Basic equations of fluid dynamics treated by pseudo-analysis

Endre Pap*,,***, Doretta Vivona******

* Department of of Mathematics and Informatics, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia, email:pap@dmi.uns.ac.rs

** Óbuda University, Becsi út 96/B, H-1034 Budapest, Hungary,

*** Educons University, Vojvode Putnika 87, 21208 Sremska Kamenica, Serbia,

**** Dipartimento di Scienze di Base e Applicate per l'Ingeneria, "Sapienza"- Universitá di Roma, Via A. Scarpa 10, 00161 Rome, Italy, e-mail:doretta.vivona@uniroma1.it

Abstract. *There is given an application of pseudo-analysis in the theory of fluid mechanics. First, the monotonicity of the components of the velocity for the solutions of Euler equations is proven, which allows to obtain the pseudo-linear superposition principle for Euler equations. This principle is proven also for the Navier-Stokes equations but with respect to two different pairs of pseudo-operations. It is shown that Stokes equations satisfy the pseudo-linear superposition principle with respect to a pair of pseudo-operations which are generated with the same function of one variable.*

Keywords: Fluid mechanics, Euler equations, Navier-Stokes equations, Stokes equations, pseudo-linear superposition principle, semiring.

.

1 Introduction

The motion of fluids was mathematically modeled in the period of more than two hundred years. The ordinary incompressible Newton Fluids are modeled by the Navier-Stokes equations and the related Euler equations. Some of the recent investigations are summarized in the two volumes of the Handbook of Mathematical Fluid Dynamics ([5, 6]).

We shall prove in this paper an important property of the three basic equations (Euler, Navier-Stokes, Stokes), the so called *pseudo-linear superposition principle*. To achieve this principle in full generality we shall neglect at this level the problem of the regularity of the solution, which is a very important part of the investigations in fluid dynamics, see ([1, 2, 3, 24]).

What we are doing, roughly speaking, is that we replace the usual field of real numbers by a semiring on a real interval $[a, b] \subset [-\infty, \infty]$ ([7, 8, 11, 12, 14]), where the corresponding operations are $\oplus$ (pseudo-addition) and $\odot$ (pseudo-multiplication). Based on the semiring structure there is developed in ([12, 13, 14, 15, 18, 19]) the so called pseudo-analysis, in an analogous way as classical analysis, introducing pseudo-measure, pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc.([15, 16, 17, 18, 20, 21, 22]). The advantage of the pseudo-analysis is that the problems (usually nonlinear) from many different fields (system theory, optimization, control theory, differential equations, difference equations, etc.) are covered with one theory, and so with unified methods. The pseudo-analysis is used for solving nonlinear equations (ODE,PDE, difference equations, etc.), based on pseudo-linear superposition principle, which means that if u_1 and u_2 are solutions of the considered nonlinear equation, then also $a_1 \odot u_1 \oplus a_2 \odot u_2$ is a solution for any numbers a_1 and a_2 from $[a, b]$. The important fact is that this approach gives also solutions in a new form, not achieved by other theories. In some cases it enables for the nonlinear equations to obtain exact solutions in a similar form as for the linear equations.

After some preliminaries in Section 2, and recalling some basic facts on the Euler equations in Section 3, we prove in Section 4 the monotonicity of the velocity for the solutions of the Euler equations. This help us to prove in Section 5 the pseudo-linear superposition principle for the Euler equations. This principle is achieved also for the Navier-Stokes with respect to two different pairs of pseudo-operations. In Section 6 it is shown that Stokes equations satisfy the pseudo-linear superposition principle but with respect to a pair of pseudo-operations which are generated with the same function of one variable.

2 Preliminary notions

We consider a fluid which occupies a 2-dimensional region, denoted by D, and we denote by ∂D the boundary of D. We denote by $\mathbf{x}$ the spatial coordinate $\mathbf{x} = (x, y)$, with t the time and with $\mathbf{u}$ the field of the velocity of each element: $\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathrm{t}) = \mathbf{u}(u(x, y, t), v(x, y, t))$. Moreover we assume that the fluid has a well-defined mass density, indicated with $\rho = \rho(\mathbf{u}, t)$.

We shall use the following notations: grad $p = (p_x, p_y, p_z)$,

$$\partial_x = \frac{\partial}{\partial x} \ , \ \partial_t = \frac{\partial}{\partial t} \ ,$$

$$\text{div } \mathbf{u} = \nabla \mathbf{u} = \partial_x u + \partial_y v, \quad (\mathbf{u}\nabla)\cdot = u\partial_x \cdot + v\partial_y \cdot \quad .$$

The expression

$$\frac{D\cdot}{Dt} = \partial_t \cdot + (\mathbf{u}\nabla)\cdot \tag{1}$$

will be called the material derivative, and we have

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = u\partial_x \mathbf{u} + v\partial_y \mathbf{u} + \partial_t \mathbf{u} = \partial_t \mathbf{u} + (\mathbf{u}\nabla)\mathbf{u} \ ,$$

$$\frac{D\rho}{Dt} + \rho \ \text{ div } \mathbf{u} = \frac{\partial \rho}{\partial t} \ + \ \text{div}(\rho \mathbf{u}) \ .$$

We consider two kinds of fluids:

– *incompressible fluid* if for any subregion W the volume is constant in t. This implies div $\mathbf{u} = 0$. From continuity equation and $\rho > 0$ it follows that the fluid is incompressible if and only if the mass density is constant: $\frac{D\rho}{Dt} = 0$;

– *homogeneous fluid* if the density ρ is constant in space.

The classical approach ([4]) is based on three assumptions:

1) *conservation of the mass* :

mass is neither created nor destroyed. The consequence of this principle is the so-called *continuity equation*:

$$\frac{\partial \rho}{\partial t} \ + \ \text{div}(\rho \mathbf{u}) \ = \ 0.$$

2) *balance of momentum* or *Newton's second law* :

$$\rho \frac{D\mathbf{u}}{Dt} = - \text{ grad p} + \mathbf{f} \ ,$$

where $\mathbf{f}$ are the forces.

3) *conservation of energy* :
energy is neither created nor destroyed.

3 Euler equations

In this paragraph we recall the equations of the motion of an incompressible fluid in 2-dimensional case. They are based on the Newton's second law, mass conservation and condition of incompressibility (*Euler equations*):

$$\rho \frac{D\mathbf{u}}{Dt} = - \text{ grad p} + \mathbf{f} \tag{2}$$

$$\frac{D\rho}{Dt} + \rho \text{ div } \mathbf{u} = 0$$

$$\text{div } \mathbf{u} = 0$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial D, \tag{3}$$

where $\mathbf{n}$ is the normal to the region D. (3) is the boundary condition.

The unknown functions of the system (2)-(3) are the components u, v of the velocity: $\mathbf{u} : I\!R \times I\!R \times I\!R^+ \to I\!R$, $\mathbf{u} = (u(\mathbf{x}, t), v(\mathbf{x}, t))$ and the pressure $p : I\!R \times I\!R \times I\!R^+ \to I\!R$. We denote by s the triple of the functions $u(\mathbf{x}, \text{t}), \text{v}(\mathbf{x}, \text{t}), \text{p}(\mathbf{x}, \text{t}) : \text{s} = (\text{u}(\mathbf{x}, \text{t}), \text{v}(\mathbf{x}, \text{t}), \text{p}(\mathbf{x}, \text{t}))$.

Without loss of generality we suppose that $\rho = 1$ and $\mathbf{f} = 0$.

Now we reformulate the equation (2) taking into account the definition of material derivative (1). We have

$$\partial_t \mathbf{u} + (\mathbf{u}\nabla)\mathbf{u} \; + \; \text{grad } p \; = 0 \tag{4}$$

$$\text{div } \mathbf{u} \; = \; 0 \tag{5}$$

$$v(x, y = 0, t) = 0 \quad . \tag{6}$$

As we have seen above, the velocity $\mathbf{u}$ depends on the variables x, y, t, in particular $y \in [0, \infty[$; the boundary condition (3) involves only the component of the velocity on the axe y, whose unit vector is $\mathbf{n}$.

Now, we project the first (vector) equation (4) on axes x and y:

$$\partial_t u + u\partial_x u + v\partial_y u + \partial_x p = 0 \tag{7}$$

$$\partial_t v + u\partial_x v + v\partial_y v + \partial_y p = 0 \quad . \tag{8}$$

We know ([2, 4, 10, 25, 26]) that the Euler equations are particular case of the Navier-Stokes equations when the viscosity ν of the fluid is zero. The solution of the Navier-Stokes equations can be well approximated by an Euler equation, when the viscosity is small, at least away from boundaries.

4 Monotonicity of the components of the velocity

Now we come back to the general discussion of the Euler equations (4)-(6). With the above notation, from the condition (5), we have for the Euler equations $\partial_x u + \partial_y v = 0$, i.e.,

$$v = -\int_0^y \partial_x u(x, y', t)\, dy'. \tag{9}$$

Proposition 4.1 . *Let* $\mathbf{u_1} = (u_1(\mathbf{x},t), v_1(\mathbf{x},t)), \mathbf{u_2} = (u_2(\mathbf{x},t), v_2(\mathbf{x},t))$ *be two velocities which satisfy the condition (5). If the function* $(u_2-u_1)(\mathbf{x},t)$ *is either non-increasing or non-decresing with respect to* $\mathbf{x}$*, then the functions* $v_1(\mathbf{x},t)$ *and* $v_2(\mathbf{x},t)$ *satisfy either the following condition* $v_1 \leqslant v_2$ *or the condition* $v_2 \geqslant v_1$*, respectively, i.e., either*

$$\partial_x(u_2 - u_1) \ \leqslant 0 \Rightarrow v_1 \leqslant v_2$$

or

$$\partial_x(u_2 - u_1) \ \geqslant 0 \Rightarrow v_1 \geqslant v_2.$$

Proof. As the function $(u_2 - u_1)(\mathbf{x},t)$ is non-increasing with respect to $\mathbf{x}$, then $\partial_x(u_2 - u_1) \ \leqslant 0$. Therefore by the the condition (9) for v we have

$$\int_0^y \partial_x(u_2 - u_1)(x, y', t)\, dy' \leqslant 0,$$

and then

$$v_2 - v_1 = -\int_0^y \partial_x u_2(x, y', t)\, dy' - \left(-\int_0^y \partial_x u_1(x, y', t)\, dy'\right) \geqslant 0,$$

i.e., $v_2 \geqslant v_1$. □

From now on we consider the following sets of functions :

$$\mathcal{U}_{ni} = \{(u_1, u_2) \mid u_1 \leqslant u_2 \text{ and } \partial_x(u_2 - u_1) \ \leqslant 0\}$$

$$\mathcal{U}_{nd} = \{(u_1, u_2) \mid u_1 \geqslant u_2 \text{ and } \partial_x(u_2 - u_1) \ \geqslant 0\}.$$

As consequence of Proposition 4.1 we have the following:

Proposition 4.2 *If the couple of functions* (u_i, v_i) $i = 1, 2$ *satisfy the condition (9) and* u_i $i = 1, 2$ *are elements either of the set* $\mathcal{U}_{ni}$ *or the set* $\mathcal{U}_{nd}$*, then* $v_1 \leqslant v_2$ *and* $v_1 \geqslant v_2$*, respectively.*

5 Pseudo-linear superposition principle

5.1 Pseudo-analysis

We shall use the approach from ([14, 15, 18]). Let $[a, b]$ be a closed (in some cases semiclosed) subinterval of $[-\infty, \infty]$. We consider here a total order $\leq$

on $[a,b]$ (although it can be taken in the general case a partial order). The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a,b] \times [a,b] \to [a,b]$ which is commutative, non-decreasing, associative and has a zero element, denoted by $\mathbf{0}$. Let $[a,b]_+ = \{x : x \in [a,b], \quad x \geqslant \mathbf{0}\}$. The operation $\odot$ (*pseudo-multiplication*) is a function $\odot : [a,b] \times [a,b] \to [a,b]$ which is commutative, positively non-decreasing, i.e. $x \leq y$ implies $x \odot z \leq y \odot z$, $z \in [a,b]_+$, associative and for which there exists a unit element $\mathbf{1} \in [a,b]$, i.e., for each $x \in [a,b]$, $\mathbf{1} \odot x = x$.

We suppose, further, $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $([a,b], \oplus, \odot)$ is called a *semiring*.

We shall use the following important cases (pairs):

$$\alpha \oplus \beta = \min(\alpha,\beta), \qquad \alpha \odot \beta = \max(\alpha,\beta),$$
$$\alpha \oplus \beta = \max(\alpha,\beta), \qquad \alpha \odot \beta = \min(\alpha,\beta),$$
$$\alpha \oplus \beta = \min(\alpha,\beta), \qquad \alpha \odot \beta = \alpha + \beta,$$
$$\alpha \oplus \beta = \max(\alpha,\beta), \qquad \alpha \odot \beta = \alpha + \beta.$$

We translate the previous operations pointwise on functions.

We use the following notations:

$$\mathbf{u_1} = (u_1(\mathbf{x},t), v_1(\mathbf{x},t), t), \mathbf{u_2} = (u_2(\mathbf{x},t), v_2(\mathbf{x},t), t),$$
$$\mathbf{s_i} = (u_i(\mathbf{x},t), v_i(\mathbf{x},t), p_i(\mathbf{x},t)),\ i = 1,2,$$

and specially for $p_1 = p_2 = p$ we take

$$\mathbf{s}_{i,p} = (u_i(\mathbf{x},t), v_i(\mathbf{x},t), p(\mathbf{x},t)),\ i = 1,2.$$

Given two triplets of solutions $\mathbf{s}_1$ and $\mathbf{s}_2$, we take

$$\mathbf{min}(\mathbf{s}_1,\mathbf{s}_2) := \Big(\min(u_1,u_2), \min(v_1,v_2), \min(p_1,p_2)\Big) \tag{10}$$

and

$$\mathbf{max}(\mathbf{s}_1,\mathbf{s}_2) := \Big(\max(u_1,u_2), \max(v_1,v_2), \max(p_1,p_2)\Big)\,. \tag{11}$$

5.2 Superposition principle for the Euler equations

In this section prove the *pseudo-linear superposition principle* for the Euler equations.

Lemma 5.1 *Let* $\mathbf{s}_{i,p} = (u_i, v_i, p), i = 1, 2$ *be two solutions of (7), (8), (5), such that both* $u_i, i = 1, 2,$ *are either elements of* $\mathcal{U}_{ni}$ *or elements of* $\mathcal{U}_{nd}, i = 1, 2.$

Then the function

$$\mathbf{s}_{1,p} \oplus \mathbf{s}_{2,p} = \mathbf{min}(\mathbf{s}_{1,p}, \mathbf{s}_{2,p}),$$

where $\mathbf{min}(\mathbf{s}_{1,p}, \mathbf{s}_{2,p})$ *is defined by (10), is again solution of (7), (8), (5).*

Proof. We consider two solutions $\mathbf{s}_{i,p} = (u_i, v_i, p), i = 1, 2,$ of (7), (8), (5) . First we shall show that $\mathbf{s}_{1,p} \oplus \mathbf{s}_{2,p}$ satisfies (4), which is written in the form of the projection (7) and (8). So we shall prove that $\mathbf{s}_{1,p} \oplus \mathbf{s}_{2,p}$ satisfies (7). Using the notation $\mathbf{u}_1 = (u_1, v_1)$ and $\mathbf{u}_2 = (u_2, v_2)$, where $\mathbf{s}_{1,p} = (u_1, v_1)$ and $\mathbf{s}_{2,p} = (u_2, v_2)$ we have

$$\begin{aligned}
& \partial_t(u_1 \oplus u_2) + (u_1 \oplus u_2)\partial_x(u_1 \oplus u_2) + (v_1 \oplus v_2)\partial_y(u_1 \oplus u_2) + \partial_x(p \oplus p) \\
= \; & \partial_t(\min(u_1, u_2)) + (\min(u_1, u_2))\partial_x(\min(u_1, u_2)) \\
& +(\min(v_1, v_2))\partial_y(\min(u_1, u_2)) + \partial_x p \\
= \; & \begin{cases} \partial_t u_1 + u_1\partial_x u_1 + v_1\partial_y u_1 + \partial_x p & \text{as } (u_1, u_2) \in \mathcal{U}_{ni} \\ \partial_t u_2 + u_2\partial_x u_2 + v_2\partial_y u_2 + \partial_x p & \text{as } (u_1, u_2) \in \mathcal{U}_{nd} \end{cases} \\
= \; & 0 \, ,
\end{aligned}$$

since by Proposition 4.2 we have for $i, j \in \{1, 2\}$ that $(u_1, u_2) \in \mathcal{U}_{ni}$, implies $v_i(x, y, t) \leq v_j(x, y, t)$, and $(u_1, u_2) \in \mathcal{U}_{nd}$ implies $v_i(x, y, t) \geq v_j(x, y, t)$. This means that $\mathbf{s}_{1,p} \oplus \mathbf{s}_{2,p}$ satisfies the equation (7).

In an analogous way we shall prove that $\mathbf{s}_{1,p} \oplus \mathbf{s}_{2,p}$ is solution of (8). Namely,

$$\partial_t(v_1 \oplus v_2) + (u_1 \oplus u_2)\partial_x(v_1 \oplus v_2) + (v_1 \oplus v_2)\partial_y(v_1 \oplus v_2) + \partial_x(p \oplus p)$$

$$= \begin{cases} \partial_t v_1 + u_1\partial_x v_1 + v_1\partial_y v_1 + \partial_x p & \text{as } (u_1, u_2) \in \mathcal{U}_{ni} \\ \partial_t u_2 + u_2\partial_x v_2 + v_2\partial_y u_2 + \partial_x p & \text{as } (u_1, u_2) \in \mathcal{U}_{nd} \end{cases}$$

$$= 0 .$$

Now we shall show that $\mathbf{s}_{1,p} \oplus \mathbf{s}_{2,p}$ is a solution of the equation (5). In fact

$$\begin{aligned} \text{div } (\mathbf{u_1} \oplus \mathbf{u_2}) &= \partial_x(\min(u_1, u_2)) + \partial_y(\min(v_1, v_2)) \\ &= \begin{cases} \partial_x u_1 + \partial_y v_1 & \text{as } (u_1, u_2) \in \mathcal{U}_{ni} \\ \partial_x u_2 + \partial_y v_2 & \text{as } (u_1, u_2) \in \mathcal{U}_{nd} \end{cases} \\ &= 0. \end{aligned}$$

So we have proved that $\mathbf{s}_{1,p} \oplus \mathbf{s}_{2,p}$ is solution of the system (7), (8) and (5). □

Lemma 5.2 *Under the same suppositions as in Lemma 5.1, we have that the function*

$$\mathbf{s}_{1,p} \oplus \mathbf{s}_{2,p} = \mathbf{max}(\mathbf{s}_{1,p}, \mathbf{s}_{2,p}),$$

defined by (11), is again a solution of the equations (7), (8) and (5).

As an immediate consequence of the previous Lemmas 5.1 and 5.2 we get the following theorems.

Theorem 5.3 *Let* $\mathbf{s}_{i,p} = (u_i, v_i, p), i = 1, 2,$ *be two solutions of (7), (8), (5) such that* (u_1, u_2) *are elements either of* $\mathcal{U}_{ni}$ *or of* $\mathcal{U}_{nd}$, *and* a_1, a_2 *two real numbers. Then the pseudo-linear combination*

$$(a_1 \odot \mathbf{s}_{1,p}) \oplus (a_2 \odot \mathbf{s}_{2,p}) = \mathbf{min}(\mathbf{max}(a_1, \mathbf{s}_{1,p}), \mathbf{max}(a_2, \mathbf{s}_{2,p}))$$

with $\oplus, \odot$ *given by (10) and (11), respectively, is again a solution of (7), (8) and (5) .*

Theorem 5.4 *Let* $\mathbf{s}_{i,p} = (u_i, v_i, p), i = 1, 2,$ *be two solutions of (7) ,(8) , (5) such that* (u_1, u_2) *are elements either of* $\mathcal{U}_{ni}$ *or of* $\mathcal{U}_{nd}$ *and* a_1, a_2 *two real numbers. Then the pseudo-linear combination*

$$(a_1 \odot \mathbf{s}_{1,p}) \oplus (a_2 \odot \mathbf{s}_{2,p}) = \mathbf{max}(\mathbf{min}(a_1, \mathbf{s}_{1,p}), \mathbf{min}(a_2, \mathbf{s}_{2,p}))$$

with $\oplus, \odot$ *given by (11) and (10), respectively, is again a solution of (7), (8) and (5) .*

We obtain, with an additional condition, the pseudo-linear superposition principle for another pair of pseudo-operations.

Theorem 5.5 *Let* $\mathbf{s}_{i,p} = (u_i, v_i, p), i = 1, 2,$ *be two solutions of (7), (8) and (5) such that* (u_1, u_2) *are elements either of* $\mathcal{U}_{ni}$ *or of* $\mathcal{U}_{nd}$*. If* $(u_i, v_i), i = 1, 2$ *satisfy the condition*

$$\partial_y u_i = \partial_y v_i \quad i = 1, 2. \tag{12}$$

then the pseudo-linear combination for two real numbers a_1, a_2

$$(a_1 \odot \mathbf{s}_{1,p}) \oplus (a_2 \odot \mathbf{s}_{2,p}),$$

where $\oplus$ *is given by (10) and* $\odot$ *is defined by*

$$\lambda \odot \mathbf{s} = \lambda \odot (u, v, p) = (\lambda + u, \lambda + v, \lambda + p), \tag{13}$$

is again a solution of (7), (8) and (5).

Proof. First, by Lemma 5.1 $\mathbf{min}(\mathbf{s}_{1,p}, \mathbf{s}_{2,p})$ is a solution of (7) and (8).

Now, it is easy to see that the trivial solution given by three constants $\lambda_i, i = 1, 2, 3$: $\mathbf{s}_c = (\lambda_1, \lambda_2, \lambda_3)$ is again a solution of (7),(8) and (5). We shall prove that for any real number λ , $\lambda \odot \mathbf{s}$ is a solution of (7). In fact,

$$\begin{aligned} & \partial_t(\lambda + u) + (\lambda + u)\ \partial_x(\lambda + u) + (\lambda + v)\ \partial_y(\lambda + u) + \partial_x(\lambda + p) \\ = \ & \partial_t u + (\lambda + u)\partial_x u + (\lambda + v)\partial_y u + \partial_x p \\ = \ & \partial_t u + u\partial_x u + v\partial_y u + \partial_x p + \lambda(\partial_x u + \partial_y u) \\ = \ & \lambda(\partial_x u + \partial_y u) = 0, \end{aligned}$$

where we have used the condition (12), which with (5) for $\mathbf{u}$, i.e., $\partial_x u + \partial_y v = 0$, implies

$$\partial_x u + \partial_y u = 0.$$

So, we have proved that $\lambda \odot \mathbf{s}$ is a solution of (7). In a analogous way we prove that it satisfies (8) and (5). □

In an analogous way we obtain the following theorem.

Theorem 5.6 *Let* $\mathbf{s}_{i,p} = (u_i, v_i, p), i = 1, 2,$ *be two solutions of (7), (8) and (5) such that* (u_1, u_2) *are elements either of* $\mathcal{U}_{ni}$ *or of* $\mathcal{U}_{nd}$*. If* $(u_i, v_i), i = 1, 2$ *satisfy the condition (12), then the pseudo-linear combination for two real numbers* a_1, a_2

$$(a_1 \odot \mathbf{s}_{1,p}) \oplus (a_2 \odot \mathbf{s}_{2,p}),$$

where $\oplus$ *is given by (11) and* $\odot$ *is defined by (13) is again a solution of (7), (8) and (5).* □

5.3 Superposition principle for Navier-Stokes equations

In this section we prove the *pseudo-linear superposition principle* to Navier-Stokes equations. We consider an incompressible homogeneous viscous flow: that means that div $\mathbf{u} = 0$, for the density $\rho = 1$, ν is the coefficient of viscosity, for the forces $\mathrm{f} = 0$. The equations of motion of this flow are the *Navier-Stokes equations*:

$$\rho \frac{D\mathbf{u}}{Dt} = - \text{ grad p} - \nu \Delta \mathbf{u} \tag{14}$$

$$\text{div } \mathbf{u} = 0$$

$$\mathbf{u} = 0 \quad \text{on} \ \ \partial D,$$

where $\Delta \mathbf{u}$ is the Laplacian of the velocity u, defined in this way:

$$\Delta \mathbf{u} = (\partial_{xx} + \partial_{yy})\mathbf{u} = (\partial_{xx} u + \partial_{yy} v),$$

as $\mathbf{u}(\mathbf{x}, t) = (u(x, y, t), v(x, y, t))$.

We consider two-dimensional incompressible flow in the upper half plane $y > 0$; so the projections of the Navier-Stokes equations (14) on axes x and y are the following:

$$\partial_t u + u\partial_x u + v\partial_y u + \partial_x p + \nu(\partial_{xx} u + \partial_{yy} u) = 0 \tag{15}$$

$$\partial_t v + u\partial_x v + v\partial_y v + \partial_y p + \nu(\partial_{xx} v + \partial_{yy} v) = 0 \tag{16}$$

$$\partial_x u + \partial_y v = 0 \tag{17}$$

$$u = v = 0 \quad \text{on} \quad \partial D. \tag{18}$$

In analogous way as in section 5.2 of the Euler equations, we obtain the following theorems.

Theorem 5.7 *Let* $\mathbf{s}_{i,p} = (u_i, v_i, p), i = 1, 2,$ *be two solutions of (15) - (18) and* a_1, a_2 *two real numbers, such that* (u_1, u_2) *are elements either of* $\mathcal{U}_{ni}$ *or of* $\mathcal{U}_{nd}$. *Then the pseudo-linear combination*

$$(a_1 \odot \mathbf{s}_{1,p}) \oplus (a_2 \odot \mathbf{s}_{2,p}) = \mathbf{min}(\mathbf{max}(a_1, \mathbf{s}_{1,p)}), \mathbf{max}(a_2, \mathbf{s}_{2,p}))$$

with $\oplus, \odot$ *given by (10) and (11), respectively, is again a solution of (15) - (18).* □

Theorem 5.8 *Let* $\mathbf{s}_{i,p} = (u_i, v_i, p), i = 1, 2,$ *be two solutions of (15) - (18) and* a_1, a_2 *two real numbers, such that* (u_1, u_2) *are elements either of* $\mathcal{U}_{ni}$ *or of* $\mathcal{U}_{nd}$. *Then the pseudo-linear combination*

$$(a_1 \odot \mathbf{s}_{1,p}) \oplus (a_2 \odot \mathbf{s}_{2,p}) = \mathbf{max}(\mathbf{min}(a_1, \mathbf{s}_{1,p)}), \mathbf{min}(a_2, \mathbf{s}_{2,p}))$$

with $\oplus, \odot$ *given by (10) and (11), respectively, is again a solution of (15) - (18).* □

Theorem 5.9 *Let* $\mathbf{s}_{i,p} = (u_i, v_i, p), i = 1, 2,$ *be two solutions of (15) - (18) such that* (u_1, u_2) *are elements either of* $\mathcal{U}_{ni}$ *or of* $\mathcal{U}_{nd}$. *which satisfy the condition (12). Then the pseudo-linear combination for two real numbers* a_1, a_2

$$(a_1 \odot \mathbf{s}_{1,p}) \oplus (a_2 \odot \mathbf{s}_{2,p}),$$

where $\oplus$ *and* $\odot$ *are given by (10) and (13), respectively, is again a solution of (15) - (18).* □

Theorem 5.10 *Let* $\mathbf{s}_{i,p} = (u_i, v_i, p), i = 1, 2,$ *be two solutions of (15) - (18) such that* (u_1, u_2) *are elements either of* $\mathcal{U}_{ni}$ *or of* $\mathcal{U}_{nd}$. *If the solutions satisfy the conditions (12) , then the pseudo-linear combination for two real numbers* a_1, a_2

$$(a_1 \odot \mathbf{s}_{1,p}) \oplus (a_2 \odot \mathbf{s}_{2,p}),$$

where $\oplus$ *and* $\odot$ *are given by (11) and (13), respectively, is again a solution of (15) - (18).* $\square$

6 Superposition principle for Stokes equations

We know ([2]) that the *Stokes equations* are approximate equations for incompressible flow:

$$\partial_t \mathbf{u} + \text{ grad p} + \nu \Delta \mathbf{u} = 0$$

$$\text{div } \mathbf{u} = 0.$$

The projections on axes x and y of the equations above are:

$$\partial_t u + \partial_x p + \nu(\partial_{xx} u + \partial_{yy} u) = 0 \tag{19}$$

$$\partial_t v + \partial_y p + \nu(\partial_{xx} v + \partial_{yy} v) = 0. \tag{20}$$

$$\partial_x u \ + \ \partial_y v \ = \ 0. \tag{21}$$

In this section we prove the *pseudo-linear superposition principle* for Stokes equations: we shall consider the application to solutions of (19)-(21), which depend only of time t.

Theorem 6.1 *Let* $\mathbf{s}_i(t) = (u_i(t), v_i(t), p_i(t)),\ i = 1, 2$ *be solutions of (19)-(21). Then the pseudo-linear combination for two real numbers* a_1, a_2

$$(a_1 \odot \mathbf{s}_1) \oplus (a_2 \odot \mathbf{s}_2),$$

where $\oplus$ *and* $\odot$ *are given with generator* g *defined by*

$$g(a) = e^{-\ c\ a},\ \ c > 0, \quad \textit{and then} \ \ g^{-1}(b) = -\frac{1}{c}\ \log\ b,$$

$$\mathbf{s}_1 \oplus \mathbf{s}_2 = (g^{-1}(g(u_1) + g(u_2)), g^{-1}(g(v_1) + (g(v_2)), g^{-1}(g(p_1) + g(p_2))),$$

and

$$\begin{aligned} a \odot \mathbf{s} &= (g^{-1}(g(a) \cdot g(u)), g^{-1}(g(a) \cdot g(v)), (g^{-1}(g(a) \cdot g(p))) \\ &= (a + u, a + v, a + p) \end{aligned}$$

is again solution of (19)-(21).

Proof. Let $\mathbf{s}_i(t) = (u_i(t), v_i(t), p_i(t))$ be solutions of the equation (19)-(21).

First we shall prove that $(u_1 \oplus u_2, p_1 \oplus p_2)$ is solution of (19), i.e.,

$$\partial_t(u_1 \oplus u_2) + \partial_x(p_1 \oplus p_2) + \nu(\partial_{xx}(u_1 \oplus u_2) + \partial_{yy}(u_1 \oplus u_2)) = 0. \tag{22}$$

Put

$$U = e^{-cu_1} + e^{-cu_2}\ ,\ P = e^{-cp_1} + e^{-cp_2}\ , \tag{23}$$

we have

$$\partial_t(u_1 \oplus u_2) = \frac{\partial_t u_1 e^{-cu_1}}{U} + \frac{\partial_t u_2 e^{-cu_2}}{U}, \partial_x(p_1 \oplus p_2) = \frac{\partial_x p_1 e^{-cp_1}}{P} + \frac{\partial_x p_2 e^{-cp_2}}{P}; \tag{24}$$

moreover

$$\partial_{xx}(u_1 \oplus u_2)$$

$$= \frac{1}{U^2}\Big(\partial_{xx}u_1 e^{-cu_1}U \ + \ \partial_{xx}u_2 e^{-cu_2}U \ - \ c(\partial_x u_1 - \partial_x u_2)^2 e^{-c(u_1+u_2)}\Big), \tag{25}$$

$$\partial_{yy}(u_1 \oplus u_2)$$

$$= \frac{1}{U^2}\Big(\partial_{yy}u_1 e^{-cu_1}U \ + \ \partial_{yy}u_2 e^{-cu_2}U \ - \ c(\partial_y u_1 - \partial_y u_2)^2 e^{-c(u_1+u_2)}\Big). \tag{26}$$

Therefore, the left side of the equation (22) is the following:

$$\frac{\partial_t u_1 e^{-cu_1}}{U} \ + \ \frac{\partial_t u_2 e^{-cu_2}}{U} \ + \ \frac{\partial_x p_1 e^{-cp_1}}{P} \ + \ \frac{\partial_x p_2 e^{-cp_2}}{P} \ + \tag{27}$$

$$\frac{\nu}{U^2}\left(\partial_{xx}u_1 e^{-cu_1}U \ + \ \partial_{xx}u_2 e^{-cu_2}U \ - \ c(\partial_x u_1 - \partial_x u_2)^2 e^{-c(u_1+u_2)}\right) +$$

$$\frac{\nu}{U^2}\left(\partial_{yy}u_1 e^{-cu_1}U \ + \ \partial_{yy}u_2 e^{-cu_2}U \ - \ c(\partial_y u_1 - \partial_y u_2)^2 e^{-c(u_1+u_2)}\right).$$

Moreover, in (27) we sum the terms which contain the function u_1 and its derivatives:

$$\frac{\partial_t u_1 e^{-cu_1}}{U} \ + \ \frac{\partial_x p_1 e^{-cp_1}}{P} \ + \ \nu\left(\frac{1}{U^2}\left(\partial_{xx}u_1 \ + \ \partial_{yy}u_1\right)e^{-cu_1}U\right), \tag{28}$$

the same for the function u_2

$$\frac{\partial_t u_2 e^{-cu_2}}{U} \ + \ \frac{\partial_x p_2 e^{-cp_2}}{P} \ + \ \nu\left(\frac{1}{U^2}\left(\partial_{xx}u_2 \ + \ \partial_{yy}u_2\right)e^{-cu_2}U\right). \tag{29}$$

Setting: $E_{ij} = e^{-c(u_i+p_j)}, i, j = 1, 2$, we have $e^{-cu_1}P = E_{11} + E_{12}$, $e^{-cp_1}U = E_{11} + E_{21}$ $(E_{12} \neq E_{21})$ and then (28) =

$$\frac{1}{U\ P}\left(\partial_t u_1 e^{-cu_1}P + \partial_x p_1 e^{-cp_1}U + \nu P\left(\left(\partial_{xx}u_1 + \partial_{yy}u_1\right)e^{-cu_1}\right)\right), \tag{30}$$

from which

$$(30) = \frac{1}{U\ P}\left(\ \left(\partial_t u_1 + p_{1x} + \nu\left(\partial_{xx}u_1 + \partial_{yy}u_1\right)\right)E_{11}\right) \ +$$

$$\frac{1}{U\ P}\left(\ \left(\partial_t u_1 + \nu\left(\partial_{xx}u_1 + \partial_{yy}u_1\right)\right)E_{12} + \partial_x p_1 E_{21}\right); \tag{31}$$

similarly (29) =

$$\frac{1}{U\ P}\left(\partial_t u_2 e^{-cu_2}P + \partial_x p_2 e^{-cp_2}U + \nu P\left(\left(\partial_{xx}u_2 + \partial_{yy}u_2\right)e^{-cu_2}\right)\right) \tag{32}$$

from which

$$(32) = \frac{1}{U\ P}\left(\ \left(\partial_t u_2 + \partial_x p_2 + \nu\left(\partial_{xx}u_2 + \partial_{yy}u_2\right)\right)E_{22}\right) \ +$$

$$\frac{1}{U\ P}\left(\ \left(\partial_t u_2 + \nu\left(\partial_{xx}u_2 + \partial_{yy}u_2\right)\right)E_{21} + \partial_x p_2 E_{12}\right) \ . \tag{33}$$

First, in (31) and (33) the coefficients of E_{11} and E_{22} are zero as u_1 and u_2 are solutions of (19). Now we sum the other terms :

$$(31) + (33) = \frac{1}{U\ P}\Big(\ (\partial_t u_1 + \nu\,(\partial_{xx}u_1 + \partial_{yy}u_1))\,E_{12}\ +\ \partial_x p_1 E_{21}\Big) +$$

$$\frac{1}{U\ P}\Big(\ (\partial_t u_2 + \nu\,(\partial_{xx}u_2 + \partial_{yy}u_2))\,E_{21}\ +\ \partial_x p_2 E_{12}\Big). \tag{34}$$

As $u_i, i = 1, 2$ are solutions of (19), we get

$$(34) = \frac{1}{U\ P}\Big(- \partial_x p_1 E_{12} + \partial_x p_1 E_{21} - \partial_x p_2 E_{21} + \partial_x p_2 E_{12}\Big) =$$

$$\frac{1}{U\ P}\Big(\partial_x(p_1 - p_2)(E_{21} - E_{12})\Big) = 0,$$

since by the supposition the functions p_i depends only on time, $\partial_x(p_1 - p_2) = \partial_x p_1 - \partial_x p_2 = 0$. In (27) it remains:

$$\frac{\nu}{U^2}\Big(- c(\partial_x u_1 - \partial_x u_2)^2 e^{-c(u_1+u_2)} -\ c(\partial_y u_1 - \partial_y u_2)^2 e^{-c(u_1+u_2)}\Big) =$$

$$\frac{-c\nu}{U^2}\Big((\partial_x u_1 - \partial_x u_2)^2 + (\partial_y u_1 - \partial_y u_2)^2\Big)\,e^{-c(u_1+u_2)} = 0,$$

since by the supposition the functions u_i depends only on time, $\partial_x(u_1 - u_2) = \partial_x u_1 - \partial_x u_2 = 0$. So, we have shown that $(u_1 \oplus u_2, p_1 \oplus p_2)$ is a solution of the equation (19).

Changing u_i with v_i, $i = 1, 2$ in the previous proof we can prove that $(v_1 \oplus v_2, p_1 \oplus p_2)$ is solution of the equation (20), and then $\mathbf{s}_1 \oplus \mathbf{s}_2$ is a solution of (19).

Now we shall prove that $\mathbf{u}_1 \oplus \mathbf{u}_2$ is a solution of the equation (21). In fact, from (23) and (24), we get

$$\mathrm{div}(\mathbf{u}_1 \oplus \mathbf{u_2}) = \partial_x(u_1 \oplus u_2) + \partial_y(v_1 \oplus v_2) =$$

$$\frac{\partial_x u_1 e^{-cu_1}}{U} + \frac{\partial_x u_2 e^{-cu_2}}{U} + \frac{\partial_y v_1 e^{-cu_1}}{U} + \frac{\partial_y v_2 e^{-cu_2}}{U} = 0.$$

As regards the product $\odot$, we note that $\partial_x(a + u) = \partial_x u$, and so on, so also $a \odot u$ is solution of (19) and (21). $\square$

7 Conclusion

In this paper it was proven the pseudo-linear superposition principle for the Euler, Navier-Stokes and Stokes equations. In order to achieve this principle for the first two equations we used the monotonicity of the velocity.

The obtained results will serve in the future for different applications, e.g., [23], and as a base for the construction of the general weak solutions as in [8, 11, 14, 17], which are in a wider class than previously considered class of monotone functions, and allow movement also in harder structures than fluid, see [9].

8 Acknowledgment

The first author was partially supported by the project MPNS-174009 and "Mathematical models of intelligent systems and their applications" by Provincial Secretariat for Science and Technological Development of Vojvodina. The second author thanks Basileus project (Erasmus Mundus) for academic years 2009-2010, 2010-2011 and the University of Novi Sad for the invitation and the kind hospitality.

References

[1] L. Caffarelli, R. Kohn, L. Nirenberg: Partial regularity of suitable weak solutions of Navier-Stokes equations, Comm. Pure and Appl. Math. 35, 771-831, 1982.

[2] A.J. Chorin, J. Marsen, A Mathamatical Intoduction to Fluid Mechanics, Springer-Verlag, 1993.

[3] P. Constantin: Open problems and Research Directions in Mathematical Study of Fluid Dynamics , in Mathematicacs Unlimited - 2001 and Beyond (Eds. B. Engquist, W. Schmid), Springer, 2001, 353-360.

[4] R. Deutray, J-L. Lions: Mathematical Analysis and Numerical Methods for Science and Tecnology, Vol.4,6, Springer-Verlag, 2000.

[5] S. Friedlander, D. Serre: Handbook of Mathematical Fluid Dynamics, Vol.1, Elsevier, 2002.

[6] S. Friedlander, D. Serre: Handbook of Mathematical Fluid Dynamics, vol.2, Elsevier, 2003.

[7] M. Grabisch, J. L. Marichal, R. Mesiar, E. Pap: Monograph: Aggregation Functions, Acta Polytechnica Hungarica 6,1 (2009), 79-94.

[8] V. Kolokoltsov, P. Maslov: Idempotent Analysis and Its Applications, Kluwer Academic Publishers, Dordrecht, 1997.

[9] M. Kuffova, P. Nečas: Fracture Mechanics Prevention: Comprehensive Approach-based Modelling?, Acta Polytechnica Hungarica 7, 5, 2010, 5-17.

[10] O. A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flows, (2nd ed.), Gordon and Breach, 1969.

[11] V.P. Maslov, S. N. Samborskij (Eds.): Idempotent Analysis, Advances in Soviet Mathematics 13, Amer. Math. Soc., Providence, Rhode Island, 1992.

[12] E. Pap, Integral generated by decomposable measure, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 20,1, 135-144, 1990.

[13] E. Pap: g-calculus, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser.Mat., 23,1, 1993, 145-150.

[14] E. Pap: Null-Additive Set Functions, Dordrecht-Boston-London, Kluwer Academic Publishers, 1995.

[15] E. Pap: Decomposable measures and nonlinear equations, Fuzzy Sets and Systems, 92, 1997, 205-222.

[16] E. Pap: Solving nonlinear equations by non-additive measures, Nonlinear Analysis, 30, 1997, 31-40.

[17] E. Pap: Applications of decomposable measures, in Handbook Mathematics of Fuzzy Sets-Logic, Topology and Measure Theory (Eds. U. Höhle, S.R. Rodabaugh), Kluwer Academic Publishers, 1999, 675-700.

[18] E. Pap: Pseudo-Additive Measures and Their Applications, in Handbook of Measure Theory,Chap.35, (Ed. E. Pap), Elsevier, North-Holland, 2002, 1403-1468.

[19] E. Pap: Applications of the generated pseudo-analysis on nonlinear partial differential equations, Proceedings of the Conference on Idempotent Mathematics and Mathematical Physics (Eds. G. L. Litvinov, V. P. Maslov), Contemporary Mathematics 377, American Mathematical Society, 2005, 239-259.

[20] E. Pap: Applications of pseudo-analysis on models with nonlinear partial differential equations, (Eds. E. Pap, J. Fodor) Proceedings of the 5th International Symposium on Intelligent Systems and Informatics, Subotica, 2007, 7-12.

[21] E. Pap, N. Ralević: Pseudo-Laplace Transform, Nonlinear Analysis 33, 1998, 533-550.

[22] E. Pap, D. Vivona: Non-commutative and non-associative pseudo-analysis and its applications on nonlinear partial differential equations, J. Math. Anal. Appl. 246, 2000, 390-408.

[23] R. M. Patel, G. Deheri, H. C. Patel: Effect of Surface Roughness on the Behavior of a Magnetic Fluid-based Squeeze Film between Circular Plates with Porous Matrix of Variable Thickness, Acta Polytechnica Hungarica 8, 5, 2011, 145-164.

[24] A. Shnirelman: On the nonuniqueness of weak solutions of the Euler equation, Comm.Pure and Appl.Math. 50, 1997, 1260-1286.

[25] M. Sammartino, R.E. Cafflisch: Zero Viscosity Limit for Analitic Solutions of the Navier-Stokes Equation on a Half-Space. I.Existence for Euler and Prandtl Equations, Commun. Math. Phys. 192, 1998, 433-461.

[26] M. Sammartino, R. E. Cafflisch: Zero Viscosity Limit for Analitic Solutions of the Navier-Stokes Equation on a Half-Space. II.Construction of the Navier-Stokes Solution, Commun. Math. Phys. 192, 1998, 465-491.