Exponential and logarithm on semi-Riemannian manifolds of constant curvature

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Abstract: The aim of this work is the unified study of the exponential and logarithm maps on semi-Riemannian manifold of constant curvature in order to prove their analytical properties and find their power series expansions. Our results generalize the corresponding Lie theoretical results on the special linear group $\mathrm{SL}(2,\mathbb{R})$.

Keywords: exponential and logarithm map, special linear group, pseudo-sphere, pseudo-hyperbolic space, semi-Riemannian manifold of constant curvature

1 Introduction

Mathematical models and algorithms of Machine Learning are often formulated as optimization problems for discrete or continuous functions defined on the edges and vertices of graphs representing hierarchically ordered data. The corresponding methods can be well used if the graphs are embedded in Euclidean or Riemannian geometries of constant curvature, so that the images of edges represent line segments of the geometry. Recently, several papers have initiated the description of data systems with complex hierarchies using semi-Riemannian manifolds (manifolds with indefinite Riemannian metrics) of constant curvature. A possible advantage of these spaces is that they contain more types of geodesics, namely they can be periodically closed or divergent curves whose covered areas are separated by pairs of straight lines. (cf. [3], [6], [7], [11], [12], [13]). The geometry of geodesics is well described by the properties of exponential and logarithmic maps; explicit analytical formulas for them in semi-Riemannian manifolds have interesting applications in computational differential geometry and mathematical physics. (cf. eg. [1], [2], [10]).

Semi-Riemannian manifolds of constant curvature are represented by pseudo-spheres and pseudo-hyperbolic spaces, which are given by the subspace geometry of central hyperquadrics in semi-Euclidean spaces. Using the observation that the point set of a geodesic on a central hyperquadric is determined by the connected component of the intersection of the hyperquadric with the 2-space spanned by the initial point and

tangent vector, T. Gao, L.-H. Lim and K. Ye, (cf. [3], Proposition 4.9 and Corollary 4.22.) expressed formulas for the exponential map on central hyperquadrics in semi-Euclidean spaces. This map induces a local bijection, the local inverse map defines the logarithm map. The obtained expressions are given by putting together different analytic functions defined on disjoint sets covered by geodesics with space-like, time-like or light-like tangent vectors, the analyticity of the map cannot be recognized at the boundary points of these sets.
The purpose of this article is to prove the real analytic properties of the exponential and logarithm maps defined on the tangent space $\mathrm{T}_I(\mathscr{S}_q)$, respectively on a neighbourhood of the point $I \in S_q$ and to find their power series expansions. Our formulas generalize analytic expressions of the corresponding maps of the special linear group $SL(2,\mathbb{R})$, introduced by J. Hilgert and K. H. Hofmann in the study Lie theory of analytic semigroups (§1. in [4], V.4.19. Theorem in [5]) by using the multiplicative group structure of $SL(2,\mathbb{R})$. Since the 4-dimensional vector space containing $\mathrm{SL}(2,\mathbb{R})$ as submanifold has a natural semi-Euclidean space structure and the induced submanifold metric coincides woth the invariant group metric on $SL(2,\mathbb{R})$, the generalization to semi-Euclidean spaces of arbitrary dimension is a natural task.
After introducing the necessary notations and concepts in §2, we examine in §3 the exponential map in detail. The main result of §4 the expression of the logarithm map with help of the functions $\arcsin t$ and $\operatorname{arsinh} t$ and to find its domain and range. §5 is devoted to the analysis of the power series decomposition of the logarithm map.

2 Preliminaries

In the following we identify the points of a vector space with their position vectors and denote them with capital letters $X,Y\dots$. A *semi-Euclidean space* $\mathbb{E}^n_\nu$ of index $0<\nu<n$ is a vector space with an indefinite nondegenerate *scalar product* $(X,Y)\mapsto\langle X,Y\rangle$ such that there is a basis $(e_1,\dots,e_n)$ in $\mathbb{E}^n_\nu$ for which the identity

$$\langle x_1e_1+\cdots+x_ne_n, y_1e_1+\cdots+x_ne_n\rangle = x_1y_1+\cdots+x_{n-\nu}y_{n-\nu}-x_{n-\nu+1}y_{n-\nu+1}-x_ny_n$$

holds. A vector $X\in\mathbb{E}^n_\nu$ is called

$$\begin{array}{lll} \textit{space-like}, & \text{if} & \langle X,X\rangle>0 \text{ or } X=0,\\ \textit{time-like}, & \text{if} & \langle X,X\rangle<0,\\ \textit{light-like}, & \text{if} & X\neq 0 \text{ and } \langle X,X\rangle=0. \end{array}$$

The semi-Euclidean space $\mathbb{E}^n_\nu$ is the disjoint union of central hyperquadrics

$$\mathscr{S}_q=\{X\in\mathbb{E}^n_\nu;\langle X,X\rangle=q\}\subset\mathbb{E}^n_\nu,\quad q\in\mathbb{R}.$$

The hyperquadric $\mathscr{S}_0$ is the *light-cone* of the space $\mathbb{E}^n_\nu$. If $q<0$ and $\nu=1$, or $q>0$ and $\nu=n-1$ the hyperquadrics $\mathscr{S}_q$ consist of two disjoint connected components, which are isometric or anti-isometric to the hyperbolic space (Riemannian space of negative constant curvature). The hyperquadrics $\mathscr{S}_q\subset\mathbb{E}^n_\nu$ are called:

$$\begin{array}{lll} \textit{pseudo-sphere of radius } r & \text{if} & 0<\nu<n-1,\ q=r^2>0,\\ \textit{pseudo-hyperbolic space of radius } r & \text{if} & 1<\nu<n,\ q=-r^2<0. \end{array}$$

Let be $\varepsilon = \pm 1$ and denote $\mathscr{S}_\nu^{n-1}(\varepsilon)$ the pseudo-sphere of radius one if $\varepsilon = 1$, and the pseudo-hyperbolic space of radius one if $\varepsilon = -1$. J. Hilgert and K. H. Hofmann in in [4], [5] introduced the analytic functions

$$\begin{aligned} \mathrm{C}(x) &= \sum_{n=0}^{\infty} \frac{x^n}{(2n)!} = 1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \cdots = \begin{cases} \cosh\sqrt{x}, & \text{if} x \geq 0, \\ \cos\sqrt{-x}, & \text{if} x < 0, \end{cases} \\ \mathrm{S}(x) &= \sum_{n=0}^{\infty} \frac{x^n}{(2n+1)!} = 1 + \frac{x}{3!} + \frac{x^2}{5!} + \frac{x^3}{7!} + \cdots = \frac{1}{\sqrt{|x|}} \begin{cases} \sinh\sqrt{x}, & \text{if} x \geq 0, \\ \sin\sqrt{-x}, & \text{if} x < 0, \end{cases} \end{aligned} \tag{1}$$

defined for all $x \in \mathbb{R}$, satisfying the identity $1 = \mathrm{C}(x)^2 - x\mathrm{S}(x)^2$. Considering the group $\mathrm{SL}(2,\mathbb{R})$ as a submanifold in the vector space $\mathrm{M}_2(\mathbb{R})$ of 2×2 matrices equipped with the scalar product

$$\langle X, Y \rangle = \frac{1}{2}(-\det(X+Y) + \det(X) + \det(Y)).$$

Then $\mathrm{M}_2(\mathbb{R})$ becomes a semi-Euclidean space $\mathbb{E}_2^4$ such that the submanifold metric of $\mathrm{SL}(2,\mathbb{R})$ agrees with the invariant metric defined by the Cartan-Killing form $\kappa(X,Y) = \frac{1}{2}\mathrm{Tr}(XY)$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$. The exponential map $\exp : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ and its inverse map $g \mapsto \mathrm{Log}(g)$ in some neighbourhood $U \subset \mathrm{SL}(2,\mathbb{R})$ of the identity element $\mathbf{1} \in \mathrm{SL}(2,\mathbb{R})$ are expressed by

$$\begin{aligned} \exp(X) &= \mathrm{C}(\kappa(X,X))\mathbf{1} + \mathrm{S}(\kappa(X,X))X, \qquad X \in \mathfrak{sl}(2,\mathbb{R}), \\ \mathrm{Log}(g) &= \frac{1}{\mathrm{S}(\mathrm{C}^{-1}(\tau(g)))}(g - \tau(g)\mathbf{1}), \quad g \in U \subset \mathrm{SL}(2,\mathbb{R}) \end{aligned}$$

where $\tau : \mathfrak{sl}(2,\mathbb{R}) \to \mathbb{R}$ and $\kappa : \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}) \to \mathbb{R}$ are the normalized trace function and Cartan-Killing form on $\mathfrak{sl}(2,\mathbb{R})$, repectively. In the following we generalize the above construction to arbitrary semi-Euclidean space and investigate power series expansion of the corresponding maps.

3 Geodesics and exponential map

Any point $P \in \mathscr{S}_q$ of a central hyperquadric $\mathscr{S}_q = \{X \in \mathbb{E}_\nu^n; \langle X, X \rangle = q\}$ in $\mathbb{E}_\nu^n$ is orthogonal to the tangent space $\mathrm{T}_P(\mathscr{S}_q)$, hence $\mathrm{T}_P(\mathscr{S}_\nu^{n-1}(r))$ is isometric to $\mathbb{E}_\nu^{n-1}$ and $\mathrm{T}_P(\mathscr{H}_{\nu-1}^{n-1}(r))$ to $\mathbb{E}_{\nu-1}^{n-1}$. We equip the submanifold $\mathscr{S}_q$ in $\mathbb{E}_\nu^{n-1}$ with the induced semi-Riemann metric of the submanifold. We distinguish a point I of $\mathscr{S}_q$, $q \in \mathbb{R}$, the vectors belonging to the tangent subspace $\mathrm{T}_I(\mathscr{S}_q) \subset \mathbb{E}_\nu^{n-1}$ of $\mathscr{S}_q$ we be denoted by lowercase bold letters $\mathbf{x}, \mathbf{y}, \ldots$. The point set of a geodesic started at $I \in \mathscr{S}_q$ with tangent vector $\mathbf{x} \in \mathrm{T}_I(\mathscr{S}_q)$ is a connected subset of the intersection $\Pi_I(\mathbf{x}) \cap \mathscr{S}_q$, where $\Pi_I(\mathbf{x})$ is the 2-dimensional subspace spanned by I and $\mathbf{x}$. The *exponential map* $\mathrm{Exp} : \mathrm{T}_I(\mathscr{S}_q) \to \mathscr{S}_q$ is defined by $\mathrm{Exp}(\mathbf{x}) = \gamma_I(1)$, where $\gamma_I(t)$, $t \in \mathbb{R}$, is the affine parametrized geodesic on $\mathscr{S}_q$ satisfying $\gamma_I(0) = I$ and $\frac{d\gamma}{dt}(0) = \mathbf{x}$.

3.1 Pseudo-sphere

We consider the pseudo-sphere $\mathscr{S}_\nu^{n-1} \subset \mathbb{E}_\nu^n$, $0 < \nu < n-1$, and a point $I \in \mathscr{S}_\nu^{n-1}$. The tangent space $\mathrm{T}_I(\mathscr{S}_\nu^{n-1})$ is isometric to the semi-Euclidean space $\mathbb{E}_\nu^{n-1}$. Since the space-like vector I is orthogonal to $\mathrm{T}_I(\mathscr{S}^{n-1}_\nu)$, the vector space $\mathbb{E}^n_\nu$ is the orthogonal direct sum $\mathbb{R}I \oplus \mathrm{T}_I(\mathscr{S}^{n-1}_\nu)$, giving for any $X \in \mathbb{E}^n_\nu$ the decomposition $X = xI + \mathbf{x}$, $x \in \mathbb{R}$, $\mathbf{x} \in \mathrm{T}_I(\mathscr{S}^{n-1}_\nu)$. The tangent hyperplane $I + \mathrm{T}_I(\mathscr{S}^{n-1}_\nu)$ of $\mathscr{S}^{n-1}_\nu$ at I contains two light-like geodesics satisfying $x = 1$, and the open domains in $\mathscr{S}^{n-1}_\nu$ given by $-1 < x < 1$ and $1 < x$ are covered by space-like and time-like geodesics through I, respectively. Space-like, time-like or light-like geodesics are ellipses, branches of hyperbolas or lines. We get easily the following reformulation of Proposition 5.38. in [9]:

Lemma 1. *(B. O'Neill) For any point $xI + \mathbf{x} \neq I$ in $\mathscr{S}^{n-1}_\nu$*

(i) I and $xI + \mathbf{x}$ are connected by a unique geodesic if and only if $-1 < x$, this geodesic is

periodic and space-like, if $-1 < x < 1$,

injective and light-like, if $x = 1$,

injective and time-like, if $x > 1$,

(ii) the point $-I$ is contained in all space-like geodesics through I,

(iii) the points I and $xI + \mathbf{x}$ are connected by a geodesic if and only if $xI + \mathbf{x} = -I$ or $x > -1$.

Now, we can examine the analytical properties of the exponential map.

3.2 Pseudo-hyperbolic space

Introducing the new scalar product $\langle X, Y\rangle^* = -\langle X, Y\rangle$ on the underlying vector space $\mathbb{V}^n$ of $\mathbb{E}^n_\nu$, $1 < \nu < n$, we get a description of the exponential map of pseudo-hyperbolic spaces $\mathscr{H}^{n-1}_{n-\nu} \subset \mathbb{E}^n_{n-\nu}$. The identity map of the space $\mathbb{V}^n$ defines a map $\Theta : \mathbb{E}^n_\nu \to \mathbb{E}^n_{n-\nu}$ changing space-like vectors with time-like vectors and the hyperquadrics $\mathscr{S}_q = \{X \in \mathbb{E}^n_\nu; \langle X, X\rangle = q\}$ with $\mathscr{S}_{-q} = \{X \in \mathbb{E}^n_{n-\nu}; \langle X, X\rangle^* = -q\}$, $q \in \mathbb{R}$. In particular, pseudo-spheres $\mathscr{S}^{n-1}_\nu \subset \mathbb{E}^n_\nu$ transform into pseudo-hyperbolic spaces $\mathscr{H}^{n-1}_{n-\nu} \subset \mathbb{E}^n_{n-\nu}$ and the selected space-like vector $I \in \mathscr{S}^{n-1}_\nu$ into a time-like vector $I \in \mathscr{H}^{n-1}_{n-\nu}$.
Hence we obtain for a pseudo-hyperbolic space $\mathscr{H}^{n-1}_\nu \subset \mathbb{E}^n_\nu$ with a fixed point $I \in \mathscr{H}^{n-1}_\nu$:

Lemma 3.1.$'$ For any point $xI + \mathbf{x} \neq I$ in $\mathscr{H}^{n-1}_\nu$

(i) I and $xI + \mathbf{x}$ are connected by a unique geodesic if and only if $-1 < x$, this geodesic is

periodic and time-like, if $-1 < x < 1$,

injective and light-like, if $x = 1$,

injective and space-like, if $x > 1$,

(ii) the point $-I$ is contained in all space-like geodesics through I,

(iii) the points I and $xI+\mathbf{x}$ are connected by a geodesic if and only if $xI+\mathbf{x}=-I$ or $x>-1$.

3.3 Computing the exponential map

Let be $\varepsilon=\pm 1$ and denote $\mathscr{S}_\nu^{n-1}(\varepsilon)=\begin{cases}\mathscr{S}_\nu^{n-1} & \text{if} \quad \varepsilon=1\\ \mathscr{H}_\nu^{n-1} & \text{if} \quad \varepsilon=-1\end{cases}$.

Proposition 1. *The exponential map* $\mathrm{Exp}_\varepsilon:\mathrm{T}_I(\mathscr{S}_\nu^{n-1}(\varepsilon))\to\mathscr{S}_\nu^{n-1}(\varepsilon)$ *is expressed by*

$$\mathrm{Exp}_\varepsilon(\mathbf{x})=\mathrm{C}(-\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)I+\mathrm{S}(-\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)\mathbf{x} \quad \textit{for all} \quad \mathbf{x}\in\mathrm{T}_I(\mathscr{S}_\nu^{n-1}(\varepsilon)), \tag{2}$$

where the functions $\mathrm{C}:\mathbb{R}\to\mathbb{R}$ *and* $\mathrm{S}:\mathbb{R}\to\mathbb{R}$ *are defined by* (1).
Moreover, $\mathrm{Exp}_\varepsilon(\mathbf{x})=-I$ *on* $\mathscr{S}_\nu^{n-1}(\varepsilon)$ *if and only if* $\langle\mathbf{x},\mathbf{x}\rangle=\varepsilon$ *and* $\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle|}=(2k+1)\pi$*,* $0\le k\in\mathbb{Z}$*.*

Proof. First, we consider the map $\mathrm{Exp}=(\mathrm{Exp}_1):\mathrm{T}_I(\mathscr{S}_\nu^{n-1})\to\mathscr{S}_\nu^{n-1}$. Let $0\neq\mathbf{x}\in\mathrm{T}_I(\mathscr{S}_\nu^{n-1})$ be a tangent vector and denote $\Gamma_I(\mathbf{x})$ the connected component of the intersection of $\mathscr{S}_\nu^{n-1}$ with the 2-space spanned by I and $\mathbf{x}$ such that $I\in\Gamma_I(\mathbf{x})$. If $\mathbf{x}$ is space-like or time-like, then $J=\frac{\mathbf{x}}{\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle|}}$ is a unit vector orthogonal to I and we get $\langle uI+vJ,uI+vJ\rangle=u^2\pm v^2$. If $\langle\mathbf{x},\mathbf{x}\rangle>0$ then $\Gamma_I(\mathbf{x})=\{\cos tI+\sin tJ; t\in\mathbb{R}\}$ is a circle, if $\langle\mathbf{x},\mathbf{x}\rangle<0$ then $\Gamma_I(\mathbf{x})=\{\cosh tI+\sinh tJ; t\in\mathbb{R}\}$ is a branch of a hyperbola. It follows

$$\mathrm{Exp}(\mathbf{x})=\begin{cases}\cos(\sqrt{\langle\mathbf{x},\mathbf{x}\rangle})I+\sin(\sqrt{\langle\mathbf{x},\mathbf{x}\rangle})J, & \text{if} \quad \langle\mathbf{x},\mathbf{x}\rangle<0,\\ \cosh(\sqrt{-\langle\mathbf{x},\mathbf{x}\rangle})I+\sinh(\sqrt{-\langle\mathbf{x},\mathbf{x}\rangle})K, & \text{if} \quad \langle\mathbf{x},\mathbf{x}\rangle>0.\end{cases}$$

If $\mathbf{x}$ is light-like, then it has the form $(J+K)\in\mathrm{T}_I(\mathscr{S}_\nu^{n-1})$, where J and K are space-like, respectively, time-like orthogonal unit vectors, and the geodesic $\Gamma_I(\mathbf{x})=\{I+t(J+K); t\in\mathbb{R}\}$ is one of a pair of parallel lines. Hence $\mathrm{Exp}(\mathbf{x})=I+\mathbf{x}$ and the formula (2) is true for any $\mathbf{x}\in\mathrm{T}_I(\mathscr{S}_\nu^{n-1})$. Clearly, $\mathrm{Exp}(\mathbf{x})=-I$ if and only if $\mathbf{x}$ is space-like and $\sqrt{\langle\mathbf{x},\mathbf{x}\rangle}=(2k+1)\pi$, $0\le k\in\mathbb{Z}$, giving the claim for pseudo-sphere. Changing the scalar product $\langle X,Y\rangle^*=-\langle X,Y\rangle$ on the underlying vector space $\mathbb{V}^n$ of $\mathbb{E}_\nu^n$ we get a description of the exponential map $\mathscr{H}_{n-\nu}^{n-1}\subset\mathbb{E}_{n-\nu}^n$. The identity map $\Theta:\mathbb{E}_\nu^n\to\mathbb{E}_{n-\nu}^n$ transforms space-like vectors into time-like vectors, and we get the proof of the assertion for the exponential map $\mathrm{Exp}_{-1}:\mathrm{T}_I(\mathscr{H}_\nu^{n-1})\to\mathscr{H}_\nu^{n-1}$ of the pseudo-hyperbolic space. □

We notice the following consequence of the previous discussion:

Remark 1. Let $\mathbb{E}_\nu^{n-1}$ be a semi-Euclidean subspace of codimension one in the vector space $\mathbb{V}^n$. For any $N\in\mathbb{V}^n\setminus\mathbb{E}_\nu^{n-1}$ the scalar product $\langle X,Y\rangle$ on $\mathbb{E}_\nu^{n-1}$ can be extended to a scalar product on $\mathbb{V}^n$ defining a semi-Euclidean space

(a) $\mathbb{E}_\nu^n$ satisfying $N\in\mathscr{S}_\nu^{n-1}(\varepsilon)\subset\mathbb{E}_\nu^n$ and $\mathrm{T}_N(\mathscr{S}_\nu^{n-1})=\mathbb{E}_\nu^{n-1}$,

(b) $\mathbb{E}_{\nu+1}^n$ satisfying $N\in\mathscr{S}_\nu^{n-1}(\varepsilon)\subset\mathbb{E}_{\nu+1}^n$ and $\mathrm{T}_N(\mathscr{H}_\nu^{n-1})=\mathbb{E}_\nu^{n-1}$.

Theorem 1. *The exponential map* $\mathrm{Exp}_\varepsilon : \mathrm{T}_I(\mathscr{S}_\nu^{n-1}(\varepsilon)) \to \mathscr{S}_\nu^{n-1}(\varepsilon)$ *for all* $\mathbf{x} \in \mathrm{T}_I(\mathscr{S}_\nu^{n-1}(\varepsilon))$ *has the absolutely convergent power series expansion*

$$\begin{aligned}\mathrm{Exp}_\varepsilon(\mathbf{x}) &= \sum_{n=0}^{\infty}\left(\frac{(\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)^n}{(2n)!}I + \frac{(\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)^n}{(2n+1)!}\mathbf{x}\right) = \\ &= \sum_{n=0}^{\infty}(\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)^n\left(\frac{1}{(2n)!}I + \frac{1}{(2n+1)!}\mathbf{x}\right) = \\ &= I+\mathbf{x}+\varepsilon\langle\mathbf{x},\mathbf{x}\rangle\big(\frac{1}{2}I+\frac{1}{3!}\mathbf{x}\big)+\langle\mathbf{x},\mathbf{x}\rangle^2\big(\frac{1}{4!}I+\frac{1}{5!}\mathbf{x}\big)+\varepsilon\langle\mathbf{x},\mathbf{x}\rangle^3\big(\frac{1}{6!}I+\frac{1}{7!}\mathbf{x}\big)\dots\end{aligned}$$

Proof. The assertion follows from the formula $\mathrm{Exp}_\varepsilon(\mathbf{x}) = \mathrm{C}(\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)I + \mathrm{S}(\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)\mathbf{x}$ using the power series (1) of the functions $\mathrm{C}(x)$ and $\mathrm{S}(x)$. □

4 Logarithm map

Proposition 2. *The domain in* $\mathrm{T}_I(\mathscr{S}_\nu^{n-1}(\varepsilon))$ *of the form* $\{\mathbf{x};\varepsilon\langle\mathbf{x},\mathbf{x}\rangle < r^2\}$ *with maximal radius r, on which the map induced by* $\mathrm{Exp}_\varepsilon : \mathrm{T}_I(\mathscr{S}_\nu^{n-1}(\varepsilon)) \to \mathscr{S}_\nu^{n-1}(\varepsilon)$ *is bijective, is*

$$\mathscr{D}_\varepsilon = \{\mathbf{x} \in \mathrm{T}_I(\mathscr{S}_\nu^{n-1}(\varepsilon)); \varepsilon\langle\mathbf{x},\mathbf{x}\rangle < \pi^2\} \subset \mathrm{T}_I(\mathscr{S}_\nu^{n-1}(\varepsilon)).$$

The image $\mathrm{Exp}_\varepsilon(\mathscr{D}_\varepsilon)$ *is* $\mathscr{R}_\varepsilon \cap \mathscr{S}_\nu^{n-1}(\varepsilon)$*, where* $\mathscr{R}_\varepsilon$ *is the half-space*

$$\mathscr{R}_\varepsilon = \{X = xI + \mathbf{x} \in \mathbb{E}_\nu^n;\ x = \varepsilon\langle X, I\rangle > -1\} \subset \mathbb{E}_\nu^n.$$

Proof. We assume $\varepsilon\langle\mathbf{x},\mathbf{x}\rangle < \pi^2$, since $\mathrm{Exp}_\varepsilon(\mathbf{x}) = -I$ if $\langle\mathbf{x},\mathbf{x}\rangle = \varepsilon$ and $\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle|} = \pi$. It follows from the inequalities $-1 < \cos\sqrt{\varepsilon\langle\mathbf{x},\mathbf{x}\rangle} \le 1$ for $0 \le \varepsilon\langle\mathbf{x},\mathbf{x}\rangle < \pi^2$ and $1 \le \cosh\sqrt{-\varepsilon\langle\mathbf{x},\mathbf{x}\rangle}$ for $\varepsilon\langle\mathbf{x},\mathbf{x}\rangle \le 0$, that $-1 < \varepsilon\langle\mathrm{Exp}_\varepsilon(\mathbf{x}), I\rangle = \mathrm{C}(-\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)$ is satisfied, consequently $\mathrm{Exp}_\varepsilon(\mathbf{x}) \in \mathscr{R}_\varepsilon \cap \mathscr{S}_\nu^{n-1}(\varepsilon)$.
Conversely, for given $xI + \mathbf{x} \in \mathscr{R}_\varepsilon \cap \mathscr{S}_\nu^{n-1}(\varepsilon)$ we want to find $\mathbf{y} \in \mathrm{T}_I(\mathscr{S}_\nu^{n-1}(\varepsilon))$ such that $xI + \mathbf{x} = \mathrm{Exp}_\varepsilon(\mathbf{y}) = \mathrm{C}(-\varepsilon\langle\mathbf{y},\mathbf{y}\rangle)I + \mathrm{S}(-\varepsilon\langle\mathbf{y},\mathbf{y}\rangle)\mathbf{y}$, or equivalently

$$x = \mathrm{C}(-\varepsilon\langle\mathbf{y},\mathbf{y}\rangle), \quad \mathbf{x} = \mathrm{S}(-\varepsilon\langle\mathbf{y},\mathbf{y}\rangle)\mathbf{y}. \tag{3}$$

Since $0 < \mathrm{C}'(t) = \frac{1}{2}\mathrm{S}(t)$ for $t > -\pi^2$ the induced map $\mathrm{C} : \{t \in \mathbb{R}; t > -\pi^2\} \to \{t \in \mathbb{R}; t > -1\}$ is bijective. Hence we can express $-\varepsilon\langle\mathbf{y},\mathbf{y}\rangle = \mathrm{C}^{-1}(x) > -\pi^2$ and $\mathbf{x} = \mathrm{S}(\mathrm{C}^{-1}(x))\mathbf{y}$ for $x > -1$, and we get

$$\mathbf{y} = \frac{1}{\mathrm{S}(\mathrm{C}^{-1}(x))}\mathbf{x}, \tag{4}$$

proving the bijectivity of $\mathrm{Exp}_\varepsilon|_{\mathscr{D}_\varepsilon} : \mathscr{D}_\varepsilon \to \mathscr{R}_\varepsilon \cap \mathscr{S}_\nu^{n-1}(\varepsilon)$. □

The expression (4) of the inverse map of $\mathrm{Exp}_\varepsilon|_{\mathscr{D}_\varepsilon}$ generalizes the formula given in V.4.19 Theorem in [5] and Theorem 1.3.a) in [4].

Definition 1. *The* logarithm map $\mathrm{Log}_\varepsilon : \mathscr{R}_\varepsilon \cap \mathscr{S}_\nu^{n-1}(\varepsilon) \to \mathscr{D}_\varepsilon$ *is the inverse of the exponential map* $\mathrm{Exp}_\varepsilon|_{\mathscr{D}_\varepsilon} : \mathscr{D}_\varepsilon \to \mathscr{R}_\varepsilon \cap \mathscr{S}_\nu^{n-1}(\varepsilon)$*, expressed by*

$$\mathrm{Log}_\varepsilon(X) = \frac{X - \varepsilon\langle X, I\rangle I}{\mathrm{S}(\mathrm{C}^{-1}(\varepsilon\langle X, I\rangle))}, \quad X \in \mathscr{R}_\varepsilon \cap \mathscr{S}_\nu^{n-1}(\varepsilon). \tag{5}$$

Proposition 3. *The map* $\mathrm{Log}_\varepsilon : \mathscr{R}_\varepsilon \cap \mathscr{S}_\nu^{n-1}(\varepsilon) \to \mathscr{D}_\varepsilon$ *can be written in the form*

$$\mathrm{Log}_\varepsilon(xI + \mathbf{x}) = \frac{\mathbf{x}}{\mathrm{S}(\mathrm{C}^{-1}(x))} = \frac{\mathbf{x}}{\sqrt{|\langle \mathbf{x}, \mathbf{x}\rangle|}} \begin{cases} \arccos x, & \text{if} \quad -1 < x < 1, \\ \operatorname{arcosh} x, & \text{if} \quad 1 \le x. \end{cases} \tag{6}$$

or equivalently,

$$\mathrm{Log}_\varepsilon(xI + \mathbf{x}) = \frac{\mathbf{x}}{\sqrt{|\langle \mathbf{x}, \mathbf{x}\rangle|}} \begin{cases} \pi - \arcsin(\sqrt{\varepsilon\langle \mathbf{x}, \mathbf{x}\rangle}), & \text{if} -1 \le x \le 0, \\ \arcsin(\sqrt{\varepsilon\langle \mathbf{x}, \mathbf{x}\rangle}), & \text{if} \quad 0 \le x \le 1, \\ \operatorname{arsinh}(\sqrt{-\varepsilon\langle \mathbf{x}, \mathbf{x}\rangle}), & \text{if} \quad 1 \le x. \end{cases} \tag{7}$$

$$\mathrm{C}(x) = \begin{cases} \cosh\sqrt{x}, & \text{if} x \ge 0, \\ \cos\sqrt{-x}, & \text{if} x < 0, \end{cases} \quad , \quad \mathrm{S}(x) = \frac{1}{\sqrt{|x|}} \begin{cases} \sinh\sqrt{x}, & \text{if} x \ge 0, \\ \sin\sqrt{-x}, & \text{if} x < 0, \end{cases}$$

Proof. According to (3) the equation $xI + \mathbf{x} = \mathrm{C}(-\varepsilon\langle \mathbf{y}, \mathbf{y}\rangle)I + \mathrm{S}(-\varepsilon\langle \mathbf{y}, \mathbf{y}\rangle)\mathbf{y}$ for given

$$X = xI + \mathbf{x} \in \mathscr{R}_\varepsilon \cap \mathscr{S}_\nu^{n-1}(\varepsilon),\ x = \varepsilon\langle X, I\rangle > -1,\ \mathbf{x} \in \mathrm{T}_I(\mathscr{S}_\nu^{n-1}(\varepsilon))$$

yields $x = \mathrm{C}(-\varepsilon\langle \mathbf{y}, \mathbf{y}\rangle)$ with $\varepsilon\langle \mathbf{y}, \mathbf{y}\rangle < \pi^2$. One has

$$x = \begin{cases} \cos(\sqrt{\varepsilon\langle \mathbf{y}, \mathbf{y}\rangle}) = \cos(\sqrt{|\langle \mathbf{y}, \mathbf{y}\rangle|}), & \text{if} \quad 0 < \varepsilon\langle \mathbf{y}, \mathbf{y}\rangle < \pi^2, \\ \cosh(\sqrt{-\varepsilon\langle \mathbf{y}, \mathbf{y}\rangle}) = \cosh(\sqrt{|\langle \mathbf{y}, \mathbf{y}\rangle|}), & \text{if} \quad \varepsilon\langle \mathbf{y}, \mathbf{y}\rangle \le 0 \end{cases} \tag{8}$$

Since $\cos : (0, \pi) \to (-1, 1)$, $\cosh : [0, \infty) \to [1, \infty)$ we get

$$\begin{aligned} -1 < x = \mathrm{C}(-\varepsilon\langle \mathbf{y}, \mathbf{y}\rangle) < 1, &\quad \text{if} \quad 0 < \varepsilon\langle \mathbf{y}, \mathbf{y}\rangle < \pi^2, \\ 1 \le x = \mathrm{C}(-\varepsilon\langle \mathbf{y}, \mathbf{y}\rangle), &\quad \text{if} \quad \varepsilon\langle \mathbf{y}, \mathbf{y}\rangle \le 0. \end{aligned}$$

Applying the maps $\arccos : [-1, 1] \to [0, \pi]$, $\operatorname{arcosh} : [1, \infty) \to [0, \infty)$ to (8) it follows

$$\sqrt{|\langle \mathbf{y}, \mathbf{y}|\rangle} = \begin{cases} \arccos x, & \text{if} -1 < x < 1, \\ \operatorname{arcosh} x, & \text{if} \quad 1 \le x. \end{cases} \tag{9}$$

Since $\mathrm{S}(-\varepsilon\langle \mathbf{y}, \mathbf{y}\rangle) > 0$ for $\varepsilon\langle \mathbf{y}, \mathbf{y}\rangle < \pi^2$ and

$$\mathbf{x} = \mathrm{S}(-\varepsilon\langle \mathbf{y}, \mathbf{y}\rangle)\mathbf{y}, \quad \sqrt{|\langle \mathbf{x}, \mathbf{x}\rangle|} = \mathrm{S}(-\varepsilon\langle \mathbf{y}, \mathbf{y}\rangle)\sqrt{|\langle \mathbf{y}, \mathbf{y}|\rangle},$$

we can express using (9) that

$$\mathbf{y} = \frac{\mathbf{x}}{\mathrm{S}(-\varepsilon\langle\mathbf{y},\mathbf{y}\rangle)} = \frac{\mathbf{x}}{\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle|}}\begin{cases}\arccos x, & \text{if} -1 < x < 1,\\ \operatorname{arcosh} x, & \text{if} \quad 1 \le x,\end{cases}$$

proving (6). The maps $\arccos : [-1,1] \to [0,\pi]$, $\operatorname{arcosh} : [1,\infty) \to [0,\infty)$ can be expressed by the maps $\arcsin : [-1,1] \to [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ and $\operatorname{arsinh} : \mathbb{R} \to \mathbb{R}$ as

$$\arccos x = \begin{cases}\pi - \arcsin(\sqrt{1-x^2}), & \text{if } -1 \le x < 0,\\ \arcsin(\sqrt{1-x^2}), & \text{if } \quad 0 \le x \le 1,\end{cases}$$
$$\operatorname{arcosh} x = \operatorname{arsinh}\sqrt{x^2-1}, \ \text{ if } 1 \le x.$$

One has $xI + \mathbf{x} \in \mathscr{S}^{n-1}_{\nu}(\varepsilon)$ if and only if $1 - x^2 = \varepsilon\langle\mathbf{x},\mathbf{x}\rangle$, hence we get

$$\mathbf{y} = \frac{\mathbf{x}}{\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle|}}\begin{cases}\pi - \arcsin(\sqrt{1-x^2}) = \pi - \arcsin(\sqrt{\varepsilon\langle\mathbf{x},\mathbf{x}\rangle}), & \text{if } -1 \le x < 0,\\ \arcsin(\sqrt{1-x^2}) \qquad = \arcsin(\sqrt{\varepsilon\langle\mathbf{x},\mathbf{x}\rangle}), & \text{if } \quad 0 \le x < 1,\\ \operatorname{arsinh}(\sqrt{x^2-1}) \qquad = \operatorname{arsinh}(\sqrt{-\varepsilon\langle\mathbf{x},\mathbf{x}\rangle}), & \text{if } \quad 1 \le x,\end{cases}$$

proving (7). □

The map $x \mapsto \varepsilon\langle\mathbf{x},\mathbf{x}\rangle$ gives a monotone decreasing correspondence $[0,\infty) \to [1,-\infty)$, hence

Corollary 1. *The map* $(xI+\mathbf{x}) \mapsto \mathrm{Log}_{\varepsilon}(xI+\mathbf{x})$ *is expressed by*

$$\mathrm{Log}_{\varepsilon}(xI+\mathbf{x}) = \frac{\mathbf{x}}{\sqrt{|\langle\mathbf{x},\mathbf{x}\rangle|}}\begin{cases}\arcsin(\sqrt{\varepsilon\langle\mathbf{x},\mathbf{x}\rangle}), & \text{if} \quad 1 \ge \varepsilon\langle\mathbf{x},\mathbf{x}\rangle > 0,\\ \operatorname{arsinh}(\sqrt{-\varepsilon\langle\mathbf{x},\mathbf{x}\rangle}), & \text{if} \quad 0 \ge \varepsilon\langle\mathbf{x},\mathbf{x}\rangle\end{cases}$$

on $\{xI+\mathbf{x} \in \mathscr{S}^{n-1}_{\nu}(\varepsilon);\ \varepsilon\langle\mathbf{x},\mathbf{x}\rangle \le 1\}$.

5 Power series of the logarithm map

Definition 2. *Let* $\mathrm{L}(t)$ *be the function defined on* $\{t \in \mathbb{R}; t < 1\}$ *by*

$$\mathrm{L}(t) = \frac{1}{\sqrt{|t|}}\begin{cases}\arcsin\sqrt{t}, & \text{if } 0 \le t \le 1,\\ \operatorname{arsinh}\sqrt{-t}, & \text{if } \quad t < 0.\end{cases}$$

Lemma 2. *The function* $\mathrm{L}(t)$ *is real analytic on the interval* $-1 < t < 1$. *Its absolutely convergent power series expansion around* 0 *is*

$$\mathrm{L}(t) = \sum_{n=0}^{\infty}\frac{(2n)!}{(2^n n!)^2}\frac{t^n}{2n+1} = 1 + \frac{1}{2}\frac{t}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{t^2}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{t^3}{7} + \dots \tag{10}$$

Proof. The absolutely convergent power series of the functions $\arcsin t$ and $\operatorname{arcosh} t$ are

$$\begin{aligned}\arcsin t =& t\sum_{n=0}^{\infty}\frac{(2n)!}{(2^n n!)^2}\frac{t^{2n}}{2n+1}=t\big(1+\frac{1}{2}\frac{t^2}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{t^4}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{t^6}{7}+\dots\big),\\ \operatorname{arcosh} t =& t\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{(2^n n!)^2}\frac{t^{2n}}{2n+1}=t\big(1-\frac{1}{2}\frac{t^2}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{t^4}{5}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{t^6}{7}\pm\dots\big),\end{aligned}$$

on the interval $-1<t<1$, (cf. 4.24.1 and 4.34.1 in [8]). Putting $\sqrt{|t|}$ into t we obtain the power series $\mathrm{L}(t)=\sum_{n=0}^{\infty}\frac{(2n)!}{(2^n n!)^2}\frac{t^n}{2n+1}$. The ratio test shows that the radius of convergence is 1, therefore this series defines a complex analytic function on the unit disk from which the claim follows. □

Theorem 2. *The logarithm map* Log_ε *has the absolutely convergent power series expansion*

$$\begin{aligned}\mathrm{Log}_\varepsilon(xI+\mathbf{x})=\mathrm{L}(\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)\mathbf{x}=\sum_{n=0}^{\infty}\frac{(2n)!}{(2^n n!)^2}\frac{(\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)^n}{2n+1}\mathbf{x}=\\ =\mathbf{x}\left(1+\frac{1}{2}\frac{\varepsilon\langle\mathbf{x},\mathbf{x}\rangle}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{(\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)^2}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{(\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)^3}{7}+\dots\right)\end{aligned}$$

around I *on the domain* $\{xI+\mathbf{x}\in\mathscr{S}_\nu^{n-1}(\varepsilon);-1<\varepsilon\langle\mathbf{x},\mathbf{x}\rangle<1\}$.

Proof. Putting $\varepsilon\langle\mathbf{x},\mathbf{x}\rangle$ into t in the function (10), we get the real analytic function $\mathrm{L}(\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)$ represented by the absolutely convergent power series

$$\sum_{n=0}^{\infty}\frac{(2n)!}{(2^n n!)^2}\frac{(\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)^n}{2n+1}=1+\frac{1}{2}\frac{\varepsilon\langle\mathbf{x},\mathbf{x}\rangle}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{(\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)^2}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{(\varepsilon\langle\mathbf{x},\mathbf{x}\rangle)^3}{7}+\dots$$

for $-1<\langle\mathbf{x},\mathbf{x}\rangle<1$. According to Corollary 1 we get the assertion. □

References

[1] B. Bertotti, R. Catenacci and C. Dappiaggi, The legacy of pseudospheres: From geometry to physics. *Riv. Nuovo Cim*, 29: 1–23, 2006.

[2] J. Gallier and J. Quaintance. *Diferential Geometry and Lie Groups: A Computational Perspective*. Springer Nature, 2020.

[3] T. Gao, L.-H. Lim and K. Ye. Semi-Riemannian Manifold Optimization. https://doi.org/10.48550/arXiv.1812.07643

[4] J. Hilgert and K. H. Hofmann. Old and new on $SL(2)$. *Manus. Math.* 54:17–52, 1985.

[5] J. Hilgert, K. H. Hofmann and J. D. Lawson, *Lie Groups, Convex Cones, and Semigroups*. Clarendon Press, Oxford, 1989.

[6] M. T. Law. Ultrahyperbolic Neural Networks. *Advances in Neural Information Processing Systems, (NeurIPS)*, 34:22058–22069, 2021.

[7] M. T. Law and J. Stam. Ultrahyperbolic Representation Learning. *Proceedings of the 34th International Conference on Neural Information Processing Systems*, NIPS'20:1668–1678, 2020.

[8] *NIST Handbook of Mathematical Functions*, edited by F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, Cambridge University Press, Cambridge. 2010.

[9] B. O'Neill. *Semi-Riemannian Geometry with Applications to Relativity*. Academic Press, New York-London, 1983.

[10] Z. Qiao and R. Dick. Matrix logarithms and range of the exponential maps for the symmetry groups $SL(2;\mathbb{R})$; $SL(2;\mathbb{C})$, and the Lorentz group. *J. Phys. Commun.* 3:1–17, 2019.

[11] A. Sim, M. Wiatrak, A. Brayne, P. Creed, S. Paliwal. Directed Graph Embeddings in Pseudo-Riemannian Manifolds. *Proceedings of Machine Learning Research*, International Conference on Machine Learning, 9681–9690, 2021.

[12] B. Xiong, S. Zhu, M. Nayyeri, C. Xu, S. Pan, C. Zhou and S. Staab. Ultrahyperbolic Knowledge Graph Embeddings. *KDD '22 : Proceedings of the 28th ACM SIGKDD Conference on Knowledge Discovery and Data Mining*, Association for Computing Machinery, 2130–2139, 2022.

[13] B. Xiong, S. Zhu, N. Potyka, S. Pan, C. Zhou, S. Staab. Pseudo-Riemannian Graph Convolutional Networks. *Advances in Neural Information Processing Systems*, 35:3488–3501, 2022.

[14] Y. Zhang, X. Wang, C. Shi, N. Liu and G. Song. Lorentzian Graph Convolutional Networks, *Proceedings of the Web Conference*, 1249–1261, 2021.