Asymmetric general Choquet integrals

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Abstract: A notion of a generated chain variation of a set function m with values in $[-1,1]$ *is proposed. The space BgV of set functions of bounded g-chain variation is introduced and properties of set functions from BgV are discussed. A general Choquet integral of bounded* $\mathcal{A}$*-measurable function is defined with respect to a set function* $m \in BgV$*. A constructive method for obtaining this asymmetric integral is considered. A general fuzzy integral of bounded g-variation, comonotone* $\oplus$*-additivite and positive* $\odot$*-homogenous is represented by a general Choquet integral. The representation of a general Choquet integral in terms of a pseudo Lebesque-Stiltjes integral is obtained.*

Keywords: symmetric pseudo-operations, non-monotonic set function, general fuzzy integral, asymmetric Choquet integral

1 Introduction

The Choquet integral is often used in economics, pattern recognition and decision analysis as nonlinear aggregation tool [4, 5, 6, 20, 21, 23, 24]. Most of the studies of non-additive set functions and integrals have been focused to the case when their values are in non-negative interval (fuzzy measures), e.g., $[0,1]$. A fuzzy measure $m : \mathcal{A} \to [0,1]$ (or $[0,\infty]$), $m(\varnothing) = 0$ is a non-decreasing set function, defined on σ-algebra $\mathcal{A}$. Integrals can be viewed as an extension of underlining measures, see [9, 10].

Choquet integral (introduced in [3]) of $\mathcal{A}$-measurable non-negative function f with respect to a fuzzy measure $m : \mathcal{A} \to [0,\infty]$ is defined by

$$C_m(f) = \int_0^\infty m\{x | f(x) \geq t\} dt.$$

The main properties of the Choquet integral are monotonicity and comonotone additivity, see [4, 18]. For a finite fuzzy measure m and $\mathcal{A}$-measurable $f : X \to \mathbb{R}$, $f^+ = f \vee 0$,

$f^- = (-f) \vee 0$ we have

$$C_m(f) = C_m(f^+) - C_{\overline{m}}(f^-),$$

where $\overline{m}$ is the conjugate set function of a fuzzy measure m, given by $\overline{m}(E) = m(X) - m(E^c)$, for $E \in \mathcal{A}$, where $E^c = X \setminus E$. The last integral is known under the name *asymmetric Choquet integral*. In [16] it has been shown that this integral is well defined on the class of bounded $\mathcal{A}$-measurable functions with respect to all real-valued set functions, $m : \mathcal{A} \to \mathbb{R}$ of bounded chain variation, such that $m(\varnothing) = 0$, even if they are non-monotonic. The asymmetric Choquet integral is linear with respect to m, hence (see [16, 18])

$$C_m(f) = C_{m_1}(f) - C_{m_2}(f).$$

Fuzzy integrals corresponding to an appropriate couple ($\oplus$,$\odot$) of pseudo-operations have been studied in [12, 13, 17, 18, 19, 25]. Symmetric pseudo-operations are introduced in [6, 7]. A construction of general fuzzy integral has been studied in [2, 10, 25]. As a special type of such integral, the Choquet-like integral, introduced in [12], is defined with respect to pseudo-operations with a generator. It can be viewed as a transformation of the Choquet integral. The Choquet-like integral related to some non-decreasing function $g : [0,1] \to [0,\infty]$, $g(0) = 0$, defined for a non-negative $\mathcal{A}$-measurable function f and a fuzzy measure m, is given by

$$C_m^g(f) = g^{-1}\left(C_{g \circ m}(g \circ f)\right) \tag{1}$$

This integral is also defined for a real-valued function f, if for g is taken its odd extension to the whole real line [12, 13], and we shall call it a general Choquet integral.

The aim of this paper is to present a general Choquet integral defined with respect to set functions of bounded g- chain variation. As we shall see, this integral is of bounded g-variation asymmetric, comonotone $\oplus$-additive and positively $\odot$-homogenous.

The paper is organized as follows. Section 2 is devoted to preliminary notions and definitions of symmetric pseudo-operations. In Section 3 we introduce a g-chain variation of set functions and we consider the space of set functions of bounded g-chain variation BgV. In Section 4 we introduce the notion of a signed $\oplus_S$-measure. A pseudo-difference representation of a signed $\oplus_S$-measure is obtained. In Section 5 we introduce a general fuzzy integral defined with respect to $m \in BgV$. We consider its relation with the asymmetric general Choquet integral, i.e., Choquet-like integral (defined by (1), w.r.t. $m \in BgV$) and present its representation in the term of a pseudo Lebesque-Stiltjes integral. As a consequence, in the case of an underlining signed $\oplus_S$-measure this integral reduces to a pseudo Lebesque integral.

2 Symmetric pseudo-operations

We recall definitions of a t-conorm and pseudo-operations according to [6, 7, 9, 10].

Definition 1 *A triangular conorm (t-conorm) is a comutative, associative, non-decreasing function $S : [0,1]^2 \to [0,1]$, with neutral element* 0.

Definition 2 *An additive generator* $s:[0,1]\to[0,\infty]$ *of a t-conorm S (if it exists) is left continuous at* 1*, increasing function, such that* $s(0)=0$*, and for all* $(x,y)\in[0,1]^2$ *we have*

$$S(x,y)=s^{(-1)}(s(x)+s(y)),$$
$$s(x)+s(y)\in Ran(s)\cup[s(1),\infty],$$

where $s^{(-1)}$ *is a pseudo-inverse function of s (see[9]).*

Definition 3 *Let* $S:[0,1]^2\to[0,1]$ *be a continuous triangular conorm.* Pseudo-addition $\oplus_S:[-1,1]^2\to[-1,1]$ *, is defined by*

$$x\oplus_S y \quad = \quad \begin{cases} S(x,y), & (x,y)\in[0,1]^2 \\ \text{-}S(|x|,|y|), & (x,y)\in[-1,0]^2 \\ a, & (x,y)\in[0,1]\times]-1,0],x\geqslant|y| \\ b, & (x,y)\in[0,1[\times[-1,0],x\leqslant|y| \\ 1\,or\text{-}1, & (x,y)\in\{(1,-1),(-1,1)\} \\ y\oplus_S x, & else, \end{cases}$$

where $a=\inf\{z\mid S(-y,z)\geqslant x\}$ *and* $b=-\inf\{z\mid S(x,z)\geq -y\}$*.*

The binary operation $\oplus_S$ is commutative, monotone, with neutral element 0. If it is associative, e.g., if S is a strict t-conorm, $\oplus_S$ can be extended to n-ary operation. Then for all n-tiple $(x_1,x_2,\ldots,x_n)\in[-1,1]^n$ we define:

$$\bigoplus_{i=1}^{n}{}_S\; x_i=\left(\bigoplus_{i=1}^{n-1}{}_S\; x_i\right)\oplus_S x_n. \tag{2}$$

Definition 4 *Let S be a continuous t-conorm.* The pseudo-difference *associated to t-conorm S is given by:*

$$x\ominus_S y=x\oplus_S(-y) \tag{3}$$

for all $(x,y)\in[-1,1]^2\setminus\{(1,1),(-1,-1)\}$*. By the convention* $1\ominus_S 1=a$*,* $a\in\{\pm1,0\}$*.*

Example 1 *For all* $(x,y)\in[-1,1]^2\setminus\{(1,1),(-1,-1)\}$ *and for maximum* $\vee$*, Yager t-conorm* S_p^Y *and Hamacher t-conorm (Einstein sum)* S_2^H *(see [10]), we have, respectively:*

(i) $x\ominus_\vee y=\text{sign}(x-y)(|x|\vee|y|)$*;*

(ii) For $p=2k-1$*,*

$$x\ominus_{S_p^Y}y=\begin{cases} -1, & x^p-y^p<-1, \\ \sqrt[p]{x^p-y^p}, & -1\leq x^p-y^p\leq 1, \\ 1, & x^p-y^p>1; \end{cases}$$

(iii) $x\ominus_{S_2^H}y=\frac{x-y}{1-xy}$.

Let S be a strict t-conorm with an additive generator $s : [0,1] \to [0,\infty]$. Let $g : [-1,1] \to [-\infty,\infty]$ be defined by:

$$g(x) \quad = \quad \begin{cases} s(x), & x \geq 0 \\ -s(-x), & x < 0 \end{cases} . \tag{4}$$

The function g is the symmetric extension of s, so it is a strictly increasing function.

A pseudo-addition $\oplus_S$ can be transformed to a binary operation U on $[0,1]$, i.e., to a generated uninorm. The results contained in the following proposition have been shown in [6, 7, 9].

Proposition 1 *Let S be a strict t-conorm with an additive generator s, pseudo-addition $\oplus_S$ and function g defined by (4), then:*

(i) for all $x,y \in [0,1]$

$$x \ominus_S y \quad = \quad g^{-1}(g(x) - g(y));$$

(ii) for all $x,y \in [-1,1]$

$$x \oplus_S y \quad = \quad g^{-1}(g(x) + g(y)); \tag{5}$$

(iii) for all $z,t \in [0,1]$

$$U(z,t) \quad = \quad u^{-1}(u(z) + u(t)),$$

where $u : [0,1] \to [-\infty,\infty]$, is given by $u(x) = g(2x-1)$, with the convention $\infty - \infty \in \{\infty, -\infty\}$.

It is clear that (i) holds for all $(x,y) \in [-1,1]^2 \setminus \{(1,1),(-1,-1)\}$. It is shown in [7] that $(]-1,1[, \oplus_S)$ is an Abelian group.

It is a well known fact that a pseudo-multiplication $\odot : [-1,1]^2 \to [-1,1]$, which is distributive with respect to $\oplus_S$, can be defined using the additive generator of pseudo-addition $\oplus_S$, i.e., for $g : [-1,1] \to [-\infty,\infty]$, $\odot$ is defined by:

$$x \odot y = g^{-1}(g(x)g(y)), \tag{6}$$

for all $(x,y) \in \,]-1,1[^2$. The pseudo-multiplication defined in this manner is commutative, associative with neutral element $e_\odot \in \,]0,1[$ and distributive with respect to pseudo-addition $\oplus_S$.

Example 2 *Let $\oplus_{S_P}$ be the pseudo-addition induced by the probabilistic sum $S_P : [0,1]^n \to [0,1]$, defined by*

$$S_P(x_1, x_2, \ldots, x_n) = 1 - \prod_{i=1}^{n} (1 - x_i).$$

The additive generator g of $\oplus_{S_P}$ is defined by:

$$g(x) = \begin{cases} -\ln(1-x), & x \geq 0 \\ \ln(1+x), & x < 0 \end{cases} .$$

Let $\odot$ be given by: $x \odot y = g^{-1}(g(x)g(y))$, for all $x, y \in \,]-1,1[$, i.e.,

$$x \odot y = sign(x \cdot y)\left(1 - e^{-\ln(1-|x|)\ln(1-|y|)}\right).$$

For all $x \in]-1,1[\setminus\{0\}$ we have:

$$x \odot e_{\odot} = x \quad i \quad x \odot x^{-1} = e_{\odot},$$

where the neutral element is given by $e_{\odot} = 1 - \frac{1}{e}$, and an inverse element, for $x \in]-1,1[\setminus\{0\}$ is given by $x^{-1} = sign(x)\left(1 - e^{\frac{1}{\ln(1-|x|)}}\right)$. Hence, $(]-1,1[\setminus\{0\}, \odot)$ is an Abelian group.

The following result was shown in [15].

Proposition 2 *Let S be a strict t-conorm, pseudo-addition $\oplus_S$ with the generating function g given by (4), and pseudo-multiplication $\odot$ is defined by (6). Then we have:*

(i) $(]-1,1[, \oplus_S, \odot)$ *is a field isomorphic to* $(\mathbb{R}, +, \cdot)$

(ii) The pseudo-multiplication has the next form

$$x \odot y = sign(x \cdot y) U^{\odot}(|x|, |y|),$$

where the uninorm $U^{\odot} : [0,1]^2 \to [0,1]$ is defined by $U^{\odot}(x,y) = s^{-1}(s(x)s(y))$ for all $x, y \in [0,1]$, with the convention:

(a) in the case $\infty \cdot 0 = 0$, $U^{\odot}$ is conjunctive,

(b) in the case $\infty \cdot 0 = \infty$, $U^{\odot}$ is a disjunctive uninorm.

It is clear now, that the couple of symmetric pseudo-operations $(\oplus_S, \odot)$ can be expressed in terms of a couple of uninorms, or as it is usual by (5) and (6).

3 Space BgV

According to [16, 18], the chain variation of a real valued set function $m : \mathcal{A} \to \mathbb{R}$, $m(\varnothing) = 0$, for all $E \in \mathcal{A}$, is defined by

$$|m|(E) = \sup\left\{\sum_{i=1}^{n} |m(E_i) - m(E_{i-1})| \mid \varnothing = E_0 \subset \ldots \subset E_n = E, \quad E_i \in \mathcal{A}, i = 1, \ldots, n\right\},$$

where supremum is taken with respect to all finite chains from $\varnothing$ to E. The chain variation $|m|$ of a real-valued set function m is positive, monotone, set function, $|m|(\varnothing) = 0$ and $|m(E)| \le |m|(E)$ for all $E \in \mathcal{A}$. We say that a real-valued set function m, $m(\varnothing) = 0$, is of bounded chain variation if $|m|(X) < \infty$, and we denote by BV the set of all set functions with the bounded chain variation, vanishing at the empty set. We refer [1, 16, 18] for an exhaustive overview of properties and results related to BV. It is proven in [1, 18] that a real-valued set function m belongs to BV if it can be represented as difference of two monotone set functions ν_1 and ν_2.

Definition 5 *[15] For a given function $g : [-1,1] \to [-\infty,\infty]$, defined by (4), g-chain variation $|m|_g$ of a set function $m : \mathcal{A} \to]-1,1[$, $m(\varnothing) = 0$, is defined by*

$$|m|_g(E) = g^{-1}\left(\sup\left\{\sum_{i=1}^{n} |g(m(E_i)) - g(m(E_{i-1}))| \right.\right.$$
$$\left.\left. \mid \varnothing = E_0 \subset \ldots \subset E_n = E, E_i \in \mathcal{A},\, i = 1,\ldots,n\right\}\right),$$

for all $E \in \mathcal{A}$ and supremum is taken with respect to all finite chains.

Using the fact that g is an odd function, we easily obtain the following result.

Proposition 3 *Let $m : \mathcal{A} \to]-1,1[$ be a set function, $m(\varnothing) = 0$, then g-chain variation has the following properties:*

(i) $0 \leqslant |m|_g(E) \leq 1, \quad E \in \mathcal{A}$.

(ii) $|m|_g(\varnothing) = 0$.

(iii) $|m(E)| \leqslant |m|_g(E), \quad E \in \mathcal{A}$.

(iv) $|m|_g$ *is a monotone set function, i.e.,*

$$|m|_g(E) \leqslant |m|_g(F),$$

for all $E \subset F$, $E, F \in \mathcal{A}$.

iv) If $m : \mathcal{A} \to [0,1]$ is a monotone set function, then

$$|m|_g(E) = m(E) \quad \text{for all} \quad E \in \mathcal{A}.$$

We say that a set function $m : \mathcal{A} \to]-1,1[$, $m(\varnothing) = 0$, is of bounded g-chain variation if $|m|_g(X) < 1$, and we denote by BgV the family of such set functions.

Proposition 4 *Let $m_1, m_2 \in BgV$. Then*

$$|m_1 \oplus_S m_2|_g(X) \leq |m_1|_g(X) \oplus_S |m_2|_g(X).$$

`Proof`: We will use the next notation

$$L = \{\emptyset = E_0 \subset E_1 \subset \ldots \subset E_n = F, \quad E_i \in \mathcal{A}, i = 1,\ldots,n\}.$$

We denote by $\mathcal{C}_F$ all finite chains from $\varnothing$ to F. We have

$$\begin{aligned} |m_1 \oplus_S m_2|_g(X) &= g^{-1}\Big(\sup_{L \in \mathcal{C}_X}\Big\{\sum_{i=1}^{n} |g((m_1 \oplus_S m_2)(E_i)) - g((m_1 \oplus_S m_2)(E_{i-1}))|\Big\}\Big) \\ &= g^{-1}\Big(\sup_{L \in \mathcal{C}_X}\Big\{\sum_{i=1}^{n} |g \circ m_1(E_i) + g \circ m_2(E_i) \end{aligned}$$

$$
\begin{aligned}
- &\quad g\circ m_1(E_{i-1}) - g\circ m_2(E_{i-1})|\Big\}\Big) \\
\leqslant &\quad g^{-1}\Big(\sup_{L\in\mathcal{C}_X}\Big\{\sum_{i=1}^{n}|g\circ m_1(E_i) - g\circ m_1(E_{i-1})| \\
+ &\quad \sum_{i=1}^{n}|g\circ m_2(E_i) - g\circ m_2(E_{i-1})|\Big\}\Big) \\
\leqslant &\quad g^{-1}\Big(g(g^{-1}(\sup_{L\in\mathcal{C}_X}\{\sum_{i=1}^{n}|g\circ m_1(E_i) - g\circ m_1(E_{i-1})|\})) \\
+ &\quad g(g^{-1}(\sup_{L\in\mathcal{C}_X}\{\sum_{i=1}^{n}|g\circ m_2(E_i) - g\circ m_2(E_{i-1})|\}))\Big) \\
= &\quad |m_1|_g(X)\oplus_S |m_2|_g(X).
\end{aligned}
$$

□

Proposition 5 *[15] A set function $m:\mathcal{A}\to]-1,1[$, $m(\varnothing)=0$, belongs to BgV if and only if it can be represented as follows*

$$m = m_1 \ominus_S m_2,$$

where $m_1, m_2 : \mathcal{A}\to[0,1]$ are two fuzzy measures.

`Proof`: We have that $m\in BgV$ if and only if $g\circ m\in BV$. By Theorem 3.10. from [18], there exist two fuzzy measures $\tilde{m}_1$ and $\tilde{m}_2$ such that $g\circ m = \tilde{m}_1 - \tilde{m}_2$. Taking $m_1 = g^{-1}\circ\tilde{m}_1$ and $m_2 = g^{-1}\circ\tilde{m}_2$ we obtain the claim. □

4 Signed $\oplus_S$-measures

In this section we consider a set function $m:\mathcal{A}\to[-1,1]$. We will define σ-$\oplus_S$-additivity of a set function m in the following manner. Let S be a strict t-conorm and $\oplus_S$ a pseudo-addition with an additive generator $g:[-1,1]\to[-\infty,\infty]$. First, we define the notion of a convergent $\oplus_S$-series $\bigoplus_{i=1}^{\infty}{}_S\, a_i$. We have the following situations:

(i) An expression $\bigoplus_{i=1}^{\infty}{}_S\, a_i$ is unambiguously defined if $a_i\geqslant 0$ for all $i=1,2\ldots$. Then $\{\bigoplus_{i=1}^{n}{}_S\, a_i\}_{n\in\mathbb{N}}$ is a monotone increasing sequence of reals from the interval $[0,1]$, hence

$$\bigoplus_{i=1}^{\infty}{}_S\, a_i := \lim_{n\to\infty}\bigoplus_{i=1}^{n}{}_S\, a_i, \tag{7}$$

i.e., the sum of $\oplus_S$-series is equal to a number from the interval $[0,1[$ and we say that $\oplus_S$-series is convergent, otherwise it diverges to 1.
(ii) In the case when $a_i\leqslant 0$, for all $i=1,2,\ldots$ we have the similar situation as in (i), i.e., the sum of $\oplus_S$-series is a number from the interval $]-1,0]$, otherwise it diverges to

-1.
(iii) For $a_i \in [-1,1]$, $i = 1,2,\ldots$, analogously as in the previous situations, we take (7), i.e., the classical limit value of the sequence of reals $\{\bigoplus_{i=1}^{n}{}_S\, a_i\}_{n\in\mathbb{N}}$, if it exists, i.e., if it is a number from the interval $]-1,1[$.

We introduce the notion of σ-$\oplus_S$-additivity as follows. A distorted signed measure μ transformed by g^{-1}, i.e., any real valued signed fuzzy measure $m = g^{-1} \circ \mu$ is σ-$\oplus_S$-additive, if g is an additive generator of pseudo-addition $\oplus_S$ and $\mu : \mathcal{A} \to [-\infty,\infty]$ is an arbitrary signed measure.

Definition 6 *A set function $m : \mathcal{A} \to [-1,1]$ is a signed $\oplus_S$-measure if there exists a signed measure $\mu : \mathcal{A} \to [-\infty,\infty]$ (μ assumes at most one of the values from $\{+\infty,\infty\}$) such that:*

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = g^{-1}\left(\sum_{i=1}^{\infty} \mu(E_i)\right)$$

is fulfilled for any sequence $\{E_i\}_{i\in\mathbb{N}}$, $E_i \in \mathcal{A}$, satisfying $E_k \cap E_j = \emptyset$ for $k \neq j$, where the series on the right side is either convergent or divergent to $+\infty$ or $-\infty$.

Obviously, we have $m(\varnothing) = 0$ and m takes on at most one of the values from $\{-1,1\}$.

Proposition 6 *Let $m : \mathcal{A} \to [-1,1]$ be a signed $\oplus_S$-measure. Then there exist unique fuzzy measures m_1 and m_2 such that*

$$m = m_1 \ominus_S m_2.$$

`Proof.` According to the classical Jordan's theorem of representation of a signed measure (see [8]), we have $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are measures. By Definition 6, for all $E \in \mathcal{A}$ we have

$$\begin{aligned} m(E) &= g^{-1}(\mu(E)) \\ &= g^{-1}(\mu^+(E) - \mu^-(E)) \\ &= g^{-1}(g(g^{-1} \circ \mu^+(E)) - g(g^{-1} \circ \mu^-(E))) \\ &= m_1(E) \ominus_S m_2(E). \end{aligned}$$

$\square$

Example 3 *Let $\mu : \mathcal{A} \to [-\infty,\infty]$ be a signed measure and let m be a set function defined on σ-algebra $\mathcal{A}$, $m : \mathcal{A} \to [-1,1]$ as follows:*

$$m(E) = \mathrm{sign}(\mu(E))\left(1 - e^{-|\mu(E)|}\right).$$

The set function m is a signed $\oplus_{S_P}$-measure.

Remark 1 *Let $m : \mathcal{A} \to [-1,1]$ be a set function such that $m \in BgV$. Then there exist m_1 and m_2 such that $m = m_1 \ominus_S m_2$. If the fuzzy measures m_1 and m_2 are S-measures, then m is a signed $\oplus_S$- measure.*

5 A general Choquet integral

Let $(X,\mathcal{A})$ be a measurable space, and $\mathcal{F}^+$ and $\mathcal{F}$ classes of $\mathcal{A}-$measurable functions given by

$$\mathcal{F}^+ = \{f \mid f : X \to [0,1],\ \sup_{x\in X} f(x) < 1\},$$
$$\mathcal{F} = \{f \mid f : X \to [-1,1],\ \sup_{x\in X} |f(x)| < 1\},$$

Let the operation $\ominus$ be given by Definition 4. For a set function $m : \mathcal{A} \to]-1,1[$, $m(\emptyset) = 0$, we define a pseudo conjugate set function $\overline{m}^{\ominus} : \mathcal{A} \to]-1,1[$ by:

$$\overline{m}^{\ominus}(E) = m(X) \ominus m(E^c),$$

for all $E \in \mathcal{A}$, where $E^c = X \setminus E$.

Proposition 7 *[15] We have*

(i) $f = f^+ \ominus f^-$, *for any* $f \in \mathcal{F}$, *where* $f^+, f^- \in \mathcal{F}^+$, $f^+ = f \vee 0$ *and* $f^- = (-f) \vee 0$.

(ii) m *is monotone if and only if* $\overline{m}^{\ominus}$ *is monotone.*

(iii) Let $m_1, m_2 : \mathcal{A} \to]-1,1[$ *such that* $m_1(X) = m_2(X)$. *Then*

$$m_1 \leqslant m_2 \text{ if and only if } \overline{m}_1^{\ominus} \geqslant \overline{m}_2^{\ominus}.$$

In the sequel, $\oplus$ and $\odot$ will denote associative pseudo-operations, defined by (5) and (6), respectively, and $\ominus$ the corresponding pseudo-difference. The measurable functions f and h on X are called *comonotone* [4] if they are measurable with respect to the same chain $\mathcal{C}$ in $\mathcal{A}$. Equivalently, comonotonicity of functions f and h can be expressed as follows: $f(x) < f(x_1) \ \Rightarrow\ h(x) \leqslant h(x_1)$ for all $x, x_1 \in X$.

Definition 7 *Let* $\mathbf{I} : \mathcal{F} \to]-1,1[$ *be a functional. We say that*

(i) $\mathbf{I}$ *is monotone if for all* $f, h \in \mathcal{F}$

$$f \leqslant h \Rightarrow \mathbf{I}(f) \leqslant \mathbf{I}(h),$$

(ii) $\mathbf{I}$ *is comonotone* $\oplus$*-additive if*

$$\mathbf{I}(f \oplus h) = \mathbf{I}(f) \oplus \mathbf{I}(h)$$

for all comonotone f *and* h *from* $\mathcal{F}$,

(iii) I *is positively* $\odot$*-homogenous if*

$$\mathbf{I}(a \odot f) = a \odot \mathbf{I}(f)$$

for all $a \in [0,1[$, $f \in \mathcal{F}$, *(iv)* $\mathbf{I}$ *is of bounded g-variation if* $G(\mathbf{I}) < 1$ *, where a g-variation* $G(\mathbf{I})$ *of* $\mathbf{I}$ *is defined by*

$$G(\mathbf{I}) = g^{-1}\left(\sup\left\{\sum_{i=1}^{n}|g(\mathbf{I}(h_i)) - g(\mathbf{I}(h_{i-1}))| \mid 0 = h_0 \leqslant \ldots \leqslant h_n = e\mathbf{1}_X, \quad h_i \in \mathcal{F}\right\}\right).$$

Remark 2 *Obviously, if* $\mathbf{I} : \mathcal{F} \to]-1,1[$ *is a monotone functional, then g-variation of I is given by* $G(\mathbf{I}) = \mathbf{I}(e\mathbf{1}_X)$.

Let $m \in BgV$ and let $s \in \mathcal{F}$ be a simple function with $Ran(s) = \{s_1, s_2, \ldots, s_n\}$. We define

$$\mathbf{I}_m(s) = s_1 \odot m(E_1) \oplus \bigoplus_{i=2}^{n}(s_i \ominus s_{i-1}) \odot m(E_i), \tag{8}$$

where $-1 < s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n < 1$ and $E_i = \{x \in X \mid s(x) \geqslant s_i\}$.

Proposition 8 *[15] Let* $\mathbf{I}_m$ *be defined by (8). For all simple functions from* $\mathcal{F}$*, and for all* $m \in BgV$ *we have:*

(i) $\mathbf{I}_m$ *satisfies the properties (ii) and (iii) given in Definition 7.*

(ii) $\mathbf{I}_m(s) = \mathbf{I}_m(s^+) \ominus \mathbf{I}_{\bar{m}^\ominus}(s^-)$.

(iii) $\mathbf{I}_m(s) = \mathbf{I}_{m_1}(s) \ominus \mathbf{I}_{m_2}(s)$*, where* m_1 *and* m_2 *are given by Proposition 5.*

(iv) $\mathbf{I}_m(a \cdot \mathbf{1}_E) = \begin{cases} a \odot m(E) & a \in [0,1[\\ a \odot \bar{m}^\ominus(E) & a \in]-1,0[\end{cases}$.

We consider now a general fuzzy integral. First we define a general fuzzy integral with respect to a monotone, non-negative function $m \in BgV$ and then with respect to an arbitrary m from BgV.

Definition 8 *A general fuzzy integral* $\mathbf{I}_m : \mathcal{F} \to]-1,1[$ *is defined by:*

(i) For a fuzzy measure m from BgV

$$\mathbf{I}_m(f) \quad = \quad \sup_{s \in \mathcal{F}^+, s \leqslant f^+} \mathbf{I}_m(s) \oplus \inf_{-s' \in \mathcal{F}^+, -s' \leqslant f^-} \mathbf{I}_m(s'). \tag{9}$$

(ii) For $m \in BgV$

$$\mathbf{I}_m(f) = \mathbf{I}_{m_1}(f) \ominus \mathbf{I}_{m_2}(f), \tag{10}$$

where m_1 *and* m_2 *are given by Proposition 5.*

A general fuzzy integral $\mathbf{I}_m : \mathcal{F} \to]-1,1[$ with respect to a fuzzy measure is monotone. $\mathbf{I}_m$ is asymmetric, i.e.,

$$\mathbf{I}_m(-f) = -\mathbf{I}_{\bar{m}^\ominus}(f),$$

for all $f \in \mathcal{F}$.

Proposition 9 *Let $\mathbf{I}_m : \mathcal{F} \to]-1,1[$ be a general fuzzy integral with respect to $m \in BgV$. We have:*

(i) $\mathbf{I}_m$ *is of bounded g-variation.*

(ii) $\mathbf{I}_m$ *satisfies the properties (ii) and (iii) given in Definition 7.*

(iii) $\mathbf{I}_m(f) = \mathbf{I}_m(f^+) \ominus \mathbf{I}_{\bar{m}^{\ominus}}(f^-)$*, for all* $f \in \mathcal{F}$*.*

Proof. (i) Let $m \in BgV$, by Proposition 5, $m = m_1 \ominus m_2$, where m_1 and m_2 are fuzzy measures from BgV. $\mathbf{I}_{m_1}, \mathbf{I}_{m_2} : \mathcal{F} \to]-1,1[$ are monotone functionals. By definition of g-variation we have $G(-\mathbf{I}) = G(\mathbf{I})$ and

$$G(\mathbf{I}_m) = G(\mathbf{I}_{m_1} \ominus \mathbf{I}_{m_2}) \leqslant G(\mathbf{I}_{m_1}) \oplus G(\mathbf{I}_{m_2}) = \mathbf{I}_{m_1}(e\mathbf{1}_X) \oplus \mathbf{I}_{m_2}(e\mathbf{1}_X) = m_1(X) \oplus m_2(X) < 1.$$

We obtain (ii) and (iii) by (8), (9), (10) and Proposition 8. □

Based on the above consideration and results proven in [2, 4, 15, 16, 18] we have the next propositions.

Proposition 10 *Let $\mathbf{I}_m : \mathcal{F} \to]-1,1[$ be a general fuzzy integral with respect to $m \in BgV$. Then*

$$\mathbf{I}_m(f) = C_m^g(f) = g^{-1}\left(C_{g \circ m}(g \circ f)\right),$$

where C_m^g is a general Choquet integral.

Proposition 11 *Let $\mathbf{I}_m : \mathcal{F} \to]-1,1[$ be a general fuzzy integral w.r.t. $m \in BgV$. Then*

$$\mathbf{I}_m(f) = g^{-1}\left(LS \int_{[-\infty,\infty]} g(t)\, d(g \circ F)(t)\right),$$

where the integral on the right-hand side is a pseudo Lebesgue-Stieltjes integral.

Proof. Let $F : [-1,1] \to [-1,1]$ be a function of bounded totally g-variation, i.e.,

$$g^{-1}\left(\sup\left\{\sum_{i=1}^{n} |g(F(t_i)) - g(F(t_{i-1}))| \mid -1 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant 1, i = 1,\ldots,n\right\}\right) < 1. \tag{11}$$

Then there exist two non-decreasing functions F^+ and F^- such that $F = F^+ \ominus F^-$ and a signed $\oplus$- measure on a σ-algebra of Borel subsets of $[-1,1]$, induced by F.

Let $\mathbf{I}_m$ be a general fuzzy integral with respect to $m \in BgV$. For $f \in \mathcal{F}$, let F be defined by

$$F(t) = -m\{x \in X | f(x) \geqslant t\}, \quad t \in [-1,1].$$

F is of bounded totally g-variation (11). $f \in \mathcal{F}$ is bounded, therefore $g \circ f$ is bounded, $\mathbf{I}_m(f) = C_m^g(f)$, and according to [16] (Appendix) we have the claim. □

Corollary 1 *Let $\mathbf{I}_m : \mathcal{F} \to]-1,1[$ be a general fuzzy integral with respect to a signed $\oplus$-measure m, $m \in BgV$. Then*

$$\mathbf{I}_m(f) = g^{-1}\left(\int g \circ f \, d\mu\right),$$

where integral on the right-hand side is g-integral, see [17, 18].

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