A Note on the Solutions to Lattice-valued Relational Equations and Inequations

Vanja Stepanović[1] and Andreja Tepavčević[2,3]

[1]University of Belgrade, Faculty of Agriculture, Nemanjina 6, 11080 Beograd, Serbia

[2]Serbian Academy of Sciences and Arts, Mathematical Institute, Kneza Mihaila 36, 11000 Beograd, Serbia

[3]Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia

e-mails: vanja@agrif.bg.ac.rs, andreja@dmi.uns.ac.rs

Abstract: The existence of the greatest solutions to some lattice-valued relational equations, inequations and their systems is proved in the case of a complete residuated codomain lattice. The aim is to structure the existing knowledge and to provide missing results related to equations and inequations, where there is only one unknown on every side of the (in)equation. We also prove the existence of the least solution, for some equations and inequations. By a counter-example, we prove that the minimal solution of some typical equations need not exist. We also prove some more general results in the case where we have more than one variable on one, or both sides, of the (in)equations. In some cases, when the greatest solution need not exist, we prove the existence of a maximal solution. A procedure yielding one such solution, is also offered in this paper.

Keywords: complete residuated lattice; lattice-valued relational equations and inequations; extremal solutions

1 Introduction

In the 1960s of the 20th Century, Zadeh defined fuzzy sets as mappings from a domain set to the interval [0,1], thus, generalizing the notion of set [18]. Fuzzy relations in this context were defined as mappings from the square of the domain set to the interval [0,1]. Together with the so-called sup-min composition of fuzzy relations, this was also a generalization of the usual, in this context called "crisp" relations and their composition. Using this generalized fuzzy relational composition and a generalized inclusion, fuzzy relational equations and inequations were defined and studied [12].

Fuzzy relational equations play a crucial role in many areas of fuzzy logic, particularly in fuzzy control systems. Since the mid-1980s, interest in fuzzy control has grown significantly, beginning with foundational theoretical research and followed by practical applications in a wide range of industrial and commercial domains.

A typical fuzzy control system comprises three main stages: input, processing, and output. The processing stage often relies on fuzzy IF-THEN rules, which form the basis of fuzzy rule-based systems. These rules commonly involve multiple antecedents, which are combined using fuzzy operators or represented as fuzzy relations. One widely used method, the Mamdani approach, constructs fuzzy controllers by deriving a fuzzy relation from real-world control behavior. This relation maps input values to output values using appropriate compositional inference rules.

In practical implementations, control objectives often require that specific output values correspond to given input conditions. Therefore, a key problem is to determine a fuzzy relation that accurately models this input-output mapping. Fuzzy relational equations provide a formal framework to address this problem, making them essential in the design and analysis of fuzzy control systems [9].

In 1967, Goguen further generalized the notions of fuzzy set and fuzzy relation, introducing lattice-valued set as a mapping from a domain set to a lattice, and lattice-valued relation as a mapping from the square of the domain set to the same lattice [4]. Some researchers started to investigate fuzzy set equations and inequations in this more general setting [3], and more recently [5] [13]. The sets of fuzzy sets and, particularly, fuzzy relations, are seen as lattices related to the component ordering, and related to that ordering, we may speak of the greatest, the least, or maximal and minimal solutions to fuzzy relational equations and inequations.

The structure of the set of solutions and, particularly, the existence of extremal solutions to some fuzzy relational equations and inequations were studied in cases where the codomain lattice is of special kinds. The most general case studied is the case of a complete codomain lattice [6] [16], or a complete lattice fulfilling some additional properties [2] [15]. Some investigations were conducted in the case of a complete residuated codomain lattice [5] [10], which is still a very general one, since Boolean algebra, complete Brouwerian lattice, Godel and Lukasiewicz algebras are special cases of the residuated lattice. Some recent investigations pointed out that the computation of minimal solutions is still a complex problem [7].

The infinite distributivity of the multiplication related to supremum in a residuated codomain lattice implies the existence of the greatest solution to some fuzzy relational equations and inequations and their systems. Here we are trying to systematize many such cases of fuzzy relational equations and inequations. Up to now, the researchers were dealing mostly with classical equations and inequations R∘X (= , ≤ , ≥) S, with an unknown X (see e.g., [8]).

We prove that the greatest solution exists in the case of equations and inequations in which there is exactly one variable (i.e., unknown) on every side of the (in)equation, whether the variable on the left side is equal to or different from the variable on the right side of the (in)equation.

If there is exactly one variable on one side and no variable on the other, the greatest solution also exists for all the inequations and all solvable equations.

By a counterexample, we show that in the case when there is more than one variable on one side of the (in)equation, the greatest solution need not exist.

The results we obtained also apply to the case of a complete Brouwerian codomain lattice, that is, a complete codomain lattice in which the infimum is infinitely distributive over the supremum.

As for the least solution, we prove that there exist the least solutions to some equations and inequations, namely those that can be expressed in terms of operators in the set of fuzzy relations.

2 Preliminaries

A lattice L is said to be complete if any subset of L has an infimum and a supremum. It is straightforward that the infimum and the supremum are unique.

A structure $(L,\vee,\wedge,\rightarrow,\otimes,0,1)$ is a residuated lattice ([1]), if the following holds:

- $(L,\vee,\wedge)$ is a complete lattice with greatest element 1 and the least element 0
- $(L,\otimes,1)$ is a commutative monoid with 1
- The operations $\rightarrow$ and $\otimes$ are connected in the following way:

  $x \otimes y \leqslant z \Leftrightarrow x \leqslant y \rightarrow z$

In a complete residuated lattice L, the "multiplication" $\otimes$ is infinitely distributive over supremum, i.e., the following holds:

$$a \otimes \bigvee_{i\in I} x_i = \bigvee_{i\in I} a \otimes x_i$$

for every $a \in L$ and every class $\{x_i |\ i \in I\} \subseteq L$.

The operation $\otimes$ in residuated lattices, fulfils the following:

$$a \otimes 0 = 0 \otimes a = 0 \quad (1)$$

Also, $\otimes$ is order-preserving related to its operands, i.e.:

$a \leqslant a_1$ and $b \leqslant b_1 \Longrightarrow a \otimes b \leqslant a_1 \otimes b_1$

If A is a set, an L-valued relation on A is a mapping from $A \times A$ to a fixed lattice L, which we call the codomain lattice. The set of all L-valued relations of A is denoted by $\mathscr{F}(A \times A)$. It is a lattice related to the following component-wise ordering:

$$P \leqslant Q \Leftrightarrow (\forall (x,y) \in A \times A) P(x,y) \leqslant Q(x,y)$$

If L is complete, $\mathscr{F}(A \times A)$ is also a complete lattice.

The composition of fuzzy relations $P, Q \in \mathscr{F}(A \times A)$, in the case of a residuated codomain lattice, is defined as follows:

$$P \circ Q(x,y) = \bigvee_{z \in A} P(x,z) \otimes Q(z,y)$$

Since $\otimes$ is order-preserving relative to its operands, we get that $\circ$ is an order-preserving operation (relative to both operands) in $\mathscr{F}(A \times A)$. It is known that $\circ$ is also an associative operation in $\mathscr{F}(A \times A)$, i.e.:

$$(P \circ Q) \circ R = P \circ (Q \circ R)$$

Hence, due to simplicity, sometimes we will skip the parentheses and write this as: $P \circ Q \circ R$

It is also known that $\circ$ is infinitely distributive in relation to supremum, from both sides, i.e., if $\{Q_i | i \in I\}$ is a family of L-fuzzy relations and P is another fuzzy relation, the following holds:

$$P \circ \bigvee_{i \in I} Q_i = \bigvee_{i \in I} (P \circ Q_i) \qquad (2)$$

$$\left(\bigvee_{i \in I} Q_i \right) \circ P = \bigvee_{i \in I} (Q_i \circ P) \qquad (3)$$

Solutions to equations may often be seen as fixed points of some operators. Therefore, we shall use the Tarski fixed-point theorem.

Theorem 1. [17] (Tarski) Let $(L, \leqslant)$ be a complete lattice, f an increasing function from L to L, and P the set of all its fixed points, $(P, \leqslant)$ is a complete, nonempty lattice, and the following holds:

$$\vee P = \vee \{x \in L | \, f(x) \geqslant x\} \in P$$

$$\wedge P = \wedge \{x \in L | \, f(x) \leqslant x\} \in P \qquad (4)$$

Tarski also proved a more general assertion about the fixed points of increasing operators on a complete lattice.

Theorem 2. [17] (Tarski) Let $(L, \leqslant)$ be a complete lattice, and $\{f_i | i \in I\}$ the set of increasing functions from L to L, such that any two of them commute, i.e. $f_i \circ f_j = f_j \circ f_i$ for all $i, j \in I$. If P is the set of all common fixed points for all f_i, $(P, \leqslant)$ is the complete lattice, and the following holds:

$\bigvee P = \bigvee\{x \in L |\ (\forall i \in I) f_i(x) \geqslant x\} \in P$

$\bigwedge P = \bigwedge\{x \in L |\ (\forall i \in I) f_i(x) \leqslant x\} \in P$

If L is a complete Brouwerian lattice, i.e., if it is a complete and infimum is infinitely distributive relative to supremum, then L is also residuated, relative to $\otimes$, which is identical to $\wedge$, and $\to$, defined by:

$$x \to y = \bigvee\{z \in L | x \wedge z \leqslant y\}$$

If infimum does not commute with arbitrary suprema - as in case of any Browerian lattice, but only with the supremum of chains, we get a weaker property, so called meet-continuity. In cases where the greatest solution of some equations (or inequations) does not exist, we will use the following lemmas to prove the existence of maximal solutions:

Lemma 3. [14] If A is a set, L a complete, meet-continuous lattice, and $\{(P_i, Q_i) | i \in I\}$ a chain of pairs of fuzzy relations from $\mathscr{F}(A \times A)$, we have:

$$\bigvee_{i \in I} P_i \circ \bigvee_{i \in I} Q_i = \bigvee_{i \in I} (P_i \circ Q_i).$$

Lemma 4. (Zorn, Kuratowski) If a partially ordered set P has the property that every chain in P has an upper bound in P, then P contains at least one maximal element.

3 Results

If not otherwise stated, L is a complete residuated lattice.

By P, Q, R, S we denote L-valued relations from $\mathscr{F}(A \times A)$, and by X and Y unknown L-valued relations. Due to the associativity of the composition, there exist only five types of terms over $\mathscr{F}(A \times A)$, that use only composition, in which we have at most one variable occurring not more than once $P \circ X, X \circ P, P \circ X \circ Q, P$ and X.

According to the infinite distributivity of relational composition related to supremum, we also have that:

$$P \circ \bigvee_{i \in I} A_i \circ Q = \bigvee_{i \in I} (P \circ A_i) \circ Q = \bigvee_{i \in I} P \circ A_i \circ Q \quad (5)$$

Now, we form fuzzy relational equations and inequations by taking expressions of the forms $P \circ X, X \circ P, P \circ X \circ Q, P$ and X on both sides of the (in)equations. Constants on the different sides of the equations are different as a rule, and unknowns on the different sides of the equations may be equal or different. All the equations and inequations using only composition and having at most one unknown on each of their sides are equivalent to one such equation or inequation.

Here, in equations and inequations with two different variables X and Y, the greatest solution is naturally the greatest with respect to the componentwise order in $\mathscr{F}(A \times A) \times \mathscr{F}(A \times A)$.

Let $P, Q, R, S \in \mathscr{F}(A \times A)$. There exists the greatest solution to the equations and inequations of the following forms (this will be demonstrated in the sequel):

$P \circ X = Q \circ X \quad P \circ X \leqslant Q \circ X$

$$P \circ X = Q \circ Y \quad P \circ X \leqslant Q \circ Y \tag{6}$$

$P \circ X = X \circ Q \quad P \circ X \leqslant X \circ Q \quad P \circ X \geqslant X \circ Q$

$$P \circ X = Y \circ Q \quad P \circ X \leqslant Y \circ Q \quad P \circ X \geqslant Y \circ Q \tag{7}$$

$P \circ X = Q \circ X \circ R \quad P \circ X \leqslant Q \circ X \circ R \quad P \circ X \geqslant Q \circ X \circ R$

$$P \circ X = Q \circ Y \circ R \quad P \circ X \leqslant Q \circ Y \circ R \tag{8}$$

$$P \circ X \geqslant Q \circ Y \circ R \tag{9}$$

$$P \circ X \leqslant Q \tag{10}$$

$$P \circ X \geqslant X \quad P \circ X = X \tag{11}$$

$$P \circ X \geqslant Y \quad P \circ X = Y \tag{12}$$

$X \circ P = X \circ Q \quad X \circ P \leqslant X \circ Q$

$$X \circ P = Y \circ Q \quad X \circ P \leqslant Y \circ Q \tag{13}$$

$X \circ P = Q \circ X \circ R \quad X \circ P \leqslant Q \circ X \circ R \quad X \circ P \geqslant Q \circ X \circ R$

$$X \circ P = Q \circ Y \circ R \quad X \circ P \leqslant Q \circ Y \circ R \tag{14}$$

$$X \circ P \geqslant Q \circ Y \circ R \tag{15}$$

$$X \circ P \leqslant Q \tag{16}$$

$$X \circ P \geqslant X \quad X \circ P = X \tag{17}$$

$$X \circ P \geqslant Y \quad X \circ P = Y \tag{18}$$

$P \circ X \circ Q = R \circ X \circ S \quad P \circ X \circ Q \leqslant R \circ X \circ S$

$$P \circ X \circ Q = R \circ Y \circ S \quad P \circ X \circ Q \leqslant R \circ Y \circ S \tag{19}$$

$$P \circ X \circ Q \leqslant R \tag{20}$$

$$P \circ X \circ Q = X \quad P \circ X \circ Q \geqslant X \tag{21}$$

$$P \circ X \circ Q = Y \quad P \circ X \circ Q \geqslant Y \tag{22}$$

It is straightforward to check that the set of solutions to the equations and inequations above is closed under the supremum. For example, if $\{X_i | i \in I\}$ is the family of solutions to the equation $P \circ X = X \circ Q$, we have that $P \circ X_i = X_i \circ Q$

for all $i \in I$; therefore:

$$\bigvee_{i\in I}(P \circ X_i) = \bigvee_{i\in I}(X_i \circ Q)$$

Now, applying (2) and (3), we get:

$$P \circ \left(\bigvee_{i\in I} X_i\right) = \left(\bigvee_{i\in I} X_i\right) \circ Q$$

which means that $\bigvee_{i\in I} X_i$ is a solution to the same equation (the greatest solution).

Using (2), (3), and (5), we prove the same for all the equations and inequations above. For those of them that contain both X and Y we prove that if $\{(X_i, Y_i) | i \in I\}$ is a set of solutions, then the following is a solution:

$$\left(\bigvee_{i\in I} X_i, \bigvee_{i\in I} Y_i\right)$$

The existence of the greatest solution to the above equations and inequations follows from the fact that $X = 0$ or - in case both X and Y occur in the equation or inequation – $(X, Y) = (0,0)$ is a solution of the above equations (the set of solutions is non-empty), and the fact that the set of solutions is closed under supremum.

We have omitted the inequations $P \circ X \leqslant X,\ X \circ P \leqslant X$ and $P \circ X \circ Q \leqslant X$, since it is obvious that the greatest fuzzy relation is the greatest solution to those inequations.

However, a modified problem with the additional condition that the solution is less than or equal to some given fuzzy relation is also of practical interest. This problem for all these inequations is solved by taking the supremum of all the solutions contained in T and such a solution is the greatest one contained in T.

Up to now, we were proving the existence of the greatest solutions, but in some cases, the solutions themselves can be easily determined.

For the inequation (10), we can "calculate" the greatest solution, following a similar argument as in [12]. Namely, since, by definition of $\circ$, we have:

$$P \circ X(z, y) = \bigvee_{x\in A}(P(z, x) \otimes X(x, y))$$

$P \circ X \leqslant Q$ is equivalent to:

$$(\forall x, y, z \in A) P(z, x) \otimes X(x, y) \leqslant Q(z, y)$$

which is further equivalent to:

$(\forall x, y, z \in A) X(x, y) \leqslant P(z, x) \rightarrow Q(z, y)$

and, finally:

$$X(x, y) \leqslant \bigwedge_{z \in A} (P(z, x) \rightarrow Q(z, y)) \tag{23}$$

So, the greatest solution to the inequation (10) is equal to $G_{P,Q}$, defined by:

$$G_{P,Q} = \bigwedge_{z \in A} (P(z, x) \rightarrow Q(z, y)) \tag{24}$$

Likewise, we conclude that the greatest solution to the inequation (16) is given by:

$$G^{P,Q} = \bigwedge_{z \in A} (P(y, z) \rightarrow Q(x, z)) \tag{25}$$

As for the inequation (20), we get the following:

$P \circ X \circ Q \leqslant R \Leftrightarrow X \circ Q \leqslant G_{P,R} \Leftrightarrow X \leqslant G^{Q,G_{P,R}}$

Thus, the greatest solution to the inequation (20) is $G^{Q,G_{P,R}}$, which is defined by the following:

$$G^{Q,G_{P,R}}(x, y) = \bigwedge_{z \in A} \left(Q(y, z) \rightarrow G_{P,R}(x, z) \right) =$$

$$\bigwedge_{z \in A} \left(Q(y, z) \rightarrow \bigwedge_{t \in A} \left(P(t, x) \rightarrow R(t, z) \right) \right) \tag{26}$$

We could also take any system of equations of the above-listed types, and using the same arguments, we conclude that there exists the greatest solution to such a system. The unknowns from the different equations of the system may differ, or be the same, or some may differ and some may be the same.

By the same arguments, it follows that there exists the greatest solution to the following equations and inequations provided that they are solvable, i.e., that there exists at least one solution:

$$P \circ X = Q \qquad P \circ X \geqslant Q \tag{27}$$

$$X \circ P = Q \qquad X \circ P \geqslant Q \tag{28}$$

$$P \circ X \circ Q = R \qquad P \circ X \circ Q \geqslant R \tag{29}$$

Actually, for an equation of the type (28), Sanchez has proved (see [12]) that it is solvable if and only if $G^{P,Q} \circ P = Q$; if that is the case, $G^{P,Q}$ is its greatest solution. He also proved that an equation of the type (27) is solvable if and only if $P \circ G_{P,Q} = Q$; if that is the case, the greatest solution to (27) is $G_{P,Q}$. The least solution to the equations of the types (27) and (28) need not exist (see [11]).

The following example shows that there might not be minimal solutions to these equations:

Example 1. We take a set $L = \{1,0,a,b\} \cup \{b_i | i \in N\}$ with the order:

$(\forall i \in N) 0 < b < b_i < a < 1$

$(\forall i,j \in N) b_i < b_j \Leftrightarrow i > j$

L is a chain. We introduce a commutative operation, denoted by $\otimes$, such that:

$(\forall x \in L)(1 \otimes x = x$ and $0 \otimes x = x \otimes 0 = 0)$

$a \otimes a = b$

$(\forall i \in N) a \otimes b_i = b$

$a \otimes b = 0$

$(\forall i,j \in N) b_i \otimes b_j = 0$

$(\forall i \in N) b \otimes b_i = 0$

$b \otimes b = 0$

Note that $\otimes$ is associative, since $(a \otimes b) \otimes c = 0 = a \otimes (b \otimes c)$ if a, b, c are different from 1. If there is an element in $\{a, b, c\}$ equal to 1, it is obvious that in this case also $(a \otimes b) \otimes c = a \otimes (b \otimes c)$.

We define the operation $\rightarrow$ such that:

$(\forall x \in L) 1 \rightarrow x = x$

$x \rightarrow y = 1$, whenever $x \leqslant y$

$(\forall i \in N) b_i \rightarrow b = a \rightarrow b_i = a$

$(\forall i,j \in N) b_i \rightarrow b_j = a \Leftrightarrow i < j$

$a \rightarrow b = a$

$a \rightarrow 0 = b$

$1 \rightarrow 0 = 0$

$(\forall i \in N) b_i \rightarrow 0 = b_1$

We can check that we've got a residuated lattice. But there is no minimal solution to the equation $a \otimes x = b$. Taking a one-element domain set $C = \{c\}$, and L as a codomain lattice, we can take $A(c,c) = a, B(c,c) = b$. There is no minimal solution to the equation $A \circ X = B$ (and $X \circ A = B$ as well), although there are solution.

Since we have that $G^{Q,G_{P,R}}$ is the greatest solution to the inequation (20), if the corresponding equation of the type (29) is solvable, all its solutions are also solutions to the inequation (20), thus the greatest solution to the equation is less than or equal to $G^{Q,G_{P,R}}$; but from the monotony of the $\circ$, $G^{Q,G_{P,R}}$ is also a solution to the equation. Thus, we have that an equation of the type (29) is solvable if and only if:

$$P \circ G^{Q,G_{P,R}} \circ Q = R \tag{30}$$

If that is the case, the greatest solution to (29) equals $G^{Q,G_{P,R}}$.

As for an inequation belonging to some of the types (27)-(29), by monotony of $\circ$ we conclude that it is solvable, if and only if the greatest fuzzy relation in $\mathscr{F}(A \times A)$ is a solution to it.

We can further calculate the greatest solution to the inequations of the type (6):

Theorem 5. The greatest solution to the inequation (6) is the pair of fuzzy relations (X, Y), such that for all $x, y \in A$, $Y(x, y) = 1$ and:

$$X(x,y) = \bigwedge_{z \in A} \left(P(z,x) \rightarrow \bigvee_{t \in A} Q(z,t) \right) \tag{31}$$

Proof: First, (31) is a solution, since by (23) we have that $P \circ X \leqslant Q \circ 1$ is equivalent to:

$$X(x,y) \leqslant \bigwedge_{z \in A} \big(P(z,x) \rightarrow (Q \circ 1)(z,y) \big)$$

and thus equivalent to:

$$X(x,y) \leqslant \bigwedge_{z \in A} \left(P(z,x) \rightarrow \bigvee_{t \in A} Q(z,t) \otimes 1(t,y) \right)$$

and finally equivalent to:

$$X(x,y) \leqslant \bigwedge_{z \in A} \left(P(z,x) \rightarrow \bigvee_{t \in A} Q(z,t) \right)$$

If (31) is not the greatest solution, since we have proved that the greatest solution exists, there is a fuzzy relation X_1 greater than:

$$\bigwedge_{z \in A} \left(P(z,x) \rightarrow \bigvee_{t \in A} Q(z,t) \right)$$

such that $P \circ X_1 \leqslant Q \circ 1$, which contradicts the equivalence we have proved.

Theorem 6. The greatest solution to the inequation (7) is the pair of fuzzy relations (X, Y), such that for all $x, y \in A$, $Y(x, y) = 1$ and

$$X(x,y) = \bigwedge_{z \in A} \left(P(z,x) \rightarrow \bigvee_{t \in A} Q(t,y) \right).$$

Theorem 7. The greatest solution to the inequation (8) is the pair of fuzzy relations (X, Y), such that for all $x, y \in A$, $Y(x, y) = 1$ and

$$X(x,y) = \bigwedge_{z \in A} \left(P(z,x) \to \bigvee_{t,s \in A} (Q(z,t) \otimes R(s,y)) \right)$$

Theorem 8. The greatest solution to the inequation (9) is the pair of fuzzy relations (X, Y), such that for all $x, y \in A$, $X(x, y) = 1$ and:

$$Y(x,y) = \bigwedge_{z \in A} \left(R(y,z) \to \bigwedge_{t \in A} \left(Q(t,x) \to \bigvee_{s \in A} P\,(t,s) \right) \right)$$

Theorem 9. The greatest solution to the inequation (12) is the pair of fuzzy relations (X, Y), such that for all $x, y \in A$, $X(x, y) = 1$ and:

$$Y(x,y) = \bigvee_{z \in A} P\,(x,z)$$

Theorem 10. The greatest solution to the inequation (13) is the pair of fuzzy relations (X, Y), such that for all $x, y \in A$, $Y(x, y) = 1$ and:

$$X(x,y) = \bigwedge_{z \in A} \left(P(y,z) \to \bigvee_{t \in A} Q(t,z) \right)$$

Theorem 11. The greatest solution to the inequation (14) is the pair of fuzzy relations (X, Y), such that for all $x, y \in A$, $Y(x, y) = 1$ and:

$$X(x,y) = \bigwedge_{z \in A} \left(P(y,z) \to \bigvee_{s,t \in A} (Q(x,s) \otimes R(t,z)) \right)$$

Theorem 12. The greatest solution to the inequation (15) is the pair of fuzzy relations (X, Y), such that for all $x, y \in A$, $X(x, y) = 1$ and:

$$Y(x,y) = \bigwedge_{z \in A} \left(Q(z,x) \to \bigwedge_{t \in A} \left(R(y,t) \to \bigvee_{s \in A} P(s,t) \right) \right)$$

Theorem 13. The greatest solution to the inequation (18) is the pair of fuzzy relations (X, Y), such that for all $x, y \in A$, $X(x, y) = 1$ and:

$$Y(x,y) = \bigvee_{z \in A} P\,(z,y)$$

Theorem 14. The greatest solution to the inequation (19) is the pair of fuzzy relations (X, Y), such that for all $x, y \in A$, $Y(x, y) = 1$ and:

$$X(x,y) = \bigwedge_{z \in A} \left(Q(x,z) \to \bigwedge_{t \in A} \left(P(t,y) \to \bigvee_{p,s \in A} \big((R(t,p) \otimes S(s,z) \big) \right) \right)$$

Theorem 15. The greatest solution to the inequation (22) is the pair of fuzzy relations (X, Y), such that for all $x, y \in A$, $X(x, y) = 1$ and:

$$Y(x, y) = \bigvee_{s,t \in A} (P(x, s) \otimes Q(t, y))$$

Up to now, we have systematically presented all the types of equations and inequations containing at most one variable on one side in connection with the existence of the greatest solutions.

The greatest solution does not exist in general for equations having more than one variable on one side. The following example illustrates this.

Example 2. Let L be a partitive set of $\{1,2\}$ (which is a complete and infinitely distributive lattice, and thus residuated), and $A = \{a, b\}$. Let P and L be an L-valued relation satisfying:

$$P(a, a) = P(b, a) = P(a, b) = P(b, b) = \{1\}$$

We consider the equation and the inequation:

$$X \circ X = P \quad X \circ X \leqslant P \tag{32}$$

Let Q and R be L-valued relations such that:

$$Q(a, a) = Q(a, b) = Q(b, b) = \{1\}; \quad Q(b, a) = \{1,2\}$$

$$R(a, a) = R(b, a) = R(b, b) = \{1\} \quad R(a, b) = \{1,2\}$$

Now we have:

$$(Q \circ Q)(a, a) = (R \circ R)(a, a) = \{1\} = (Q \circ Q)(a, b) = (R \circ R)(a, b);$$

$$(Q \circ Q)(b, a) = (R \circ R)(b, a) = \{1\} = (Q \circ Q)(b, b) = (R \circ R)(b, b).$$

Thus, Q and R are solutions to the (in)equation (32). If there exists the greatest solution to the (in)equation, it would have to contain $Q \vee R$. But:

$$(Q \vee R)(a, a) = (Q \vee R)(b, b) = \{1\}$$

$$(Q \vee R)(a, b) = (Q \vee R)(b, a) = \{1,2\}$$

Clearly, $(Q \vee R) \circ (Q \vee R) > P$, and if S were the greatest solution, we would have:

$$S \circ S \geqslant (Q \vee R) \circ (Q \vee R) > P$$

since $S \geqslant Q \vee R$ and since the composition is order-preserving operation in $\mathscr{F}(A \times A)$, which contradicts the assumption that S is the greatest solution to $X \circ X = P$, or the greatest solution to $X \circ X \leqslant P$.

Thus, the greatest solution to $X \circ X = P$ does not exist, nor does the greatest solution to the corresponding inequation $X \circ X \leqslant P$.

The same holds for the equation $X \circ Y \leqslant P$, the greatest solution does not exist, because (R, R) and (Q, Q) are solutions, and the greatest solution (S_1, S_2) would contain $(R \vee Q, R \vee Q)$, but we get a contradiction:

$$S_1 \circ S_2 \geqslant (R \vee Q) \circ (R \vee Q) > P$$

However, there exists a maximal solution to the equations and inequations $X \circ X \leqslant A$, $X \circ X = A$, $X \circ Y \leqslant A$ and $X \circ Y = A$. We get it applying the following analogue of Lemma 3:

Lema 16. If A is a set, L a complete, residuated lattice, and $\{(P_i, Q_i) | i \in I\}$ a chain of pairs of fuzzy relations from $\mathscr{F}(A \times A) \times \mathscr{F}(A \times A)$, we have:

$$\bigvee_{i \in I} P_i \circ \bigvee_{i \in I} Q_i = \bigvee_{i \in I} (P_i \circ Q_i).$$

Proof: Since the operation $\circ$ is order-preserving with respect to both arguments, we have that for all $i \in I$:

$$P_i \circ Q_i \leqslant \bigvee_{i \in I} P_i \circ \bigvee_{i \in I} Q_i$$

thus,

$$\bigvee_{i \in I} (P_i \circ Q_i) \leqslant \bigvee_{i \in I} P_i \circ \bigvee_{i \in I} Q_i$$

To prove the opposite inequality, we use the infinite distributivity of $\otimes$ over supremum and the fact that it is order-preserving:

$$\bigvee_{i \in I} P_i \circ \bigvee_{i \in I} Q_i \,(x, y) = \bigvee_{z \in A} \left(\bigvee_{i \in I} P_i(x, z) \otimes \bigvee_{j \in I} Q_j(z, y) \right) =$$

$$\bigvee_{z \in A} \bigvee_{j \in I} \left(\bigvee_{i \in I} P_i(x, z) \right) \otimes Q_j(z, y) = \bigvee_{z \in A} \bigvee_{(i,j) \in I^2} (P_i\,(x, z) \otimes Q_j(z, y)) \leqslant$$

$$\bigvee_{z \in A} \bigvee_{(i,j) \in I^2} (\max\{P_i, P_j\}\,(x, z) \otimes \max\{Q_i, Q_j\}(z, y)) =$$

$$\bigvee_{(i,j) \in I^2} \bigvee_{z \in A} (\max\{P_i, P_j\}\,(x, z) \otimes \max\{Q_i, Q_j\}(z, y)) =$$

$$\bigvee_{(i,j) \in I^2} (max\,\{P_i, P_j\} \circ max\,\{Q_i, Q_j\}) \leqslant \bigvee_{i \in I} (P_i \circ Q_i)$$

As for the least solution, using Theorem 1, analogously as in [16], we can prove that there exists the least solution to equations and inequations (11), (17) and (21). The least relation in $\mathscr{F}(A \times A)$ is the least solution of all these (in)equations (due to

(1)). In the sequel, we discuss solutions under the condition that the solution is greater than or equal to a given fuzzy relation.

Theorem 17. Let $U \in \mathscr{F}(A \times A)$ be a fuzzy relation such that $P \circ U \geqslant U$. There exists the least solution to the inequation $P \circ X \leqslant X$ containing U, and the least solution to the equation $P \circ X = X$ containing U, and the two of them coincide.

Proof: We define $\varphi:\mathscr{F}(A \times A) \to \mathscr{F}(A \times A)$ by

$\varphi(X) = P \circ X.$

Since $\circ$ is order preserving relative to both arguments, φ is an increasing operator in the complete lattice $\mathscr{F}(A \times A)$, and since $P \circ U \geqslant U$, we have $P \circ X \geqslant U$ for all $X \geqslant U$, $\psi = \varphi|_{[U,1]}$ is also an increasing operator in the segment $[U, 1]$ of $\mathscr{F}(A \times A)$. Its fixed points are solutions to the equation $P \circ X = X$. Using Theorem 1, we conclude that the set of solutions to $P \circ X = X$ is a nonempty, complete lattice relative to the inclusion in $[U, 1]$, and it has the least fixed point. It is the least solution to the equation $P \circ X = X$ containing U.

Now, applying assertion (4) of Theorem 1 to ψ, we get that there exists the least solution to $P \circ X \leqslant X$ containing U, and it is also the least fixed point of ψ, i.e. the solution to the equation $P \circ X = X$.

Applying the same arguments for operators $\psi_1(X) = X \circ P$ and $\psi_2(X) = P \circ X \circ Q$, we get more results:

Theorem 18. Let $U \in \mathscr{F}(A \times A)$ be a fuzzy relation such that $U \circ P \geqslant U$. There exists the least solution to the inequation $X \circ P \leqslant X$ containing U and the least solution to the equation $X \circ P = X$ containing U, and the two of them coincide.

Theorem 19. Let $U \in \mathscr{F}(A \times A)$ be a fuzzy relation such that $P \circ U \circ Q \geqslant U$. There exists the least solution to the inequation $P \circ X \circ Q \leqslant X$ containing U and the least solution to the equation $P \circ X \circ Q = X$ containing U, and the two of them coincide.

We may also apply the generalized Tarski theorem (Theorem 2) to systems of equations. Due to the associativity of $\circ$, we have that $(P \circ X) \circ Q = P \circ (X \circ Q)$, and the operators $\varphi(X) = P \circ X$ and $\psi(X) = X \circ Q$ commute; therefore, we get that the system of equations $P \circ X = X$ and $X \circ Q = X$ does not just have the greatest solution, but also the least one. But since the least fuzzy relation in $\mathscr{F}(A \times A)$ is – due to (1) – the solution to the system, we add the conditions $P \circ U \geqslant U$ and $U \circ Q \geqslant U$ for a fuzzy relation U to get another meaningful result.

Theorem 20. Let $U \in \mathscr{F}(A \times A)$ be a fuzzy relation such that $P \circ U \geqslant U$ and $U \circ Q \geqslant U$. There exists the least solution to the system of inequations containing U:

$$P \circ X = X$$

$$X \circ Q = X$$

4 Additional Results

In this part, we generalize some of the obtained results.

We consider equations obtained by more complex terms. The terms are obtained by constant relations and variables, using only one binary symbol $\circ$. Since the interpretation of this binary symbol is an associative operation of composition, we will exclude all the parentheses for simplicity.

Let the language consist of constant symbols (constants) $A_1, A_2, \dots, B, \dots$ and symbols of variables (variables) $X_1, X_2, \dots, Y, Z, \dots$ and one symbol of a binary operation $\circ$.

Here is the inductive definition of the term in this language:

1. Variables and constants are terms
2. If $T_1$and T_2 are terms, then $T_1 \circ T_2$ is a term
3. All terms are obtained exactly, with finitely many applications of rules 1,2

An equation is an expression $T_1 = T_2$ where T_1 and T_2 are terms, and there is at least one variable in at least one of the terms T_1 and T_2.

Let L be a complete residuated lattice.

The solution of the equation $T_1 = T_2$ on the set A with fixed constants (relations) from $\mathscr{F}(A \times A)$ and the set of variables $\{X_1, \dots, X_n\}$ is the ordered n-tuple of relations $(R_1, \dots, R_n)$ such that the equality $T_1 = T_2$ is true for all elements from A, when variables $\{X_1, \dots, X_n\}$ are exchanged with the relations $(R_1, \dots, R_n)$ respectively and the binary symbol $\circ$ is exchanged with the composition of relations.

The next example shows that for some equations, there are no solutions.

Example 3

Let L be a three-element chain $\{0, p, 1\}$ with $0 < p < 1$. Let $A = \{a, b\}$.

We can easily check that the equation $X \circ X = Q$ does not have a solution with

$Q(a,a) = 0, Q(a,b) = p, Q(b,a) = 0, Q(b,b) = 0.$

Indeed, from the system of equations:

$$(X(a,a) \wedge X(a,a)) \vee (X(a,b) \wedge X(b,a)) = 0$$

$$\big(X(a,a) \wedge X(a,b)\big) \vee \big(X(a,b) \wedge X(b,b)\big) = p$$

$$\big(X(b,a) \wedge X(a,a)\big) \vee \big(X(b,b) \wedge X(b,a)\big) = 0$$

$$(X(b,a) \wedge X(a,b)) \vee (X(b,b) \wedge X(b,b)) = 0$$

we have that $X(a,a) = 0$ and $X(b,b) = 0$ (from the first and the fourth equations, respectively). So the left side of the second equation is 0, which is a contradiction. Hence, there is no solution to this equation.

The next theorem gives conditions under which there is a solution to a type of equation.

Theorem 21. Let $T_1 = T_2$ be an equation with variables $\{X_1, \dots, X_n\}$, such that there is at least one variable on each side of the equation. Then, there is the least solution $(0,0,\dots,0)$, and every solution is contained in a maximal solution.

Proof: Obviously, $(0,0,\dots,0)$ is always one solution to the equation, and it is the least solution.

By a generalization of Lemma 16, if $\{(R_1^i, \dots, R_n^i) \big| i \in I\}$ is a chain of solutions, then $(\bigvee_{i\in I} R_1^i, \dots, \bigvee_{i\in I} R_1^i)$ is also a solution, and it is an upper bound of the chain in the set of solutions. Since every chain in the set of solutions has an upper bound, by Lemma 4, there exists a maximal solution, and every solution is contained in a maximal one.

Example 4. Let $P \circ X \circ X \circ Y = S \circ P \circ X \circ R$ be an equation, where P, S, R are constants and X and Y variables. Then this equation has the least solution $(X, Y) = (0,0)$ and the maximal solutions.

Now, we proceed with the case when there is a side of the equation $T_1 = T_2$ without variables.

Procedure for solving an equation in case there is a side of the equation $T_1 = T_2$ without variables, and there is a variable appearing only once.

Let $T_1 = T_2$ be an equation in which X is a variable appearing only once in T_1, T_2 is the term without variables and $Y_1, \dots, Y_n$ are all other variables.

Now, we take $Y_1, \dots, Y_n$ to be all equal to 1 ($Y_i(x,y) = 1$ for all $x, y \in A$ and all $i \in \{1,\dots,n\}$.)

After composing all the constant elements in T_1 and T_2, we obtain one of three equations: $P \circ X = Q$, $X \circ P = Q$ and $P \circ X \circ Q = R$.

Necessary and sufficient conditions for their solvability are given as in [12] for the first two equations, and in formula (30) for the third equation. If the conditions are met, the greatest solutions to the equations coincide with the greatest solutions to the corresponding inequations, i.e., with $G_{P,Q}$, $G^{P,Q}$ $G^{Q,G_{P,R}}$ (see (24)-(26)) respectively. Let the solution for some of the equations $P \circ X = Q$, $X \circ P = Q$ and $P \circ X \circ Q = R$ exist and let U be the greatest solution of the equation. Then, a maximal solution of the starting equation would be $(U, 1,1,\dots,1)$.

Example 5. Let L be a four-element Boolean algebra $\{0, p, q, 1\}$, with $0 < p < 1$ and $0 < q < 1$. Let $A = \{a, b\}$ and let the equation be given with:

$P \circ X \circ Y \circ R \circ Y = Q$, with two variables X and Y the constant relations $P, Q, R \in \mathscr{F}(A \times A)$ given in tables 1-3.

Table 1
Relation P

P	a	b
a	1	p
b	p	1

Table 2
Relation Q

Q	a	b
a	p	p
b	1	1

Table 3
Relation R

R	a	b
a	q	0
b	o	p

To solve the equation, we take $Y = 1$ and we obtain the equation $P \circ X \circ S = Q$, with S given in Table 4.

Table 4
Relation S

S	a	b
a	1	1
b	1	1

Finally, using the formula (26) to find the greatest solution of the corresponding inequation $P \circ X \circ S \leq Q$ and checking by the formula (30) that it is also a solution to the considered equation, we obtain the greatest solution (Table 5).

Table 5
Solution

X	a	b
a	p	p
b	1	1

and a maximal solution for the starting equation is $(X, 1)$.

We can note here that this procedure does not always give us a solution, even if it exists. In case all the variables appear more than once in the term, this procedure cannot be applied.

We can, likewise, consider general inequations, of the type $T_1 \leqslant T_2$. In case when T_1 contains a single variable X occurring only once, we may prove the existence of the greatest solution. By $T(X_1, \dots, X_n)$ we shall denote a term - as defined above - in which all the occurring variables belong to the set $\{X_1, \dots, X_n\}$.

Theorem 22. There exists the greatest solution to inequations of the type $T_1(X) \leqslant T_2(X, X_1, \dots, X_n)$.

Proof: Let $T_1(X) \leqslant T_2(X, X_1, \dots, X_n)$ be an inequation of the above type. It is solvable, since $(0,0, \dots ,0)$ is a solution. Let $\{(X^i, X_1^i, \dots, X_n^i) | i \in I\}$ be the set of its solutions (X^i and X_k^i are constant relations), meaning that:

$T_1(X^i) \leqslant T_2(X^i, X_1^i, \dots, X_n^i)$

Then we have:

$$T_1\left(\bigvee_{i\in I} X^i\right) = \bigvee_{i\in I} T_1(X^i) \leqslant \bigvee_{i\in I} T_2(X^i, X_1^i, \dots, X_n^i)$$

Since $\circ$ is order-preserving in $\mathscr{F}(A \times A)$, by induction on the complexity of terms, we can prove that terms - seen as functions on their variables - are order-preserving. Thus, for all $i \in I$:

$$T_2(X^i, X_1^i, \dots, X_n^i) \leqslant T_2\left(\bigvee_{i\in I} X^i, \bigvee_{i\in I} X_1^i, \dots, \bigvee_{i\in I} X_n^i\right)$$

and, consequently:

$$\bigvee_{i\in I} T_2(X^i, X_1^i, \dots, X_n^i) \leqslant T_2\left(\bigvee_{i\in I} X^i, \bigvee_{i\in I} X_1^i, \dots, \bigvee_{i\in I} X_n^i\right)$$

which further implies

$$T_1\left(\bigvee_{i\in I} X^i\right) \leqslant T_2\left(\bigvee_{i\in I} X^i, \bigvee_{i\in I} X_1^i, \dots, \bigvee_{i\in I} X_n^i\right)$$

This means that the supremum of all the solutions to the above inequation is also a solution, actually, the greatest one.

Conclusions

In our study, as a first step, we structured the existing knowledge and provided some missing results, related to the existence of solutions (the greatest and the least) in lattice-valued relational equations and inequations, with only one unknown on each side of the equation (inequation). We used the obtained results to find solutions to equations and inequations, with more unknowns.

Our next step in the future work is to continue solving the more complex cases, in this class of equations and inequations, with more unknowns.

Acknowledgements

This research was supported by the Science Fund of the Republic of Serbia, Grant no 6565, Advanced Techniques of Mathematical Aggregation and Approximative Equations Solving in Digital Operational Research - AT-MATADOR. The authors also gratefully acknowledge the financial support of the Ministry of Science, Technological Development and Innovation of the Republic of Serbia (Grants No. 451-03-137/2025-03/200116, 451-03-137/2025-03/200125, 451-03-136/2025-03/200125 and 451-03-136/2025-03/200029)

References

[1] R. Bělohlávek, V. Vychodil: Fuzzy Equational Logic. In: Fuzzy Equational Logic. Studies in Fuzziness and Soft Computing, Vol. 186, Springer, Berlin, Heidelberg

[2] J. C. Díaz-Moreno, J. Medina, E. Turunen: Minimal solutions of general fuzzy relation equations on linear carriers, An algebraic characterization, Fuzzy Sets Syst., Vol. 311, 2017, pp. 112-123

[3] J. Drewniak: Fuzzy relation equations and inequalities, Fuzzy Sets Syst., Vol. 14, 1984, pp. 237-247

[4] J. A. Goguen: L-Fuzzy Sets, Journal of Mathematical Analysis and Applications, Vol. 18, 1967, pp. 145-174

[5] J. Ignjatović, M. Ćirić, B. Šešelja, A. Tepavčević, Fuzzy relational inequalities and equalities, fuzzy quasi-orders, closures and openings of fuzzy sets, Fuzzy Sets Syst., Vol. 260, 2015, pp. 1-24

[6] J. Jimenez, S. Montes, B. Šešelja, A. Tepavčević: Fuzzy correspondence inequations and equations, Fuzzy Sets Syst, Vol. 239, 2014, pp. 81-90

[7] D. Lobo, J. Medina, T. Camillo Merkl, R. Pichler: Minimal solutions of fuzzy relation equations via maximal independent elements, Information Sciences Vol. 690, 2025, 121558

[8] A. di Nola, A. Lettieri: Relation equations in residuated lattices, Rend. Circ. Mat. Palermo 38, Vol. 246-256, 1989

[9] A. Di Nola, S. Sessa, W. Pedrycz, E. Sanchez: Fuzzy Relation Equations and Their Applications to Knowledge Engineering, in: Theory and Decision Libr., Ser. D, Kluwer Acad. Publ., Dordrecht, 1989

[10] I. Perfilieva: Fuzzy function as an approximate solution to a system of fuzzy relation equation, Fuzzy Sets Syst, Vol. 147, 2004, pp. 363-383

[11] W. Pedrycz: Fuzzy relational equations: bridging theory, methodology and practice, International Journal of General Systems, Vol. 29(4), 2007, pp. 529-554, 2007

[12] E. Sanchez: Resolution of composite fuzzy relation equations, Information and Control, Vol. 30(1), 1976, pp. 38-48

[13] I. Stanković, M. Ćirić, J. Ignjatović: Fuzzy relation equations and inequalities with two unknowns and their applications, Fuzzy Sets and Syst, Vol. 322, 2017, pp. 86-105

[14] V. Stepanović: Two sufficient conditions for having a maximal solution to fuzzy relational equations, Special Issue of ESCIM 2024, Springer Series: Studies in Computational Intelligence, to appear

[15] V. Stepanović: Fuzzy set inequations and equations with a meet-continuous codomain lattice, Journal of Intelligent and Fuzzy Systems, Vol. 34(6), 2018, pp. 4009-4021

[16] V. Stepanović and A. Tepavčević: Fuzzy sets (in)equations with a complete codomain lattice, Kybernetika, Vol. 58, 2022, pp. 145-162

[17] A. Tarski: A lattice-theoretical fixpoint theorem and its applications, Pacific Journal of Mathematics, Vol. 5, 1955, pp. 285-309

[18] L. A. Zadeh: Fuzzy sets, Information and Control, Vol. 8, 1965, pp. 338-353