divisor.

The reverse statement is obvious.

□

In the following we will show how to construct a nullfunction and a unifunction based on the given underlying functions. For the sake of simplicity, we will assume that underlying overlap and underlying grouping functions are immediately defined on the respective subintervals.

Theorem 21

Let $a \in [0,1]$, $G: [0,a]^2 \to [0,a]$ *be a grouping function and* $O: [a,1]^2 \to [a,1]$ *be an overlap function. Then* $NF_a: [0,1]^2 \to [0,1]$ *is the nullfunction with the underlying functions* G *and* O *if and only if*

$$NF_a(x,y) = \begin{cases} G(x,y) & \text{if } (x,y) \in [0,a]^2, \\ O(x,y) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{otherwise.} \end{cases} \tag{1}$$

We skip the proof of Theorem 21 since it is obvious.

For unifunctions the situation is much more complicated, but similarly as for uninorms we propose a construction method based on an additive generator.

Theorem 22

Let $e \in \,]0,1[$, $O: [0,e]^2 \to [0,e]$ *be an overlap function and* $G: [e,1]^2 \to [e,1]$ *be a grouping function. Moreover, let* $f: [0,1] \to [-\infty,\infty]$ *be a continuous increasing function such that* $f(e) = 0$ *and* $f(0) \neq -\infty$ *or* $f(1) \neq \infty$. *Define* $g(x) = G(x,e)$ *and* $o(x) = O(x,e)$ *for* $x \in [0,1]$. *Then,* UF_e *given by*

$$UF_e(x,y) = \begin{cases} O(x,y) & \text{if } (x,y) \in [0,e]^2, \\ G(x,y) & \text{if } (x,y) \in [e,1]^2, \\ f^{-1}(f(o(x) + f(g(y))) & \text{if } (x,y) \in [0,e[\times]e,1], \\ f^{-1}(f(o(y) + f(g(x))) & \text{if } (x,y) \in]e,1] \times [0,e[. \end{cases} \tag{2}$$

is a unifunction with underlying functions O *and* G, *respectively.*

PROOF: We will check all properties of unifunctions item by item (see Definition 8).

1. We will only show that $UF_e(x_0,\cdot)$ is non-decreasing for $x_0 \leq e$, otherwise the proof is analogous. For $y_1 < y_2 \leq e$ the result obviously holds since O is non-decreasing. Similarly for $e < y_1 < y_2$ we obtain the desired result since f and thus f^{-1} is increasing bijection. Moreover, g as defined is a non-decreasing function. Therefore, $f^{-1}(f(o(x_0) + f(g(\cdot)))$ is a non-decreasing function on $]e,1[$. Now, only the case when $y_1 \leq e < y_2$ is necessary to check. In this case,

$$f^{-1}(f(o(x_0) + f(g(y_2))) \geq f^{-1}(f(o(x_0) + f(g(e))) = f^{-1}(f(o(x_0) + f(e)) = f^{-1}(f(o(x_0) + 0) = f^{-1}(f(o(x_0)) = o(x_0) \geq O(x_0,y_1).$$

2. UF_e is symmetric by Definition 8.
3. Follows from the structure of O.
4. Follows from the structure of G.

5. Follows from the structure of O and G.
6. We need only to check continuity on the neighbourhood of (x,e) and (e,x) for $x \in [0,1]$. We will check only the first case for $x \in [0,e]$ since otherwise we would proceed analogously. Observe that O is continuous on $[0,e]^2$ and $f^{-1}(f(o(\cdot)+f(g(\cdot)))$ is continuous on $[0,e]\times[e,1]$, because it is a combination of continuous functions. Therefore, if $O(x,e)=f^{-1}(f(o(x)+f(g(e)))$ then the continuity of UF_e is satisfied. Now,

$$f^{-1}(f(o(x)+f(g(e)))=f^{-1}(f(o(x)+f(e))=f^{-1}(f(o(x)+0)=o(x)\\ =O(x,e),$$

which concludes the proof.

□

Remark 23

1. *When $f(0),f(1)\in\mathbb{R}$ then $0<UF_e(0,1)=UF_e(1,0)<1$.*
2. *When $f(0)=-\infty$ then $UF_e(0,1)=UF_e(1,0)=0$, i.e., f generates a conjunctive unifunction.*
3. *When $f(1)=\infty$ then $UF_e(0,1)=UF_e(1,0)=1$, i.e., f generates a disjunctive unifunction.*
4. *Observe that if both $f(0)=-\infty$ and $f(1)=\infty$ then the continuity of UF_e is violated and therefore in that case UF_e is not a unifunction.*

Example 24

Consider an overlap function O on $[0,\frac{1}{2}]^2$ given by $O(x,y)=\sqrt{xy}$ and a grouping function G on $[\frac{1}{2},1]^2$ given by $G(x,y)=max(x,y)$. Then for any additive generator f fulfilling the condition of Theorem 22, the generated unifunction is an idempotent unifunction with underlying functions O and G.

1. *If $f(x)=x-e$ then* $UF^1_{\frac{1}{2}}=\begin{cases}\sqrt{xy} & if\,(x,y)\in[0,\frac{1}{2}]^2,\\ max(x,y) & if\,(x,y)\in[\frac{1}{2},1]^2,\\ \sqrt{\frac{x}{2}}+y-\frac{1}{2} & if\,(x,y)\in[0,\frac{1}{2}]\times[\frac{1}{2},1]\\ x+\sqrt{\frac{y}{2}}-\frac{1}{2} & if\,(x,y)\in[0,\frac{1}{2}]\times[\frac{1}{2},1].\end{cases}$
2. *If $f(x)=ln(2x)$ then* $UF^2_{\frac{1}{2}}=\begin{cases}\sqrt{xy} & if\,(x,y)\in[0,\frac{1}{2}]^2,\\ max(x,y) & if\,(x,y)\in[\frac{1}{2},1]^2,\\ y\sqrt{2x} & if\,(x,y)\in[0,\frac{1}{2}]\times[\frac{1}{2},1]\\ x\sqrt{2y} & if\,(x,y)\in[0,\frac{1}{2}]\times[\frac{1}{2},1].\end{cases}$

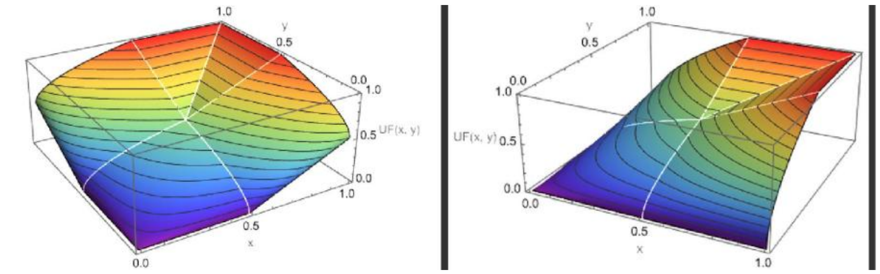

Figure 1
Visualization of Example 24. Unifunction $UF^1_{\frac{1}{2}}$ n the left side and unifunction $UF^2_{\frac{1}{2}}$ on the right side.

4 Generalized Frank's Equation and 1-Lipschitz Property

In this section we want to study whether some pairs of unifunctions and nullfunctions solve the Frank's equation [8]. Original Frank's equation shows the connection between a subclass of t-norms and t-conorms. Initial motivation for defining nullnorms was the aim to solve generalized Frank's equation for uninorms [4]. Due to that no uninorm is continuous (and thus 1-Lipschitz), there was no such connection between these two classes established. We will show that for certain types of unifunctions and nullfunctions such connection does exist. Consider the generalized Frank's equation for unifunction and nullfunctions given by

$$UF_e(x,y) + NF_a(x,y) = x + y \quad (3)$$

for any $(x,y) \in [0,1]^2$.

Proposition 25

Let $a, e \in [0,1]$. If a pair of a unifunction UF_e and a nullfunction NF_a is a solution to equation (3) then, both functions are 1-Lipschitz and $a = e$.

PROOF: First of all, we will show that $a = e$. Equation (3) and the fact that a is the annihilator of NF_a imply

$$NF_a(a,e) + UF_e(a,e) = a + UF_e(a,e) = a + e,$$

and thus $UF_e(a,e) = e$. By Definition 8 we get that $e = a$. Really, if $a < e$ we get $UF_e(a,e) < e$ and if $a > e$ then $UF_e(a,e) > e$, and in both cases equation (3) is violated.

The fact that both, UF_e and NF_a are 1-Lipschitz is due to the fact that the function at the right-hand-side of equation (3) is 1-Lipschitz and that $UF_e(x,y) \geq 0$ and $NF_a(x,y) \geq 0$ for all $(x,y) \in [0,1]^2$. □

Together with the results in [2], where 1-Lipschitz overlap functions were characterized, we can characterize all 1-Lipschitz unifunctions and 1-Lipschitz nullfunctions.

Proposition 26

Let $NF_a: [0,1]^2 \to [0,1]$ be a nullfunction, then it is 1-Lipschitz if and only if both its underlying functions are 1-Lipschitz.

PROOF: The necessity is obvious. Hence, we prove only sufficiency. Choose $(x_1, y_1) \in [0, a]^2$ and $(x_2, y_2) \in [a, 1]^2$. Then

$$NF_a(x_2, y_2) - NF_a(x_1, y_1) = NF_a(x_2, y_2) - a + a - NF_a(x_1, y_1) =$$
$$NF_a(x_2, y_2) - NF_a(x_2, a) + NF(a, y_1) - NF(x_1, y_1) \leq$$
$$k|y_2 - a| + k|a - x_1| \leq k|y_2 - y_1| + k|x_2 - x_1|.$$

All the other cases can be shown analogously.

□

Theorem 27

Let $UF_e: [0,1]^2 \to [0,1]$ be a 1-Lipschitz binary function then it has the following form.

$$UF_e(x, y) = \begin{cases} e\, O(\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [0, e]^2 \\ e + (1 - e)\, G(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x, y) \in [e, 1]^2 \\ x + y - e & \text{otherwise,} \end{cases} \quad (3)$$

where O and G is a 1-Lipschitz overlap and grouping function, respectively.

PROOF: Clearly O and G have to be 1-Lipschitz in order to preserve the 1-Lipschitz property on $[0, e]^2$ and $[e, 1]^2$, respectively. We will show that UF_e has a neutral element e. Choose $x \leq e \leq y$ then on the one hand

$$UF_e(e, y) - UF_e(e, e) = UF_e(e, y) - e \leq |y - e|,$$

i.e., $UF_e(e, y) \leq y$ and on the other hand

$$UF_e(e, 1) - UF_e(e, y) = 1 - UF_e(e, y) \leq |1 - y|,$$

i.e., $UF_e(e, y) \geq y$. Hence, $UF_e(e, y) = y$. Similarly, we can show that $UF_e(x, e) = x$.

For $UF_e(x, y)$ we show that

$$UF_e(x, y) - UF_e(x, e) = UF_e(x, y) - x \leq y - e$$
$$UF_e(x, y) \leq x + y - e$$

and

$$UF_e(e, y) - UF_e(x, y) = y - UF_e(x, y) \leq e - x$$
$$UF_e(x, y) \geq x + y - e.$$

So, $UF_e(x, y) = x + y - e$ for $(x, y) \in A(e)$.

□

Observe that this solution exists for strong unifunctions with 1-Lipschitz underlying functions and with an additive generator $f(x) = x - e$.

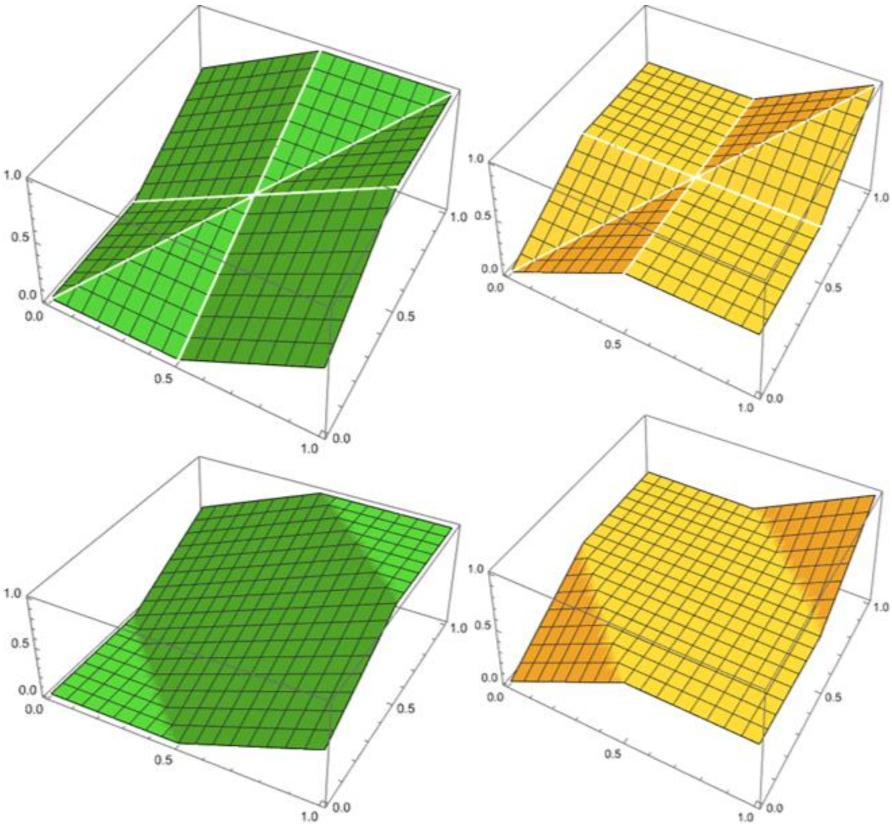

Figure *2*
From left to right - unifunction $UF_{\frac{1}{2}}$, nullfunction $NF_{\frac{1}{2}}$ in the top row and $GUF_{\frac{1}{2}}$ and $GNF_{\frac{1}{2}}$ are visualized in the bottom row

Example 28

1. *Consider*

$$UF_e(x, y) = \begin{cases} min(x, y) & if\ (x, y) \in [0, e]^2, \\ max(x, y) & if\ (x, y) \in [e, 1]^2, \\ x + y - e & otherwise, \end{cases}$$

and

$$NF_e(x,y) = \begin{cases} max(x,y) & if\,(x,y) \in [0,e]^2, \\ min(x,y) & if\,(x,y) \in [e,1]^2, \\ e & otherwise, \end{cases}$$

is a pair solving equation (3). *Observe that UF_e is a strong unifunction and NF_e is a nullfunction and an idempotent nullnorm, too.*

2. *Consider the pair of functions given by $GUF_e = med(0, x+y-e, 1)$ and $GNF_e = med(x+y, e, 1+e-x-y)$.*

   This pair of functions solves equation (3), *but in this case GUF_e is not a unifunction and GNF_e is not a nullfunction. These functions are non-decreasing and continuous, even* 1*-Lipschitz. The reason why they are not a unifunction and nullfunction, respectively, is due to the underlying functions of GUF_e and GNF_e are neither overlap nor grouping functions. However, the underlying functions are pairs of Frank's t-norm and t-conorm solving equation* (3).

Conclusions

In this paper, we introduced the notion of unifunction and nullfunction motivated by uninorms and nullnorms. These operators provide a useful framework for bipolar aggregation where continuity is essential and associativity is not required. Moreover, we presented some examples and showed possibility to construct them. The attention was then focused to solving the well-known Frank's equation (3). Further, we investigated some basic properties of unifunctions and nullfunctions.

Future research will focus on a deeper classification of these functions, investigation of multi-argument extensions, and their integration into fuzzy decision models. Moreover, practical applications in preference modeling and AI reasoning systems will be explored to validate their utility in real-world contexts.

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