Bessel–sampling restoration of stochastic signals

Tibor K. Pogány

Óbuda University, John von Neumann Faculty of Informatics, Institute of Applied Mathematics, Bécsi út 96/b, Budapest Hungary
e-mail: tkpogany@gmail.com *and*

University of Rijeka, Faculty of Maritime Studies, Studentska 2, Rijeka, Croatia
e-mail: poganj@pfri.hr

Abstract: *The main aim of this article is to establish sampling series restoration formulae in for a class of stochastic L^2-processes which correlation function possesses integral representation close to a Hankel-type transform which kernel is either Bessel function of the first and second kind J_ν, Y_ν respectively. The results obtained belong to the class of irregular sampling formulae and present a stochastic setting counterpart of certain older results by Zayed* [25] *and of recent results by Knockaert* [13] *for J–Bessel sampling and of currently established Y–Bessel sampling results by Jankov Maširević* et al. [7]. *The approach is twofold, we consider sampling series expansion approximation in the mean–square (or L^2) sense and also in the almost sure (or with the probability* 1*) sense. The main derivation tools are the Piranashvili's extension of the famous Karhunen–Cramér theorem on the integral representation of the correlation functions and the same fashion integral expression for the initial stochastic process itself, a set of integral representation formulae for the Bessel functions of the first and second kind J_ν, Y_ν and various properties of Bessel and modified Bessel functions which lead to the so–called Bessel–sampling when the sampling nodes of the initial signal function coincide with a set of zeros of different cylinder functions.*

Keywords: *Almost sure convergence, Bessel functions of the first and second kind J_ν, Y_ν, correlation function, harmonizable stochastic processes, Karhunen–Cramér–Piranashvili theorem, Karhunen processes, Kramer's sampling theorem, mean–square convergence, sampling series expansions, sampling series truncation error upper bound, spectral representation of correlation function, spectral representation of stochastic process.*

MSC(2010): 42C15, 60G12, 94A20.

1 Introduction

The development and application of sampling theory in technics, engineering but in parallel in pure mathematical investigations was rapid and continuous since the middle of the 20th century [4, 6, 9, 15, 16, 17]. It is one of the most important mathematical techniques used in communication engineering and information theory, and it is also widely represented in many branches of physics and engineering, such as signal analysis, image processing, optics, physical chemistry, medicine etc. [9, 25]. In general sampling theory can be used where functions need to be restored from their discretized–measured–digitalized sampled values, usually from the values of the functions and/or their derivatives at certain points. Here we are focused to a kind of Bessel–sampling restoration of finite second order moment stochastic processes (signals), which correlation function possesses Hankel–transform type integral representation. In the Bessel sampling procedure the sampling nodes we take to be the positive zeros $j_{\nu,k}, y_{\nu,k}$ of the Bessel functions J_ν, Y_ν respectively, depending on the appearing Bessel function in the kernel of the integral expression representing the correlation function of the considered initial stochastic signal.

The results obtained form a stochastic setting counterpart to recent results by Zayed [25, 26, 24, 27], Knockaert [13] and Jankov *et al.* [7].

This paper is organized as follows: in the sequel we give a short account in correlation and spectral theory of stochastic signals, which consists from a necessary introductionary knowledge about different kind stochastic processes appearing in the engineering literature together with associated mathematical models. Secondly, J–Bessel and Y–Bessel deterministic sampling theorems are recalled together with their ancestor result, that is the Kramer's sampling theorem. In Section 2 we prove our main results on the Bessel sampling restoration of stochastic signals in both mean–square and almost sure manner. Finally, we proceed with restoration error analysis, presenting associated results in finding the uniform upper bounds for newly derived truncated sampling series, which is a counterpart of deterministic results which has been considered in a number of publications in the mathematical literature, consult for instance [7, 8, 9] and the appropriate references therein. In Conclusion section we give an overview of the exposed matter together with new research directions and improvement possibilities. The exhaustive references list finishes the exposition.

1.1 Brief invitation to correlation theory of stochastic processes

Let $(\Omega, \mathfrak{A}, \mathsf{P})$ a standard fixed probability space and consider the random variables $\xi\colon T \times \Omega \mapsto \mathbb{C}, T \subseteq \mathbb{R}$; the double–indexed infinite family of random variables $\{\xi(t) \equiv \xi(t,\omega)\colon t \in \mathbb{T}, \omega \in \Omega\}$ is a *stochastic process*. Here T is the *index set* of the process ξ. Denote $L^2(\Omega, \mathfrak{A}, \mathsf{P})$ [abbreviated to $L^2(\Omega)$ in the sequel] be the space of all finite second order complex–valued random variables defined on $(\Omega, \mathfrak{A}, \mathsf{P})$, equipped with the norm $\sqrt{\mathsf{E}|\cdot|^2} := \|\cdot\|_2$, where E means the expectation operator. Notice that $L^2(\Omega)$ is a Hilbert–space with the inner (or scalar) product $\mathsf{E}\xi\,\overline{\eta}$ endowed. However, it is enough to restrict ourselves to the linear mean–square–closure $\mathscr{H}_t(\xi) := \overline{\{L^2\{\xi(s)\colon s \le t\}}$ spanned by all finite linear combinations and/or their *in medio* limits generated by the family $\{\xi(s)\colon s \le t\}, t \in \mathbb{R}$, which is the linear subspace of the Hilbert space $L^2(\Omega)$. It is well-known that $\mathscr{H}_\infty(\xi) \equiv \mathscr{H}(\xi)$ possesses also a Hilbert–space structure, keeping the norm and inner product of

$L^2(\Omega)$. We recall that when $\bigcap_{t\in\mathbb{R}} H_t(\xi) = \emptyset$, then ξ is *purely indeterministic*[1], say; moreover in the case $\bigcap_{t\in\mathbb{R}} H_t(\xi) = \mathscr{H}(\xi)$, process ξ is *purely deterministic*[2].

The function $m_\xi(t) = \mathsf{E}\xi(t)$ is the *expectation function*. Let us assume throughout that the considered stochastic processes are centered, that is $m_\xi(t) \equiv 0, t \in \mathbb{R}$[3]. The function $B_\xi(t,s) = \mathsf{E}\xi(t)\overline{\xi(s)}$ is the *correlation function* (or autocorrelation function) of the centered process ξ at two "times values" $t,s \in T$. By the Cauchy–Buniakovskiy-Schwarz inequality it is straightforward that

$$|B_\xi(t,s)|^2 \le B_\xi(t,t)B_\xi(s,s), \qquad t,s,\in T\,, \tag{1}$$

being ξ with the finite second order moment rv, with any fixed $t \in T$. The function $\mathsf{D}\xi(t) := B_\xi(t,t)$ is the *variance* of the process ξ[4].

Very wide class of stochastic processes has been introduced by Piranashvili [18]. He has studied the sampling reconstruction of a class of nonstationary processes, which correlation function (and *a fortiori* the initial process itself) possess spectral representations in a form of a double integral. In fact Piranashvili extended the Karhunen-Cramér theorem for a wider class stochastic processes; see the works of Karhunen [11] and Cramér [3], also see [29, p. 156].

Theorem A. [Karhunen–Cramér–Piranashvili Theorem] *Let a centered stochastic $L^2(\Omega)$–process ξ has correlation function (associated to some domain $\Lambda \subseteq \mathbb{R}$ with some sigma–algebra $\sigma(\Lambda)$) in the form:*

$$B(t,s) = \int_\Lambda\int_\Lambda f(t,\lambda)\overline{f(s,\mu)}\,F_\xi(\mathrm{d}\lambda,\mathrm{d}\mu), \tag{2}$$

with analytical exponentially bounded kernel function $f(t,\lambda)$, while F_ξ is a positive definite measure on $\mathbb{R}^2$ provided the total variation $\|F_\xi\|(\Lambda,\Lambda)$ of the spectral distribution function F_ξ *satisfies*

$$\|F_\xi\|(\Lambda,\Lambda) = \int_\Lambda\int_\Lambda \big|F_\xi(\mathrm{d}\lambda,\mathrm{d}\mu)\big| < \infty.$$

Then, the process $\xi(t)$ has the spectral representation as a Lebesgue integral

$$\xi(t) = \int_\Lambda f(t,\lambda)Z_\xi(\mathrm{d}\lambda); \tag{3}$$

in (2) *and* (3)

$$F_\xi(S_1,S_2) = \mathsf{E}Z_\xi(S_1)\overline{Z_\xi(S_2)}, \qquad S_1,S_2 \subseteq \sigma(\Lambda),$$

and vice versa.

1 In the Western terminology; however, according to the Eastern, Soviet/Russian probabilistic terminology this kind process is *regular*.

2 Singular. It is worth to mention that we deal here with a class of weakly stationary singular processes.

3 Otherwise we pick up the so–called centered process $\xi_0(t) = \xi(t) - m_\xi(t)$, which expectation function is obviously zero.

4 By (1) we see, that $\mathsf{D}\xi(t) \le \sup_{u\in\mathbb{R}} B_\xi^2(u,u) := \mathfrak{B}_\xi^2 < \infty$.

Note that in the case of finite Λ we will talk on processes *bandlimited to* Λ.

If F_ξ of (2) concentrates of diagonal $\lambda = \mu$, that is $F_\xi(\lambda,\mu) = \delta_{\lambda,\mu}F_\xi(\lambda)$, then the resulting correlation is called of *Karhunen class*, and B_ξ becomes

$$B_\xi(t,s) = \int_\Lambda f(t,\lambda)\,\overline{f(s,\lambda)}\,F_\xi(d\lambda).$$

The spectral representation of the resulting *Karhunen process* $\xi(t)$ remains of the form given by (3).

Also, putting $f(t,\lambda) = e^{it\lambda}$ in (2) one gets the *Loève-representation*:

$$B(t,s) = \int_\Lambda\int_\Lambda e^{i(t\lambda - s\mu)}F_\xi(\mathrm{d}\lambda,\mathrm{d}\mu).$$

Then, the Karhunen process with the Fourier kernel $f(t,\lambda) = e^{it\lambda}$ we recognize as the *weakly stationary stochastic process* having covariance

$$B(\tau) = \int_\Lambda e^{i\tau\lambda}F_\xi(\mathrm{d}\lambda), \qquad \tau = t-s.$$

The stochastic processes having correlation function expressible in the form (2) we call *harmonizable*. Further reading about different kind harmonizabilities present the works [10, 20, 21] and the appropriate references therein. Finally, when $\Lambda = (-w,w)$ for some finite $w>0$ in this considerations, we get the *band–limited* variants of the above introduced processes. So, for $\xi(t)$, being weak sense stationary band–limited to $w>0$, there holds the celebrated Whittaker–Kotel'nikov–Shannon sampling theorem:

$$\xi(t) = \sum_{k\in\mathbb{Z}} \xi\left(\frac{\pi}{w}k\right)\frac{\sin(wt-k\pi)}{wt-k\pi}, \tag{4}$$

uniformly convergent on all compact t–subsets of $\mathbb{R}$, in both mean–square and almost sure sense; the latter has been proved by Belyaev [2].

1.2 Kramer's theorem and Bessel sampling

Here we recall three theorems which will help us to derive our first set of Bessel sampling restoration results for a class of harmonizable stochastic processes having Karhunen representable correlation functions.

Theorem B. [Kramer's Theorem], [12, 13] *Let $K(x,t)$ be in $L^2[a,b], -\infty<a<b<\infty$ a function of x for each real number t and let $E=\{t_k\}_{k\in\mathbb{Z}}$ be a countable set of real numbers such that $\{K(x,t_k)\}_{k\in\mathbb{Z}}$ is a complete orthogonal family of functions in $L^2[a,b]$. If*

$$f(t) = \int_a^b g(x)K(x,t)\,\mathrm{d}x,$$

for some $g \in L^2[a,b]$*, then* f *admits the sampling expansion*

$$f(t) = \sum_{k\in\mathbb{Z}} f(t_k)\, S^\star(t,t_k)\,,$$

where

$$S^\star(t,t_k) = \frac{\displaystyle\int_a^b K(x,t)\overline{K(x,t_k)}\,\mathrm{d}x}{\displaystyle\int_a^b |K(x,t_k)|^2\,\mathrm{d}x}\,.$$

Remark 1. *Annaby reported, that points* $\{t_k\}_{k\in\mathbb{Z}}$*, which are for practical reasons preferred to be real, can also be complex,* [1, p. 25].

Obviously, the function f*, having above integral representation property bandlimited to the region* $\Lambda = [a,b]$*.*

Now we give the two Bessel–sampling theorems, the J–Bessel derived e.g. by Zayed [25, p. 132], but the J–Bessel sampling method was known already by Whittakers [22, 23], Helms and Thomas [5] and Yao [30].

Theorem C. *It there is some* $G \in L^2(0,b)$ *with a finite Hankel–transform*

$$f(\lambda) = \frac{2^\nu\,\Gamma(\nu+1)}{b^{\nu+\frac{1}{2}}\,\lambda^\nu} \int_0^b G(x)\sqrt{x}J_\nu(x\lambda)\,\mathrm{d}x\,, \tag{5}$$

then there holds

$$f(t) = \frac{2J_\nu(bt)}{b^\nu z,t^\nu} \sum_{k\geq 1} \frac{j_{\nu,k}^{\nu+1}\, f(a^{-1}\,j_{\nu,k})}{(b^2t^2 - j_{\nu,k}^2)\,J_\nu'(j_{\nu,k})}\,,$$

where the series converges uniformly on any compact subset of the complex t–plane. Here λ_k *denote the kth zero of* $J_\nu(b\sqrt{\lambda})$*.*

In turn the Y–Bessel sampling theorem has been recently derived by Jankov Maširević *et al.* in [7, p. 81, Theorem 4].

Theorem D. *Let for some* $G \in L^2(0,a), a > 0$*, function* f *possesses a finite Hankel–transform*

$$f(t) = \int_0^a G(x)\sqrt{x}Y_\nu(tx)\,\mathrm{d}x\,, \tag{6}$$

then, for all $t \in \mathbb{R}$*,* $\nu \in [0,1)$*, the function* f *admits the sampling expansion*

$$f(t) = 2Y_\nu(at) \sum_{k\geq 1} f(b^{-1}y_{\nu,k})\, \frac{y_{\nu,k}}{(y_{\nu,k}^2 - a^2t^2)\,Y_{\nu+1}(y_{\nu,k})}\,,$$

where $y_{\nu,k}$*,* $k \in \mathbb{N}$ *are the positive real zeros of the Bessel function* $Y_\nu(t)$*. Here the convergence os uniform in all compact t–subsets of* $\mathbb{C}$*.*

2 Main results

Although formula (4), Theorem C and Theorem D yield an explicit restoration of bandlimited weakly stationary stochastic process $\xi(t)$ by the WKS sampling theorem, and Hankel-transformable $f(t)$ by either J–Bessel or Y–Bessel sampling procedures respectively, these results are usually considered to be of theoretical interest only, because the restoration procedures require computations of infinite sums. In practice, we truncate the sampling expansion series. The sampling size N is determined by the relative error accepted in the reconstruction. Thus the error analysis plays a crucial role in setting up the interpolation formula, and it is of considerable interest to find sampling series truncation error upper bounds (the exact value of the truncation error is in general a "mission impossible") which vanishes with the growing sampling size.

Here and in what follows we will concentrate to a class of harmonizable stochastic processes having spectral representation of the form (3) with the kernel function

$$f(t,\lambda) \in L^2(0,b), \qquad b>0,$$

with respect to the time–parameter t.

According to these requirements, we introduce the notations for both kind Bessel sampling procedures:

$$\begin{aligned} \mathscr{S}_N^J(\mathfrak{G};t) &:= \frac{2J_\nu(bt)}{b^\nu t^\nu} \sum_{k=1}^N \frac{j_{\nu,k}^{\nu+1}\, \mathfrak{G}(b^{-1} j_{\nu,k})}{(b^2t^2 - j_{\nu,k}^2)\, J_\nu'(j_{\nu,k})} \\ \mathscr{S}_N^Y(\mathfrak{G};t) &:= 2Y_\nu(bt) \sum_{k=1}^N \frac{y_{\nu,k}\, \mathfrak{G}(b^{-1} y_{\nu,k})}{(y_{\nu,k}^2 - b^2t^2)\, Y_{\nu+1}(y_{\nu,k})}, \end{aligned}$$

for the truncated (partial) Bessel sampling series expansions either of $L^2(0,b)$–bandlimited signal f, or for the stochastic process ξ, that is $\mathfrak{G} \in \{f,\xi\}$. Next, we introduce the sampling series restoration truncation error, read as follows

$$\begin{aligned} \mathscr{T}_N^J(\mathfrak{G};t) &:= \mathfrak{G}(t) - \mathscr{S}_N^J(\mathfrak{G};t) = \frac{2J_\nu(bt)}{b^\nu t^\nu} \sum_{k\ge N+1} \frac{j_{\nu,k}^{\nu+1}\, \xi(b^{-1} j_{\nu,k})}{(b^2t^2 - j_{\nu,k}^2)\, J_\nu'(j_{\nu,k})} \qquad (7) \\ \mathscr{T}_N^Y(\mathfrak{G};t) &:= \mathfrak{G}(t) - \mathscr{S}_N^Y(\mathfrak{G};t) = 2Y_\nu(bt) \sum_{k\ge N+1} \frac{y_{\nu,k}\, \xi(b^{-1} y_{\nu,k})}{(y_{\nu,k}^2 - b^2t^2)\, Y_{\nu+1}(y_{\nu,k})}, \end{aligned}$$

Our main goal in that stage of investigation is to establish as sharp as possible mean square truncation error upper bounds in both Bessel–sampling procedures, that is for

$$\Delta_N^{\mathscr{B}}(\xi;t) = \mathsf{E}\big|\xi(t) - \mathscr{S}_N^{\mathscr{B}}(\xi;t)\big|^2 = \mathsf{E}\big|\mathscr{T}_N^{\mathscr{B}}(\xi;t)\big|^2, \qquad \mathscr{B} \in \{J,Y\}.$$

Firstly, we establish the spectral representation formula for $\mathscr{S}_N^J(\xi;t)$.

Theorem 1. *Let $\xi(t), t \in T \subseteq \mathbb{R}$ a harmonizable stochastic process of Piranashvili class, that is*

$$\xi(t) = \int_\Lambda f(t,\lambda)\, Z_\xi(\mathrm{d}\lambda)$$

with the kernel function $f(t,\lambda) \in L^2(0,b)$ with respect to t and any fixed $\lambda \in \Lambda$. Then we have

$$\mathscr{S}_N^{\mathscr{B}}(\xi;t) = \int_\Lambda \mathscr{S}_N^{\mathscr{B}}(f;t)\, Z_\xi(\mathrm{d}\lambda), \qquad \mathscr{B} \in \{J,Y\}.$$

Moreover, there holds true

$$\mathscr{T}_N^{\mathscr{B}}(\xi;t) = \int_\Lambda \mathscr{T}_N^{\mathscr{B}}(f;t)\, Z_\xi(\mathrm{d}\lambda), \qquad \mathscr{B} \in \{J,Y\};$$

both formulae are valid in the mean square sense.

Proof. The sampling series expansion of the kernel function $f(t,\lambda)$ which appears in the representation (3), when truncated to the terms indexed by N becomes $\mathscr{S}_N^J(f;t)$. Now, by (7) we get

$$\begin{aligned}
\mathscr{S}_N^J(\xi;t) &= \frac{2J_\nu(bt)}{b^\nu t^\nu} \sum_{k=1}^N \frac{j_{\nu,k}^{\nu+1}\, \xi(b^{-1} j_{\nu,k})}{(b^2t^2 - j_{\nu,k}^2)\, J_\nu'(j_{\nu,k})} \\
&= \frac{2J_\nu(bt)}{b^\nu t^\nu} \sum_{k=1}^N \frac{j_{\nu,k}^{\nu+1}}{(b^2t^2 - j_{\nu,k}^2)\, J_\nu'(j_{\nu,k})} \int_\Lambda f(a^{-1} j_{\nu,k},\lambda)\, Z_\xi(\lambda) \\
&= \int_\Lambda \left\{ \frac{2J_\nu(bt)}{b^\nu t^\nu} \sum_{k=1}^N \frac{j_{\nu,k}^{\nu+1}}{(b^2t^2 - j_{\nu,k}^2)\, J_\nu'(j_{\nu,k})} f(b^{-1} j_{\nu,k},\lambda) \right\} Z_\xi(\lambda);
\end{aligned}$$

here all equalities are in the mean square sense used. This is exactly the statement for $\mathscr{B} = J$. The case of Y–Bessel sampling we handle in the same way.

The second assertion we prove directly:

$$\begin{aligned}
\mathscr{T}_N^J(\xi;t) &= \xi(t) - \mathscr{S}_N^J(\xi;t) = \int_\Lambda f(t,\lambda)\, Z_\xi(\mathrm{d}\lambda) - \int_\Lambda \mathscr{S}_N^J(f;t)\, Z_\xi(\mathrm{d}\lambda) \\
&= \int_\Lambda \big\{ f(t,\lambda) - \mathscr{S}_N^J(f;t) \big\}\, Z_\xi(\mathrm{d}\lambda) \\
&= \int_\Lambda \mathscr{T}_N^J(f;t)\, Z_\xi(\mathrm{d}\lambda).
\end{aligned}$$

The equalities are also in the mean square sense used. The rest is clear. □

Theorem 2. *Let the situation be the same as in* Theorem 1. *Then we have*

$$\Delta_N^{\mathscr{B}}(\xi;t) = \int_\Lambda \int_\Lambda \mathscr{T}_N^{\mathscr{B}}(f;t)\, \overline{\mathscr{T}_N^{\mathscr{B}}(f;t)}\, F_\xi(\mathrm{d}\lambda,\mathrm{d}\mu), \qquad \mathscr{B} \in \{J,Y\}, \tag{8}$$

in the mean square sense.

The proof is a straightforward consequence of the Karhunen–Cramér–Piranshvili Theorem A and the spectral representation formulae of stochastic process ξ, therefore we omit it.

Remark 2. Obviously Theorem 2 is devoted to the case of Piranashvili processes. For the Karhunen processes this result reduces to

$$\Delta_N^{\mathscr{B}}(\xi;t) = \int_\Lambda \left|\mathscr{T}_N^{\mathscr{B}}(f;t)\right|^2 F_\xi(\mathrm{d}\lambda), \qquad \mathscr{B} \in \{J, Y\}. \tag{9}$$

Denote $L^2(\Lambda;F_\xi)$ the class of square–integrable on the support domain Λ, complex functions with respect to the measure $F_\xi(\mathrm{d}\lambda)$, i.e.

$$L^2(\Lambda;F_\xi) := \left\{\varphi\colon \int_\Lambda |\varphi|^2 F_\xi(\mathrm{d}\lambda) < \infty\right\}.$$

This class form also a Hilbert–space and the correspondence $\xi(t) \longleftrightarrow f(t,\lambda)$ defines an isomorphism between $\mathscr{H}(\xi)$ and $L^2(\Lambda;F_\xi)$. Therefore by the existing isometry, we conclude (9).

Next, a special case of the Karhunen process is the weakly stationary stochastic process[5]. Choosing $\Lambda = (-w,w)$, we arrive at

$$\Delta_N^{\mathscr{B}}(\xi;t) = \int_{-w}^{w} \left|\mathscr{T}_N^{\mathscr{B}}(\mathrm{e}^{\mathrm{i}t\lambda})\right|^2 F_\xi(\mathrm{d}\lambda), \qquad \mathscr{B} \in \{J, Y\}.$$

Now, we are ready to state our Bessel–sampling series finding for stochastic processes.

Theorem 3. *Let* $\{\xi(t)\colon t \in \mathbb{T} \subseteq \mathbb{R}\}$ *a Piranashvili process* (3) *with a kernel function* $f(t,\lambda) \in L^2(0,b)$ *which possesses a Hankel–transform representation either of the form* (5) (*J–Bessel sampling*) *or* (6) (*Y–Bessel sampling*). *Then we have*

$$\xi(t) = \mathscr{S}^J(\xi;t) = \frac{2J_\nu(bt)}{b^\nu t^\nu} \sum_{k\ge 1} \frac{j_{\nu,k}^{\nu+1}\,\xi(b^{-1}j_{\nu,k})}{(b^2t^2 - j_{\nu,k}^2)\,J_\nu'(j_{\nu,k})}$$

$$\xi(t) = \mathscr{S}^Y(\xi;t) = 2Y_\nu(bt) \sum_{k\ge 1} \frac{y_{\nu,k}\,\xi(b^{-1}y_{\nu,k})}{(y_{\nu,k}^2 - b^2t^2)\,Y_{\nu+1}(y_{\nu,k})},$$

respectively. Both equalities hold in the mean square sense.

Proof. Having in mind that (8)

$$\Delta_N^{\mathscr{B}}(\xi;t) = \mathsf{E}|\mathscr{T}_N^{\mathscr{B}}(\xi : t)|^2 = \int_\Lambda\int_\Lambda \mathscr{T}_N^{\mathscr{B}}(f;t)\,\overline{\mathscr{T}_N^{\mathscr{B}}(f;t)}\,F_\xi(\mathrm{d}\lambda,\mathrm{d}\mu),$$

and $\mathscr{T}_N^{\mathscr{B}}(f;t)$ vanishes pointwise and uniformly [25, p. 132] (J–Bessel sampling), that is [7, p. 83, Theorem 4] (Y–Bessel sampling) with the growing N, we deduce

$$\lim_{N\to\infty} \Delta_N^{\mathscr{B}}(\xi;t) = 0, \qquad \mathscr{B} \in \{J, Y\},$$

which completes the proof. □

[5] Also known as *stationary in the Khintchin sense*.

3 Truncation error bounds for Y–Bessel sampling of Karhunen processes

In this section we would derive uniform upper bound for the truncation error for the Y–Bessel sampling expansion of the Karhunen process $\xi(t), t \in T \subseteq \mathbb{R}$:

$$\mathscr{S}^Y(\xi;t) = 2Y_\nu(t)\sum_{k=1}^{N} \frac{y_{\nu,k}\,\xi(y_{\nu,k})}{(y_{\nu,k}^2 - t^2)\,Y_{\nu+1}(y_{\nu,k})}\,,$$

setting for the sake of simplicity $b = 1$, $\nu \in [0,1)$ and the function f has a band–region contained in $(0,1)$. Having in mind (9) exposed in Remark 2, we specify:

$$\Delta_N^Y(\xi;t) = \int_\Lambda \big|\mathscr{T}_N^Y(f;t)\big|^2 F_\xi(\mathrm{d}\lambda)\,. \tag{10}$$

The truncation error upper bound has been already calculated in under the polynomial decay condition (see e.g. [14])

$$|f(t)| \le \frac{A}{|t|^{r+1}}, \qquad A > 0,\, r > 0\,, t \neq 0. \tag{11}$$

The corresponding truncation error upper bound [7, p. 83, Theorem 5] for all

$$\nu \in [0,1), \quad t \in (\nu, y_{\nu,2}), \quad \min\{A,r\} > 0, \quad N \ge 2$$

reads as follows

$$\mathscr{T}_N^Y(f;t) < \frac{2AH(t)M_N(\nu)}{\pi^2 L_{N+1}(\nu)} := U_N^Y(t)\,,$$

where

$$H(t) = 1 + \frac{2t}{\pi(t^2-\nu^2)}$$

$$M_N(\nu) = \exp\left\{\left(N + \frac{1-\pi+2(\nu-y_{\nu,2})}{2\pi}\right)^{-1}\right\} - 1$$

$$L_{N+1}(\nu) = \frac{2}{\sqrt{\pi}}\, y_{\nu,N+1}^r \left\{\frac{y_{\nu,N+1}^2 - (2\nu+3)(2\nu+7)}{(4y_{\nu,N+1}-\nu-1)^{\frac{3}{2}} + \mu^*}\right\}^{\frac{1}{2}}$$

and $\mu^* = (2\nu+3)(2\nu+5)$.

Moreover, for any fixed $t \in (\nu, y_{\nu,2})$ and growing N the following the asymptotic behavior results holds [7, p. 83, Eq. (15)]

$$\mathscr{T}_N^Y(f;t) = \mathscr{O}\left(N^{-r-\frac{5}{4}}\right).$$

Now, we are ready to formulate our next main result.

Theorem 4. *Let $\xi(t), t \in \mathbb{R}$ a Karhunen process with the kernel function f satisfying polynomial decay condition* (11)*. Then for all $\nu \in [0,1)$, for all $t \in (\nu, y_{\nu,2})$,* $\min\{A, r\} > 0$ *and all $N \geq 2$, we have*

$$\Delta_N^Y(\xi;t) \leq \frac{A^2 \, \|F_\xi\|(\Lambda) \, (\pi\nu t+2)^2 \, [(4y_{\nu,N+1}-\nu-1)^{\frac{3}{2}}+(2\nu+3)(2\nu+5)]}{\pi^5 \, \nu^2 \, t^2 \, y_{\nu,N+1}^{2r} [y_{\nu,N+1}^2-(2n+3)(2n+7)]} \times \left(\exp\left\{ \left(N + \frac{1-\pi+2(\nu-y_{\nu,2})}{2\pi} \right)^{-1} \right\} - 1 \right)^2 ,$$

where $\|F_\xi\|(\Lambda)$ stands for the total variation of the spectral distribution function F_ξ. Moreover, the decay magnitude of the truncation error is

$$\Delta_N^Y(\xi;t) = \mathscr{O}\left(N^{-2r-\frac{5}{2}}\right) . \tag{12}$$

Proof. Because of the spectral representation formula (10) and the functional truncation error upper bound (11) by Jankov Maširević *et al.* we have

$$\Delta_N^Y(\xi;t) = \int_\Lambda \left|\mathscr{T}_N^Y(f;t)\right|^2 F_\xi(\mathrm{d}\lambda) \leq \int_\Lambda \left|U_N^Y(f;t)\right|^2 F_\xi(\mathrm{d}\lambda) .$$

Now routine calculations lead to the statement. Relation (12) is the immediate consequence of this upper bound result. □

Next, we consider the almost sure convergence in the Y–Bessel sampling series restoration of the Karhunen process.

Theorem 5. *Let $\xi(t)$ a Karhunen process with the kernel function f satisfying polynomial decay condition* (11)*. Then for all $\nu \in [0,1)$, for all $t \in (\nu, y_{\nu,2})$,* $\min\{A, r\} > 0$ *and all $N \geq 2$, we have*

$$\mathsf{P}\left\{ \lim_{N\to\infty} \mathscr{S}_N^Y(\xi;t) = \xi(t) \right\} = 1 .$$

Proof. Firstly, for some positive ε we evaluate the probability

$$\mathsf{P}_N = \mathsf{P}\left\{ \left|\xi(t) - \mathscr{S}_N^Y(\xi;t)\right| \geq \varepsilon \right\} .$$

Applying the Čebyšev inequality, then Theorem 3 we conclude th estimate

$$\mathsf{P}_N \leq \varepsilon^{-2}\, \mathsf{E}\left|\mathscr{T}_N^Y(\xi;t)\right|^2 = \mathscr{O}\left(N^{-2r-\frac{5}{2}}\right) .$$

For certain enough large absolute constant C the following bound follows in terms of the Riemann Zeta function:

$$\sum_{N\geq 2} \mathsf{P}_N \leq C \sum_{N\geq 2} N^{-2r-\frac{5}{2}} = C\left[\zeta\left(2r+\tfrac{5}{2}\right)-1\right] ,$$

and the series converges, being $r > 0$. Now, by the Borel–Cantelli lemma it follows the a.s. convergence, which completes the proof. □

4 Final remarks

In the footnote 2 it was mentioned that we work throughout with singular, or purely deterministic processes. Indeed, having in mind that the initial input process of Piranashvili type $\xi(t)$ possesses spectral representation (3) in which the kernel function is a Hankel transform of some convenient $G \in L^2(0,b)$, we deduce

$$\begin{aligned}\xi(t) &= \int_\Lambda f(t,\lambda) Z_\xi(\mathrm{d}\lambda) \\ &= \frac{2^\nu \Gamma(\nu+1)}{b^{\nu+\frac{1}{2}}} \int_\Lambda \left\{ \frac{1}{\lambda^\nu} \int_0^b G(x)\sqrt{x} J_\nu(x\lambda)\,\mathrm{d}x \right\} F_\xi(\mathrm{d}\lambda) \\ &= \frac{2^\nu \Gamma(\nu+1)}{b^{\nu+\frac{1}{2}}} \int_0^b G(x)\sqrt{x} \left\{ \int_\Lambda \frac{J_\nu(x\lambda)}{\lambda^\nu} F_\xi(\mathrm{d}\lambda) \right\} \mathrm{d}x \\ &= \frac{2^\nu \Gamma(\nu+1)}{b^{\nu+\frac{1}{2}}} \int_0^b G(x)\sqrt{x}\,\Psi_\nu(x)\,\mathrm{d}x.\end{aligned}$$

Obviously $\xi(t)$ is bandlimited to $(0,b)$. (We mention that the sample function $\xi(t) \equiv \xi(t,\omega_0)$ and $f(t,\lambda)$ possess the same exponential types [2, Theorem 4], [18, Theorem 3], and also by the Paley–Wiener theorem we conclude that $\xi(t)$ is bandlimited to the support set $(0,b)$).

The Kolmogorov–Krein analytical singularity criterion states that the singular processes possesses divergent integral:

$$\int_{\mathbb{R}} \frac{\log \frac{\mathrm{d}}{\mathrm{d}\lambda} F_\xi(\mathrm{d}\lambda)}{1+\lambda^2}\,\mathrm{d}\lambda = -\infty,$$

which is obviously true, being the Radon–Nikodým derivative (or in other words spectral density) in the integrand equal to zero on a set of positive Lebesgue measure.

References

[1] M. H. ANNABY, Sampling Integrodifferential Transforms Arising from Second Order Differential Operators, *Math. Nachr.* 216 (2000), 25–43.

[2] YU. K. BELYAEV, Analytical random processes, *Teor. Verojat. Primenen.* **IV** (1959), No. 4, 437–444. (in Russian)

[3] H. CRAMÉR, A contribution to the theory of stochastic processes, *Proc. Second Berkely Symp. Math. Stat. Prob.*, 329–339, 1951.

[4] B. DRAŠČIĆ, Sampling Reconstruction of Stochastic Signals - The Roots in the Fifties, *Austrian J. Statist.* **36** (2007), No.1, 65–72.

[5] H. D. HELMS, J. B. THOMAS, On truncation error of sampling theorem expansion, *Proc. I.R.E.* **50** (1962), 179–184.

[6] J. R. HIGGINS, *Sampling Theory in Fourier and Signal Analysis*, Clarendon Press, Oxford, 1996.

[7] D. JANKOV MAŠIREVIĆ, T. K. POGÁNY, Á. BARICZ, A. GALÁNTAI, Sampling Besssel functions and Bessel sampling, Proceedings of IEEE 8th International Symposium on Applied Computational Intelligence and Informatics, May 23-25, Timişoara (2013) 79-84.

[8] A. J. JERRI, I. A. JOSLIN, Truncation error for the generalized Bessel type sampling series, *J. Franklin Inst.*, **314** (1982), No. 5, 323–328.

[9] A. J. JERRI, The Shannon sampling theoremits various extensions and applications: A tutorial review, *Proc. IEEE* **11** (1977), 1565-1596.

[10] Y. KAKIHARA, *Multidimensional Second Order Stochastic Processes*, World Scientific, Singapore, 1997.

[11] K. KARHUNEN, Über lineare Methoden in der Wahrscheinlichkeitsrechnung, *Ann. Acad. Sci. Fennicae, Ser A, I* **37** (1947), 3–79.

[12] H. P. KRAMER, A generalized sampling theorem, *J. Math. Phys.* **38** (1959), 68–72.

[13] L. KNOCKAERT, A class of scaled Bessel sampling theorems, *IEEE Trans. Signal Process.* **59** (2011), No. 11, 5082–5086.

[14] X. M. LI, Uniform Bounds for Sampling Expansions, *Journal of approximation theory* **93** (1998), 100–113.

[15] A. Y. OLENKO, T. K. POGÁNY, Time shifted aliasing error upper bounds for truncated sampling cardinal series, *J. Math. Anal. Appl.* **324** (2006), 262–280.

[16] A. Y. OLENKO, T. K. POGÁNY, Universal truncation error upper bounds in irregular sampling restoration, *Appl. Anal.* **90(3–4)** (2011), 595–608.

[17] A. Y. OLENKO, T. K. POGÁNY, Average sampling reconstruction od harmonizable processes, *Commun. Stat. Theor. Methods* **40** (2011) , No. 19-20, 3587-3598.

[18] Z.A. PIRANASHVILI, On the problem of interpolation of random processes, *Teor. Ver. Primenen.* **XII** (1967), No. 4, 708–717. (in Russian)

[19] T. POGÁNY, Almost sure sampling reconstruction of band–limited stochastic signals, Chapter 9. in *Sampling Theory in Fourier and Signal Analysis. Advanced Topics*, Higgins, J.R. & Stens, R.L. (eds.), Oxford University Press, Oxford, 209-232, 284–286, 1999.

[20] M. B. PRIESTLEY, *Non–Linear and Non–Stationary Time Series*, Academic Press, London, New York, 1988.

[21] M. M. RAO, Harmonizable processes: structure theory. *Einseign. Math. (2)* **28**(1982), No. 3–4, 295–351.

[22] E. WHITTAKER, On the functions which are represented by the expansion of the interpolation theory, *Proc. Roy. Soc. Edinburgh Sect. A* **35** (1915), 181-194.

[23] J. WHITTAKER, *Interpolatory Function Theory*, Cambridge University Press, Cambridge, 1935.

[24] A. I. ZAYED, On Kramer's sampling theorem associated with general Sturm-Liouville problems and Lagrange interpolation, SIAM *J. Appl. Math.* **51** (1991), 575-604.

[25] A. I. ZAYED, *Advances in Shannon's Sampling Theory*, CRC Press, New York, 1993.

[26] A. I. ZAYED, A proof of new summation formulae by using sampling theorems, *Proc. Amer. Math. Soc.* **117(3)** (1993), 699–710.

[27] A. ZAYED, G. HINSEN, P. BUTZER, On Lagrange interpolation and Kramer-type sampling theorems associated with Sturm-Liouville problems, SIAM *J. Appl. Math.* **50** (1990), 893-909.

[28] A. M. YAGLOM, *Correlations Theory of Stationary and Related Random Function I. Basic Results*, Spriner–Verlag, 1987.

[29] A. M. YAGLOM, *Correlations Theory of Stationary and Related Random Function II. Supplementary Notes and References*, Spriner–Verlag, 1987.

[30] K. YAO, Application of reproducing kernel Hilbert spaces–band-limited signal models, *Inf. Contr.* **11** (1967), 427–444.