On Horizontal Fuzzy Relations and Hypotheses Testing

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Abstract: Making decisions in conditions of uncertainty is one of today's everyday problems. Various statistical models have found their place in practical applications, however working with imprecise, quite often fuzzy valued, data is still a challenge. In this paper, a new fuzzy relation is presented that enables the comparison of triangular fuzzy numbers along the horizontal axis, and a proposal is given for the generalization of the statistical hypothesis for data in the form of triangular fuzzy numbers. The proposed method allows a fuzzy form for both the registered value of the test statistic and the reference critical value and defines the degree of acceptance of the null hypothesis.

Keywords: Fuzzy sets; fuzzy numbers; fuzzy relations; comparing fuzzy values; hypotheses testing

1 Introduction

Real-life problems often deal with a certain dose of imprecision and uncertainty which can cause some difficulties while constructing classical "crisp" mathematical models. For a notable number of cases, this can be overcome with the use of fuzzy sets and corresponding nonclassical mathematical tools. The focus of this paper is on the problem of how to extend a well-known statistical method to a fuzzy surrounding. One of the methods for extending the classical hypotheses testing from crisp to fuzzy-valued data was introduced by Wu in [18]. In this approach, all observed variables were fuzzy random variables, and the hypotheses' statements had to be phrased with fuzzy variables. However, the conclusion was obtained by comparing crisp values that were extracted from α-cuts of the analyzed fuzzy data. This paper aims to generalize the method from [18] by allowing the comparison of fuzzy values in the decision process. For that purpose, a new horizontal fuzzy relation is introduced, and it is used to express to which extent the registered value of a test statistic diverts from a threshold given by a critical value. Both compared values are now in the form of a fuzzy number, and the proposed method measures the horizontal deviation of those two fuzzy values.

The paper is of the following structure. An overview of basic notions regarding fuzzy sets is given in the second section. The new horizontal fuzzy relation is introduced in Section 3. The fourth section presents a generalization of the method from [18]. Discussion of possibilities for further research and some concluding remarks are given in the fifth section.

2 Preliminaries – Fuzzy Sets

This section provides some basic information on elementary notions related to the fuzzy set theory (e.g. see [9, 12, 16, 19, 20, 22]).

In general, the universe X can be an arbitrary nonempty set. However, for the purpose of the method introduced later on, let's assume $X = \mathbb{R}$.

Definition 1: [20] A **fuzzy set** A *on* X, *is a collection of all ordered pairs* $(x, m_A(x))$ *where* $x \in X$ *and*

$$m_A : X \to [0,1].$$

Function $m_A : X \to [0,1]$ is the **membership function**, and $m_A(x)$ provides information to which extend an element x from X belongs to the fuzzy set A. For $X = \mathbb{R}$, A is called a fuzzy subset of the set of reals, or of the real line.

The connection between fuzzy sets and classical sets can be nicely expressed with well-known α-cuts. The $\boldsymbol{\alpha}$**-cut** of a fuzzy set A, for $\alpha \in (0,1]$, given by

$$[A]^\alpha = \{x \in X \mid m_A(x) \geq \alpha\},$$

is a classical set of all elements from the universe X that belong to A to the extent of at least α. The support of a fuzzy set A is

$$Supp(A) = [A]^{0^+} = \{x \in X \mid m_A(x) > 0\},$$

and its closure is denoted with $[A]^0$.

Based on the well-known Decomposition theorem for fuzzy sets, each fuzzy set can be characterized by its α-cuts, i.e., it can be represented as a union of specific fuzzy sets formed from α-cuts (see [12]). Therefore, the form of α-cuts is an important characteristic of fuzzy sets. The focus of this paper is on **fuzzy numbers** that are fuzzy sets with closed real intervals as α-cuts, and **left and right shoulder fuzzy sets** that are supersets of fuzzy numbers with semiclosed unbounded real intervals as α-cuts.

Definition 2: [22] A **fuzzy number A** is a fuzzy subset of the real line such that its membership function $\mathrm{m_A}: \mathbb{R} \to [0,1]$ is a normalized, convex, and continuous function with a bounded support.

For this specific fuzzy set, α-cuts are closed real intervals given by

$$[A]^{\alpha} = [a_l(\alpha), a_r(\alpha)], \quad \alpha \in (0,1],$$

where $\mathrm{a_l}, \mathrm{a_r}: (0,1] \to \mathbb{R}$ are continuous functions, known as border functions. It is possible to include 0 in the domains of both $\mathrm{a_l}$ and $\mathrm{a_r}$ by acquiring values of the left and the right border of $[\mathrm{A}]^{0^+}$, respectively.

Especially, if the membership function of a fuzzy number consists of linear segments, this fuzzy number is called a triangular fuzzy number ($m_A(x) = 1$ for exactly one $x \in \mathbb{R}$) or triangular fuzzy interval ($m_A(x) = 1$ on an interval):

• Membership function for a triangular fuzzy number A is

$$m_A(x) = \begin{cases} 0, & x < a \ \ or \ \ x > c, \\ \dfrac{x-a}{b-a}, & a \le x \le b, \\ \dfrac{c-x}{c-b}, & b \le x \le c \end{cases}$$

and shorter notation is $A = (a, b, c)$.

• Membership function for a triangular fuzzy interval A is

$$m_A(x) = \begin{cases} 0, & x < a \ \ or \ \ x > d, \\ \dfrac{x-a}{b-a}, & a \le x \le b, \\ 1, & b \le x \le c, \\ \dfrac{d-x}{d-c}, & c \le x \le d \end{cases}$$

and shorter notation is $A = (a, b, c, d)$.

Remark 3: *Based on the fact that α-cuts are closed intervals, operations with fuzzy numbers are derived from corresponding arithmetic operations with intervals:*

$$[A * B]^{\alpha} = [A]^{\alpha} * [B]^{\alpha}, \quad \alpha \in (0,1]$$

where A and B are two fuzzy numbers and $[A * B]^\alpha$ are α-cuts of the resulting fuzzy value. Additionally, it is possible to extend $\min$ and $\max$ to fuzzy numbers in the same manner ([9]).

The arithmetic of fuzzy numbers (values) is also based on the well-known Zadeh's extension principle. Both approaches coincide for continuous fuzzy numbers ([16, 19, 22]).

Also, the ranking of fuzzy numbers and expectation of fuzzy numbers are interesting topics that are being intensively investigated ([2, 4, 7, 8, 11]).

Another type of fuzzy set needed throughout this paper is the fuzzy set with a monotone membership function that is closest to an observed fuzzy set A.

Definition 4:[1] For any fuzzy set A, LTR(A) and RTL(A) are fuzzy sets given by the following membership functions:

$$m_{LTR(A)}(x) = sup\{m_A(y), y \leq x\}, \quad (1)$$

and

$$m_{RTL(A)}(x) = sup\{m_A(y), x \leq y\}. \quad (2)$$

Obviously, $LTR(A)$ is the superset of A with minimal deviation from A and a non-decreasing membership function, and $RTL(A)$ is the superset of A with minimal deviation from A, and a non-increasing membership function. Additionally, for A being a fuzzy number, sets $LTR(A)$ and $RTL(A)$ belong to the class of **left and right shoulder fuzzy sets**, and α-cuts are semiclosed unbounded real intervals, see Figure 1 and Figure 2. This type of fuzzy set is used for modeling real-life concepts that are bounded from one side, for example, low temperature or high salary.

Of course, for two fuzzy sets A and B on the same universe X holds $A \subseteq B$ whenever $m_A(x) \leq m_B(x)$ for all $x \in X$.

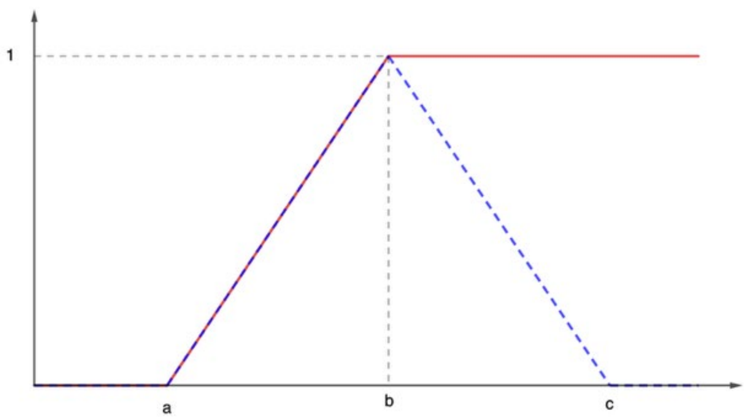

Figure 1
Triangular fuzzy number A=(a,b,c) and fuzzy set LTR(A)

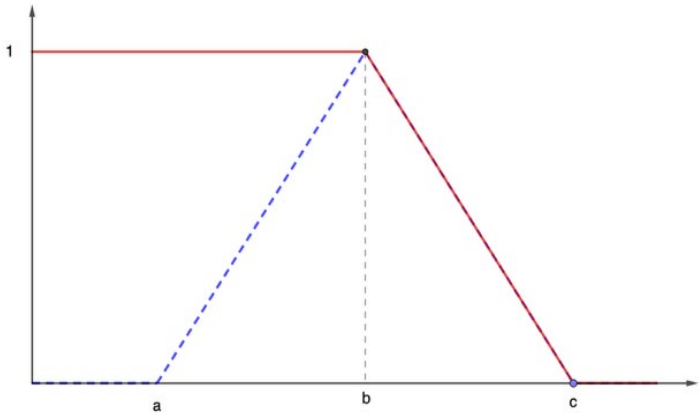

Figure 2
Triangular fuzzy number A=(a,b,c) and fuzzy set RTL(A)

3 Horizontal Fuzzy Relation $\preceq_F$

In general, if two universes X and Y are observed, the **fuzzy relation *R*** is a fuzzy subset of the Cartesian product $X \times Y$ with a membership function

$$m_R: X \times Y \rightarrow [0,1]$$

where $m_R(x, y)$ represents the strength of the observed relation between $x \in X$ and $y \in Y$.

The fuzzy relation introduced here generalizes the previous concept to families of all fuzzy subsets of the universe X and the universe Y. Now, for some A that is a fuzzy subset of X and some B that is a fuzzy subset of Y, $m_R(A, B)$ represents the strength of the relation between A and B. More precisely, for the construction given here, the universe is the real line, F the family of all triangular fuzzy numbers (or intervals) on ℝ, and fuzzy relation is given with the membership function

$$m_R: F \times F \rightarrow [0,1]$$

The main idea is, very roughly speaking, that the more fuzzy number A is to the left with respect to a fuzzy number B, the smaller it is compared to B, and the obtained value $m_R(A, B)$ is closer to zero.

Example 5: It is easy to intuitively accept that the triangular fuzzy number $A = (2,3,4)$ representing 3 ± 1, is smaller than the triangular fuzzy number $B = (5,6,7)$, representing 6 ± 1. This conclusion remains also for, e.g., $A = (1,2,3)$. However, it is not that obvious if A is 5 ± 1 ($A = (4,5,6)$) or 5.5 ± 1 ($A = (4.5,5.5,6.5)$). Some information to which degree A is smaller than B is needed.

The following subsection defines the **horizontal fuzzy relation "smaller" on triangular fuzzy numbers** with the step-by-step construction, which has roots in [17]. Afterward, some examples are given that illustrate both the advantages and disadvantages of an ordering of fuzzy numbers that are introduced through the horizontal fuzzy relation. One problem with this approach is that ordering obtained in such a manner is not a total ordering.

3.1 Construction

Let the universe X be the real line, and let F be the family of all triangular fuzzy numbers.

The first step consists of the introduction of two non-commutative operators S_L and S_R on $F \times F$ given by

$$S_L(A,B) = \begin{cases} 0, & LTR(A) \subseteq LTR(B) \\ sup_{\{x \mid m_{LTR(A)}(x) > m_{LTR(B)}(x)\}} m_B(x), & otherwise, \end{cases}$$

and

$$S_R(A,B) = \begin{cases} 0, & RTL(B) \subseteq RTL(A) \\ sup_{\{x \mid m_{RTL(B)}(x) > m_{RTL(A)}(x)\}} m_B(x), & otherwise, \end{cases}$$

where A and B are from F. Of course,

$$S_L(B,A) = \begin{cases} 0, & LTR(B) \subseteq LTR(A) \\ sup_{\{x \mid m_{LTR(B)}(x) > m_{LTR(A)}(x)\}} m_A(x), & otherwise, \end{cases}$$

and

$$S_R(B,A) = \begin{cases} 0, & RTL(A) \subseteq RTL(B) \\ sup_{\{x \mid m_{RTL(A)}(x) > m_{RTL(B)}(x)\}} m_A(x), & otherwise, \end{cases}$$

And $S_L(A,B)$ and $S_R(A,B)$, in general, do not coincide with $S_L(B,A)$ and $S_R(B,A)$, respectively.

The second step is calculation of the proximity index from the left side FL_{AB} and the proximity index from the right side FR_{AB}:

$$FL_{AB} = \begin{cases} S_L(A,B), & S_L(A,B) \leq S_L(B,A), \\ 1 - S_L(B,A), & otherwise, \end{cases}$$

and

$$FR_{AB} = \begin{cases} S_R(A,B), & S_R(A,B) \le S_R(B,A), \\ 1 - S_R(B,A), & otherwise. \end{cases}$$

Again, it has to be stressed that $\mathrm{FL_{AB}}$ and $\mathrm{FR_{AB}}$ do not coincide with $\mathrm{FL_{BA}}$ and $\mathrm{FR_{BA}}$, respectively. Values $\mathrm{FL_{AB}}$ (FR_{AB}) provides information on to which extent the left part (the right part) of a triangular fuzzy number A is smaller than the left part (the right part) of B, while FL_{BA} (FR_{BA}) provides information on to which extent the left part (the right part) of a triangular fuzzy number B is smaller than the left part (the right part) of A.

The third step is the definition of the horizontal fuzzy relation "smaller".

Definition 6: Let the universe X be the real line, and let F be the family of all triangular fuzzy numbers. The **horizontal fuzzy relation $\preceq_\mathbf{F}$**, where $\preceq_\mathrm{F}: \mathrm{F} \times \mathrm{F} \rightarrow [0,1]$, is

$$\preceq_F (A,B) = A \preceq_F B = 0.5(FL_{AB} + FR_{AB}). \qquad (3)$$

The previously defined relation can be interpreted as **information to which degree "$\boldsymbol{A}$ is smaller than $\boldsymbol{B}$"**. From the construction can be easily seen that the obtained fuzzy relation need not be symmetrical, i.e.

$$A \preceq_F B \neq B \preceq_F A,$$

for some $A, B \in F$. However, they complement each other up to 1.

Proposition 7: Let the universe X be the real line, F be the family of all triangular fuzzy numbers, and $\preceq_\mathrm{F}$ the horizontal fuzzy relation given by (3). For all $\mathrm{A}, \mathrm{B} \in \mathrm{F}$ holds

$$A \preceq_F B = 1 - B \preceq_F A.$$

3.2 Examples

Examples in this section illustrate how the degree of "A is smaller than B" changes with the horizontal shift of fuzzy number B.

Example 8: Let $A = (0,5,6)$ and $B = (0.5,2.5,5.5)$ (Figure 3) For calculating to which degree "A is smaller than B", the following is needed

$$S_L(A,B) = \frac{1}{6}, \quad S_L(B,A) = 1, \quad and \quad FL_{AB} = \frac{1}{6},$$

and

$$S_R(A,B) = 0, \quad S_R(B,A) = 1, \quad and \quad FR_{AB} = 0.$$

Now,

$$A \preceq_F B = 0.5(FL_{AB} + FR_{AB}) = \frac{1}{12}.$$

From the obtained values it is possible to calculate the reverse information as well, i.e., to which degree "B is smaller than A":

$$FL_{BA} = \begin{cases} S_L(B,A), & S_L(B,A) \leq S_L(A,B) \\ 1 - S_L(A,B), & otherwise \end{cases} = \frac{5}{6},$$

$$FR_{BA} = \begin{cases} S_R(B,A), & S_R(B,A) \leq S_R(A,B) \\ 1 - S_R(A,B), & otherwise \end{cases} = 1,$$

and

$$B \preceq_F A = 0.5(FL_{BA} + FR_{BA}) = \frac{11}{12}.$$

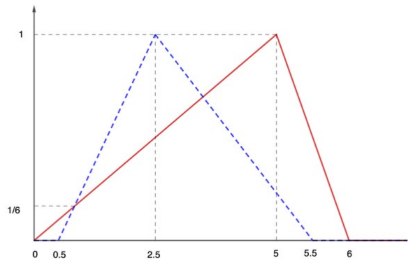

Figure 3
Triangular fuzzy numbers from Example 8.

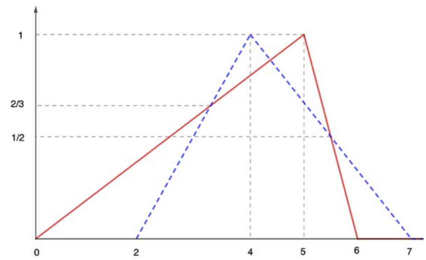

Figure 4
Triangular fuzzy numbers from Example 9.

Example 9: Let $A = (0,5,6)$ and $B = (2,4,7)$ (Figure 4) For calculating to which degree "*A is smaller than B*", the following is needed

$$S_L(A,B) = \frac{2}{3}, \quad S_L(B,A) = 1, \quad and \quad FL_{AB} = \frac{2}{3},$$

and

$$S_R(A,B) = \frac{1}{2}, \quad S_R(B,A) = 1, \quad and \quad FR_{AB} = \frac{1}{2}.$$

Now,

$$A \preceq_F B = 0.5(FL_{AB} + FR_{AB}) = \frac{7}{12}.$$

Again, from the obtained values it is possible to calculate the reverse information. Now,

$$FL_{BA} = \frac{1}{3}, \quad FR_{BA} = \frac{1}{2} \quad and \quad B \preceq_F A = \frac{5}{12}.$$

The previous two examples provide nontrivial values for $\preceq_F$. It can be observed that membership functions of the given fuzzy numbers have at least two intersections for $x \in Supp(A) \cup Supp(B)$. If there is only one $x \in Supp(A) \cup Supp(B)$ such that $m_A(x) = m_B(x) \neq 1$, or no intersection at all, results are 1 or 0, which is illustrated by the following example.

Example 10:

1. Let $A = (0,5,6)$ and $B = (5,7,10)$ (Figure 5). Now,

$$A \preceq_F B = 1 \quad and \quad B \preceq_F A = 0.$$

2. Let $A = (0,5,6)$ and $B = (7,9,12)$ (Figure 6). Now,

$$A \preceq_F B = 1 \quad and \quad B \preceq_F A = 0.$$

In both cases of the previous example, fuzzy number A can be considered to be smaller than B with degree 1, or "truly smaller than B".

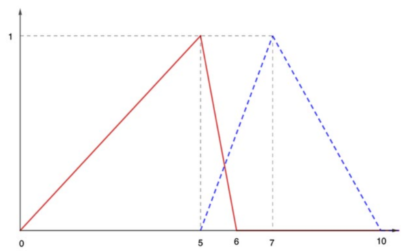

Figure 5
Triangular fuzzy numbers from Example 10, 1.

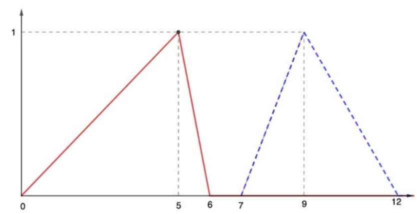

Figure 6
Triangular fuzzy numbers from Example 10, 2.

It is interesting to see how horizontal fuzzy relation deals with situations when there is one $x \in Supp(A) \cup Supp(B)$ such that $m_A(x) = m_B(x) = 1$.

Example 11:

1. Let $A = (0,5,6)$ and $B = (4,5,5.5)$ (Figure 7). Now,

$$A \preceq_F B = \frac{1}{2} \quad and \quad B \preceq_F A = \frac{1}{2}.$$

2. Let $A = (0,5,6)$ and $B = (4,5,7)$ (Figure 8). Now,

$$A \preceq_F B = 1 \quad and \quad B \preceq_F A = 0.$$

It can be observed that in this case the obtained value for $A \preceq_F B$ and $B \preceq_F A$ is $1/2$ if $Supp(A) \subset Supp(B)$ (or $Supp(B) \subset Supp(A)$). Otherwise, if the left slope of A is horizontal "more left" than the left slope of B, A can be again considered to be smaller than B with degree 1, or "truly smaller than B".

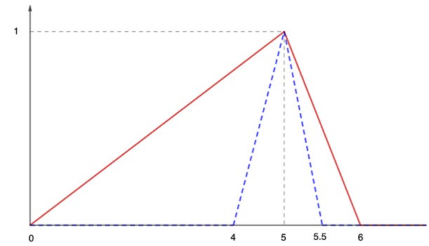

Figure 7
Triangular fuzzy numbers from Example 11, 1.

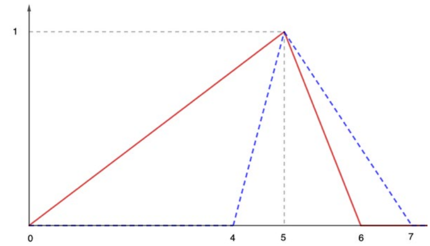

Figure 8
Triangular fuzzy numbers from Example 12, 2.

4 Acceptance Degree of a Statistical Hypothesis

Over the years, different approaches to hypothesis testing and other statistical techniques in a fuzzy setting have been developed ([3, 8, 11, 13, 15, 21, 23]). A particular method that is of interest for this paper has been introduced by Wu in [18] and it provides a transition from crisp data to fuzzy-valued data. In [18] the observed variables are fuzzy random variables, hypotheses' statements are expressed through fuzzy values, and all fuzzy values are in the form of fuzzy numbers. Since α-cuts of fuzzy numbers are closed intervals, in [18], at a certain point, the focus from fuzzy values is shifted to borders of α-cuts, and the classical methods are applied to crisp data. The method proposed here generalizes the one from [18] in such a manner that it allows forming a conclusion based on fuzzy values without going back to crisp data.

Since a classical random variable X is a measurable function from a sample space Ω to a measurable space E (e.g. $\mathbb{R}$), where (Ω, Σ, P) is a probability space, a **fuzzy random variable** $\tilde{X}$ is a measurable fuzzy-number-valued function from a sample space Ω ([10, 14]). Also, by [14], $\tilde{X}$ is a fuzzy random variable if and only if $\tilde{X}_l(\alpha)$ and $\tilde{X}_r(\alpha)$ are random variables for all $\alpha \in [0,1]$. Problems of distribution and independence for fuzzy random variables are solved by transferring to the distribution and independence of classical random variables $\tilde{X}_l(\alpha)$ and $\tilde{X}_r(\alpha)$ ([6]):

- Fuzzy random variables $\tilde{X}$ and $\tilde{Y}$ are independent if and only if all random variables from $\{\tilde{X}_l(\alpha), \tilde{X}_r(\alpha) | \alpha \in [0,1]\}$ are independent of all random variables from $\{\tilde{Y}_l(\alpha), \tilde{Y}_r(\alpha) | \alpha \in [0,1]\}$.
- Fuzzy random variable $\tilde{X}$ has $N(\tilde{\theta}, \sigma^2)$ distribution where $\tilde{\theta}$ is a fuzzy number and σ^2 a known variance, if random variables $\tilde{X}_l(\alpha)$ and $\tilde{X}_r(\alpha)$ have normal distributions $N(\theta_{l,\alpha}, \sigma^2)$ and $N(\theta_{r,\alpha}, \sigma^2)$, where

  $$\theta_{l,\alpha} = E(\tilde{X}_l(\alpha)), \quad \theta_{r,\alpha} = E(\tilde{X}_r(\alpha)) \qquad and \qquad [\tilde{\theta}]^\alpha = [\theta_{l,\alpha}, \theta_{r,\alpha}],$$

  and E is the classical expectation of a random variable.

Different types of tests for the mean value are described in detail in [18], and the focus here is on the two-sided test with the known variance. Now, the level of significance for tests is denoted with ξ, and α remains a notation for α-cuts. Having in mind the previously described setting, the problem of hypotheses testing is now of the following form ([18]):

Let $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n$ be independent fuzzy random variables with distribution $N(\tilde{\theta}, \sigma^2)$, i.e., let $(\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n)$ be a fuzzy random sample. Let $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ be observations of $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n$ obtained in the form of fuzzy numbers. The statistical test problem is

$$FH_0: \tilde{\theta} = \tilde{\theta}_0 \quad vs \quad FH_1: \tilde{\theta} \neq \tilde{\theta}_0,$$

where $\tilde{\theta}_0$ is a fixed fuzzy number.

The main idea from [18] is to apply the classical setting to $(\tilde{x}_1)_l(\alpha), (\tilde{x}_2)_l(\alpha), \dots, (\tilde{x}_n)_l(\alpha)$ and $(\tilde{x}_1)_r(\alpha), (\tilde{x}_2)_r(\alpha), \dots, (\tilde{x}_n)_r(\alpha)$ by introducing the following real values

$$x_l(\alpha) = \frac{1}{n}\sum_{i}^{n} (\tilde{x}_i)_l(\alpha) - core(\tilde{\theta}_0) \quad and$$

$$x_r(\alpha) = \frac{1}{n}\sum_{i}^{n} (\tilde{x}_i)_r(\alpha) - core(\tilde{\theta}_0),$$

where $core(\tilde{\theta}_0)$ is the center of $Supp(\tilde{\theta}_0)$, and deducing a conclusion in the following form:

- FH_0 is accepted in the α-cut sense if $-z_{\xi/2} \cdot \frac{\sigma}{\sqrt{n}} < x_l(\alpha) < z_{\xi/2} \cdot \frac{\sigma}{\sqrt{n}}$ and $-z_{\xi/2} \cdot \frac{\sigma}{\sqrt{n}} x_r(\alpha) < z_{\xi/2} \cdot \frac{\sigma}{\sqrt{n}}$,
- FH_0 is rejected in the α-cut sense if

  $x_l(\alpha) \geq z_{\xi/2} \cdot \frac{\sigma}{\sqrt{n}}$ and $x_r(\alpha) \geq z_{\xi/2} \cdot \frac{\sigma}{\sqrt{n}}$, or $x_l(\alpha) \leq -z_{\xi/2} \cdot \frac{\sigma}{\sqrt{n}}$ and $x_r(\alpha) \leq -z_{\xi/2} \cdot \frac{\sigma}{\sqrt{n}}$,

  where $z_{\xi/2}$ is $(1 - \frac{\xi}{2})$ - quantile of normal $N(0,1)$ distribution ([5]).

The generalization presented here is given for data in the form of triangular fuzzy numbers, and it allows the variance to be a fuzzy value as well. It requires the application of fuzzy arithmetics and the answer provides only a **degree of acceptance.**

Now, let $(\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n)$ be a fuzzy random sample from $N(\tilde{\theta}, \tilde{\sigma}^2)$ distribution, and let $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ be observations of $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n$ in the form of triangular fuzzy numbers. The statistical test problem is

$$FH_0 : \tilde{\theta} = \tilde{\theta}_0 \quad vs \quad FH_1 : \tilde{\theta} \neq \tilde{\theta}_0,$$

where $\tilde{\theta}_0$ is a fixed fuzzy number, and $\tilde{\sigma}$ is a fuzzy number as well.

Definition 12: The acceptance degree of a hypothesis $\boldsymbol{FH_0}$ *is*

$$AD_{FH_0} = \max(\tilde{\theta}_0 - \hat{y} \preceq_F \frac{z_{\xi/2}}{\sqrt{n}} \cdot \tilde{\sigma}, \quad \frac{-z_{\xi/2}}{\sqrt{n}} \cdot \tilde{\sigma} \preceq_F \tilde{\theta}_0 - \hat{y}),$$

where $\hat{y}$ is fuzzy arithmetic mean of $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$, $z_{\xi/2}$ is $(1 - \frac{\xi}{2})$ - quantile of normal $N(0,1)$ distribution.

It should be noted that the acceptance degree is a crisp number from the interval $[0,1]$. Of course, a **rejection degree** of a hypothesis FH_0 can be considered as

$$RD_{FH_0} = 1 - AD_{FH_0}.$$

Conclusions

A new fuzzy relation on fuzzy numbers that is sensitive to the horizontal positions of the observed values is introduced. Also, a method for hypotheses testing that incorporates this new fuzzy relation is proposed through the introduction of a notion of the acceptance degree of a null hypothesis. The authors hope that the concepts presented here will inspire further research on this topic. Especially interesting will be incorporating a wider class of fuzzy values, as well as crisp values interpreted as fuzzy sets.

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