On the Optimal Representation of a Continuous Signal

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Abstract: The optimal representation of a continuous signal is discussed. The underlying dynamics is presented by an operator acting on the domain, which makes the optimal frame to be wavelets. The wavelet-domain hidden Markov model has been constituted following statistics of the measurement process. Information contained in causal states is the global complexity, which is a measure of the representation optimality.

Keywords: delta function; wavelets; hidden Markov model; complexity; optimality; multiresolution analysis

1 Introduction

Processing continuous signals raises the issue of its representation. The porblem actually occurs due to an interpretation of the continuum in terms of an unenumerable set that should represent the domain of a signal space. In that regard, conversance of the signal implies an unenumarable values attributed to particular elements of the domain which is not only practically unfeasable but completely pointless considering functional analysis of the Lebesgue space such as L^2. It is constituted by the finite energy requirement, which menas spanning by a discrete base whose choice becomes crucial for faithful representation.

The issue therefore relates to recognition of a domain which is the signal defined upon. An interpretation by means of set theory termed *extensional view* is the least effective solution as it has stated above. On the other hand, such a domain is interpreted to be the time continuum which implies an *intensional view* in terms of a procedure that has generated the signal. In that manner, a processual designation comes to the fore concerning its generation by a measurement process.

A preconsideration of the problem, founded upon the microcannonical cascade formalism, was presented in [1] and [2]. The optimal representation is defined by maximization of mutual information transferred at successive scales of the wavelet decomposition. The method does not address denoising aspect.

The current paper is also based upon wavelets that should implement the domain of a signal. The wavelet-domain hidden Markov model has captured statistics of a measurement process unfolding over time, which is regarded to be an operator on the signal space. The time operator, which is a definition of the complex system has generalized multiresolution analysis playing a fundamental role in the representation [3]. The global complexity that is the minimal information required for the optimal prediction indicates an increase of the local complexity, which is the definition of self-organization in complex systems [4]. Optimality aligns with maximal self-organization, reinforcing an intensional view rather than some extensional values. Denoising procedure, which is an inherent component of the model has proven advantageous over other methods. Using the optimal representation results in enhanced performances, due to separation of structural information from irreducible randomness [5].

2 Frames and Bases of the Signal Space

The concept of the delta function plays a significant role in signal processing. In discrete settings, it was introduced by Leopold Kronecker and a generalization to the continuous domain was implemented by Paul Dirac [6]. The function captures a concept of the signal being evaluated at a point, and it is often used to designate the identity in convolution and operator theory.

One should consider the two-variable matrix $\delta_x(y)$, which corresponds to a distribution encoding the propositional value $x = y$ (Fig. 1). It underlies the decomposition of a signal F through the action of an adjoint operator

$$F(x) = \delta_x^\dagger F \tag{1}$$

with reconstruction given by

$$F = \delta\delta^\dagger F \tag{2}$$

In the extensional view, the signal F is characterized by the pointwise evaluation $F(x)$ forming the graph of a function

$$x \mapsto \delta_x F \tag{3}$$

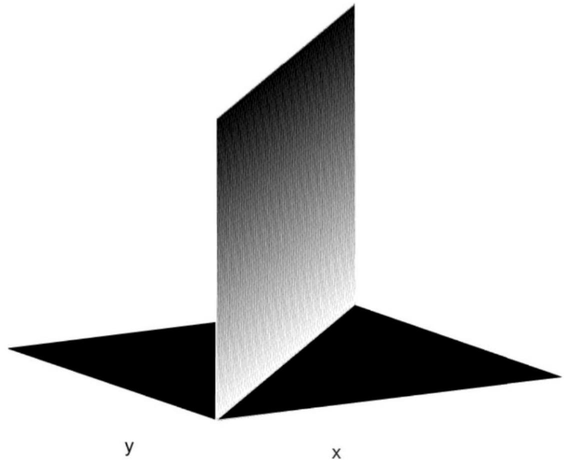


Figure 1
The two-variable matrix $\delta x(y)$ corresponding to a distribution of the propositional value $x = y$

Although the representation is trivially valid, it is suboptimal in terms of signal modeling and analysis. The trivial identification of a signal with its graph leads to the relation

$$\delta = \delta\delta^{\dagger} \tag{4}$$

which makes δ an orthoprojector corresponding to the identity operator. Such an interpretation regards signal to be a pure concept, deprived of any processual designation [7].

The intensional view on the other hand aims to represent a signal not merely by its values, but through the process that has generated it. From such a perspective, the optimal representation should capture a procedure of its generation. Signal processing has often neglected such an underlying dynamics [8], resulting in the extensional model that is a set of mere points which cannot be the domain of a contonuous signal. Consequently, the formulation (1) is not any pointwise expression but an almost everywhere holding statement

$$F = \delta^{\dagger}F \tag{5}$$

The representation of a signal F is demanded in the form $\Psi^{\dagger}F$ satisfying

$$\delta = \widetilde{\Psi}\Psi^{\dagger} \tag{6}$$

wherein $\widetilde{\Psi}$ is a dual frame. In particular, if

$$\delta = \Psi\Psi^{\dagger} \tag{7}$$

it is a *Parseval frame* which implies self-duality that is a straightforword generalization of (4). A more general requirement

$$a\delta \leq \Psi\Psi^{\dagger} \leq b\delta \tag{8}$$

concerns the exact reconstruction implying positive constants a amd b which are termed *frame bounds*. In that instance, the canonical dual is given by

$$\widetilde{\Psi} = (\Psi\Psi^{\dagger})^{-1}\Psi \tag{9}$$

and the operator in brackets is invertible because of its positivity.

If in addition

$$\delta = \Psi^{\dagger}\widetilde{\Psi} \tag{10}$$

such a frame is *biorthonormal base* of the signal space. It is orthonormal if

$$\Psi^{\dagger} = \Psi^{-1} \tag{11}$$

which means that the representation operator is unitary.

Frames are geometrically interpreted to be sequences that dilate to bases of the extended space [9]. It means that there is the base which has restricted to a frame in the initial space, whereby the Parseval frame is restriction of an orthonoraml base. In that manner, the biorthonormal base has restricted to a frame by neglecting environment out of the scope [10]. The frame therefore relates to an open system that is partially described by the measurement process [11]. It is a reason to focus onto the base Ψ and its dual $\widetilde{\Psi}$ satisfying (6) and (10). A generalization to frame representations should be pointed in the discussion of multiresolution analysis.

3 Wavelets and the Measurement Process

A commensuration of magnitudes by the Euclidean algorithm gives rise to continued fraction

$$\cfrac{1}{n_1+\cfrac{1}{n_2+\cfrac{1}{\ddots}}} \tag{12}$$

which is an intensional procedure taking place over time. Due to the question mark function by Minkowski

$$?:\cfrac{1}{n_1+\cfrac{1}{n_2+\cfrac{1}{\ddots}}} \mapsto \frac{1}{2^{n_1-1}} - \frac{1}{2^{n_1+n_2-1}} + \cdots \tag{13}$$

which is an automorphism of the time continuum, it corresponds to the binary code

$$x = \underbrace{0.0\ldots0}_{n_1}\underbrace{1\ldots1}_{n_2}\underbrace{0\ldots0}_{\ldots}\ldots \tag{14}$$

producing a real number of the unit interval [5].

Devices of the measurement process relate to a base Ψ_i which is enumerated by dyadic fractions

$$i = \frac{2k-1}{2^j} \quad (15)$$

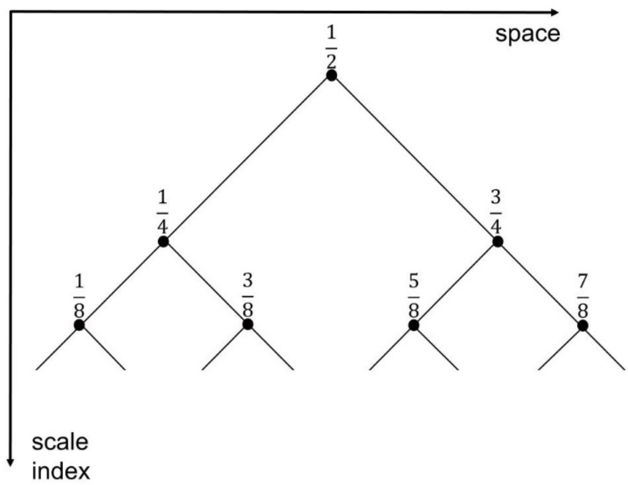
between 0 and 1, forming nodes the binary tree. Measurement states correspond to a dual $\widetilde{\Psi}_i$, enumarated by dyadic fractions (15) as well. A step of the process is given by the operator

$$U: \delta_x \mapsto \delta_{2x(-1)} \quad (16)$$

which is a shift leftwards in terms of the binary code (14). It acts onto states in the manner of

$$U: \widetilde{\Psi}_i \mapsto \widetilde{\Psi}_{2i(-1)} \quad (17)$$

mapping each node of binary tree to its precursor (Fig. 2). Such an evolution (17) makes the frame $\widetilde{\Psi}_i$ to be wavelets [11].



The binary tree that implies a step mapping each node except the root to its precursor

The evolutionary opeartor U extends to an invertible one whcih is denoted by the same symbol. The extension depends upon the base and its dual in order to retain the property (17). The time operator of wavelets

$$T: \widetilde{\Psi}_i \mapsto j\widetilde{\Psi}_i \quad (18)$$

is extended as well, in regard that the commutator relation

$$[U, T] = U \quad (19)$$

remains satisfied [5].

Projectors

$$P_i = \widetilde{\Psi}_i \Psi_i^\dagger \quad (20)$$

generate a Boolean algebra in respect to addition and multiplication. Due to the Stone representation theorem, it is isomorphic to a measurable space which is regarded to be the domain of a signal [7]. The evolution of the domain which maps P_i into $P_{2i(-1)}$ is governed by the superoperator $V: F \mapsto UFU^\dagger$ that preserves positivity since

$$\rho = FF^\dagger \geq 0 \Rightarrow V\rho = (UF)(UF)^\dagger \geq 0 \quad (21)$$

The commutator relation (19) holds for V as well, involving the time operator of wavelets (18).

4 Wavelet-Domain Hidden Markov Model

A defining feature of complex systems is the time operator T satisfying the commutator relation in respect to the operator V that is positivity preserving. Preservation of positivity (21) implies that it governs the evolution of a density, which is reversible since the inverse operator has also preserved positivity. Random variables over the same domain evolve by an adjoint operator $V^\dagger$.

Existence of the time operator provides a change in representation

$$\Lambda = \lambda(T) \quad (22)$$

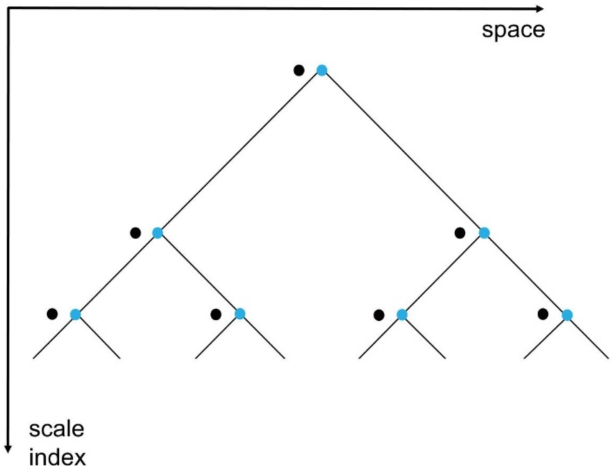
which is an operator function of time [12]. The inverse operator W^{-1} does not preserve positivity and therefore the adjoint

$$W^\dagger = \Lambda^{-1\dagger} V^\dagger \Lambda^\dagger \quad (23)$$

should govern the Markov process applied to the transfigured base $\Omega_i = \Lambda^{-1\dagger}\Psi_i$, whose dual $\Lambda\widetilde{\Psi}_i$ evolves by W. In such a base, the signal F corresponds to hidden variables

$$\Omega_i^\dagger F = \Psi_i^\dagger \Lambda^{-1} F \quad (24)$$

which are attributed one per each node of the binary tree (Fig. 3). The wavelet-domain hidden Markov model has been constituted in that manner, following statistics of the measurement process [13].



The wavelet-domain hidden Markov model wherein black nodes correspond to a signal and blue ones to hidden states

Information of the signal

$$H(F) = H(\Lambda^{-1}F) + H(F|\Lambda^{-1}F) \tag{25}$$

separates into a sum of two terms wherein the second one $H(F|\Lambda^{-1}F)$ is an irreducible randomness persisting even after none correlation has remained. Maximizing such a term achieves $H(\Lambda^{-1}F)$ to be the minimal information required for the optimal prediction, which is the global complexity. It makes hidden variables $\Omega_i^{\dagger}F$ to be causal states containing information, which is the local complexity. The global complexity is an indicator of increasing the local complexity over time [4]. Such a complexity increase is the definition of self-organization in complex systems [14].

In order to determine local and global complexities, one should estimate parameters of the statistical model. For that purpose, an iterative expectation maximization is performed [15]. Applied to the hidden Markov model, the procedure is known to be the Baum-Welch algorithm. Starting from initial parameters ϑ_0, it estimates in each successive step l a repaired value ϑ_l. The algorithm converges to a value ϑ maximizing the likelihood function $p(F|\vartheta)$, due to the fact that the graph is cordal since there are no cycles in the binary tree. Having an intelligent initialization ϑ_0, the convergence should occur in as few as ten iteration for a simple model that implies a locally two-state causal structure [13].

An intrinsic component of the model is a denoising procedure that has been proven advantageous over other methods. Adding white noise to a signal convolves the distribution density and denoising concerns a deconvolution, using the optimal representation which results in enhanced performances [4]. The denoising procedure does not affect complexity $H(\Lambda^{-1}F)$ but only an irreducible randomness $H(F|\Lambda^{-1}F)$ and in that regard the equality (25) has separated structural information from noise.

5 Multiresolution Analysis

In the light of the above, maximization of global complexity corresponds to a measure of the representation optimality. It aims to distinguish the time operator corresponding to an intensional procedure that has generated the signal. Such an underlying dynamics is presented by multiresolution analysis, which is a term that has originated from optics and computer vision [16]. An instance of multiresolution analysis is presented by the approximation of an image at successive scales (Fig. 4).

It concerns a sequence of wandering subspaces D_j which are eigenspaces of the time operator. One should satisfy axioms:

(i) $UF \in D_j \Leftrightarrow F \in D_{j+1}$;

(ii) D_0 consists of constants only;

(iii) $\overline{\oplus D_j}$ involves any signal;

(iv) there is $\widetilde{\Psi}_{\frac{1}{2}}$ which is the base of D_1.

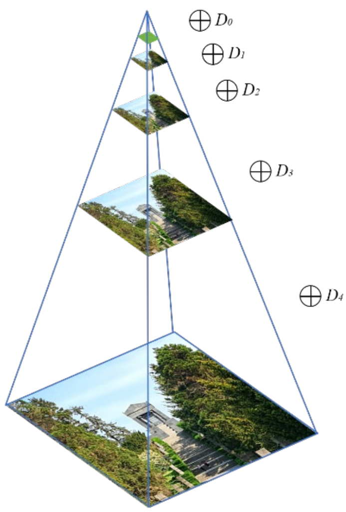


Figure 4
An instance of multiresolution analysis that is presented by the approximation of an image at successive scales which are obtained due to adding detail subspaces

A subspace D_j corresponds to the projector $\sum \widetilde{\Psi}_i \Psi_i^\dagger$ implying a summation over dyadic fractions (15) related to the time instance j. In that manner, the domain (20) is temporally arranged to be the time contonuum offering an intensional view of the signal space.

A generalization to frames is somewhat demanding, due to the general measurement which relates to an open system that is partially described by the process [11]. In that regard, the base has restricted to a frame of the signal space by neglecting an environment that has remained out of the scope [9]. The Naimark dilatation theorem implies a method analogous to heterodyne detection in communication engineering: the signal to be observed combines with another one, which is termed *ancilla* [17]. Thereafter, a measurement process is performed on the space which has dilated by the environment. The amount of information gained in that manner might be greater than if the observation were limited to the measurement without ancilla [10]. One concludes that the optimal representation does not restrict to bases only, involving as well a general measurement ,which are related to frames [11].

Conclusion

Addressing the optimality issue in the representation of continuous signals has demonstrated that the extensional view which concerns the delta function is the least effective solution. On the contrary, it is proposed the optimal representation emerging from an underlying dynamics which is the intensional procedure that generates a signal.

The paper offers a novel perspective which has presented the measurement process generating the domain of a continuous signal. The underlying dynamics is an operator acting on the domain, which makes optimal frames to be wavelets. Frames are sequences that dilate into bases of an extended space, relating therefore to an open system partially described by the process.

The underlying dynamics is implemented by means of multiresolution analysis. Such a concept derived from optics and computer vision refers to wandering subspaces which are eigenspaces of the time operator. Its existence provides a change in representation, transfiguring reversible to irreversible evolution, which is Markovian. The wavelet-domain hidden Markov model is constituted in that manner, following statistics of the measurement process. The optimality corresponds to maximization of global complexity concerning information contained in causal states.

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