Design Different TP Model Type Polytopic Representations of the Extended Identifiable Virtual Patient Model via Advanced TP Model Transformation Approaches

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Abstract: The paper is situated within the research domain of modeling Identifiable Virtual Patient (IVP) models for subsequent control design using Linear Matrix Inequalities (LMIs). In terms of the modeling methodology employed, the study also aligns with the research domain of the TP (Tensor Product) model transformation. The proposed method has two attributes that are important in automated insulin delivery. The effect of insulin is nonlinear. The glucose dynamics are different from patient to patient, and can also change between days and during the day. Thus, it is important to represent the uncertainty in the parameters of the model in controller design. The central research question addressed in this paper is twofold: first, whether and how alternative state-space TP model-type polytopic representations of the Quasi Linear Parameter Varying (QLPV) state-space model of the IVP model can be derived; and second, what types of convex hulls, defined by the vertices of these TP model-type polytopic models, can be generated using TP model transformation techniques in both cases. The motivation behind this manipulation stems from the fact that the effectiveness of the subsequent Linear Matrix Inequality-based control design optimization is highly dependent on the shape and the number of the vertices of the convex hull. Therefore, - as an answer *to the research question - the paper applies the most up-to-date variant of the TP model transformation in combination with the recently published polytopic model transition method to derive the polytopic representation with a minimal number of vertices and showcases how to manipulate the convex hull. As a result, the paper derives both tight and loose convex hulls and defines the transition between these two leading to a series of convex hulls to support further control design optimization. Additionally, the paper details all the steps of the numerical implementation.*

Keywords: Diabetes mellitus, T1DM model, T2DM model, Identifiable Virtual Patient (IVP), qLPV model, polytopic model, TP model transformation, TP model transition

1 Introduction

Diabetes mellitus (DM) is a chronic metabolic disorder associated with the hormone insulin. Type 1 DM (T1DM) is characterized by a complete deficiency of insulin, resulting from the autoimmune destruction of insulin-producing β-cells within the pancreatic islets of Langerhans. Insulin plays a critical role in regulating various metabolic pathways [1, 2].

In the context of T1DM modeling and control, a crucial consideration is determining the optimal balance between robustness and personalization. Robustness is essential to develop generalized models and controllers that can accommodate a larger patient population under uniform treatment options (e.g., insulin pumps). Typically, these devices are calibrated to manage 'average patients' (e.g., well-trained, middle-aged adults), particularly when such devices are designed for widespread application [3].

Personalization is critical to ensure that each patient receives the most effective therapy tailored to their individual needs. Personalization can be achieved during usage, as modern devices possess adaptive capabilities — enabling limited adjustments to meet the patient's evolving requirements over time [4].

To achieve both robust and personalized therapy, the quasi Linear Parameter Varying (qLPV) state-space modeling and control frameworks [5] are employed in the related literature. These frameworks allow us to address the challenges encountered in T1DM treatment, such as the inherent nonlinearity of physiological processes, patient-specific variations over time, and periodic signals related to food intake, insulin administration, and endogenous insulin secretion. Furthermore, certain internal physiological processes are not directly observable, necessitating the integration of specific state observers within the model. Managing intra- and inter-patient variability is also challenging due to potential fluctuations in the model's variable parameters [6–8] .

1.1 Goal of the paper

The overarching goal of this paper is to investigate how TP model transformation techniques can be utilized to derive various alternative TP model-type polytopic representations of the IVP model, enabling the direct execution of subsequent LMI-based controller design. This foundational work will support future studies aimed at applying LMI control strategies to develop effective IVP control laws. By developing different IVP model representations (with different convex hulls) helps us to select the most appropriate one for a more optimized LMI-based controller design, thus, better performance can be achieved to meet various requirements, directly impacting patient treatment outcomes, reducing treatment costs, and optimizing the administered insulin dosage.

1.2 Research questions of the paper

The research question of this paper is how to derive alternative TP model-type polytopic representations of the state-space qLPV model of the Extended IVP model using TP model transformation techniques, while ensuring that these representations hold the following characteristics:

- **Minimal form**

  The resulting TP model-type polytopic model is designed to have a minimal number of vertices.

- **Series of convex hulls**

  The resulting TP model type-polytopic models have different vertices that define distinct convex hulls. Moreover, these convex hulls serve as intermediaries, effectively representing a continuous transition between distinct convex hull configurations.This step helps the selection of the optimal convex hull during subsequent control design.

1.3 Motivation

The motivation behind the aforementioned goal stems from the fact that it has been demonstrated that the effectiveness of Linear Matrix Inequality (LMI) design applied to TP model type-polytopic representations, and the resulting control performance, are highly dependent on the following characteristics of the TP model type-polytopic models [9, 10].

- **Number of vertices**

  The efficiency of LMI solvers is highly sensitive to the number of LMIs that must be solved simultaneously. The number of LMIs increases with the number of vertices, and the computational load on the LMI solvers grows exponentially as the number of LMIs increases. Furthermore, the likelihood of finding the optimal solution diminishes with a higher number of vertices. For further information see [9, 10].

- **Shape of the convex hull**

  It has recently been demonstrated that different convex hulls defined by the vertices can lead to varying control performances. Some convex hulls result in improved control performance, while in other cases, the selected vertices may render an LMI-based design infeasible. As no direct relationship between the model, convex hull, and LMI has been established in the literature, one may systematically derive a set of convex hulls and subsequently select the one that yields the optimal solution. However, based on experimental tests, a general rule of thumb suggests that the tighter the convex hull, the better the solution tends to be. For further information see [9, 10].

Consequently, the primary motivation behind deriving a set of distinct convex hulls for the polytopic model is to identify the minimal number of vertices that enhance LMI design, while exploring different convex hull shapes to optimize the results of the subsequent LMI design.

In the present case of the IPV design it means that if we derive a set of TP model-type polytopic model then we can execute the LMI design on all.

1.4 Novel contribution of the paper

The novelty of this paper lies in its application of the most recent variant of the TP model transformation - published in 2023 [10] - in combination with the newly introduced interpolation technique for polytopic models, also published in 2024 [11]. This application is presented as a framework for similar cases and serves as a case study to demonstrate the effectiveness of combining these methods.

In this paper, this framework is applied to the Extended Identifiable Virtual Patient Model to derive its TP model-type polytopic representation and to demonstrate how an infinite number of distinct convex hulls can be easily generated in a systematic way.

A case study utilizing the combination of these two recent methods has not been previously published. Furthermore, such polytopic representations have not yet been derived for the Extended Identifiable Virtual Patient Model.

1.5 Structure of the paper

The paper begins by presenting the notation and key concepts employed throughout, to facilitate comprehension of subsequent sections. Following this, the qLPV model of the Identifiable Virtual Patient (IVP) Model is derived in Section III. Section IV introduces the essential steps of the TP model transformation, along with the TP model-type polytopic model interpolation technique. Section V applies the theory introduced in Section IV to the IVP and details the steps of the numerical execution of the TP model transformation and interpolation method. The results are demonstrated in Figures. Section VI concludes the paper.

2 Notation and basic concepts

This section defines the notation and fundamental concepts employed in the following. Moreover, it expounds on the characters and letters that hold specific roles within the context of the TP model transformation.

- $a \in \mathbb{R}, \mathbf{a} \in \mathbb{R}^I, \mathbf{A} \in \mathbb{R}^{I_1 \times I_2}, \mathscr{A} \in \mathbb{R}^{I^N}$ denote, scalars, vectors, matrices and tensors, respectively, where the notation $\mathbb{R}^{I^N}$ is equivalent to $\mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$.
- $\mathbf{1}$ denotes a vector whose all elements are 1.
- $\mathbf{I}$ denotes the identity matrix.
- $g, i, j, n, \ldots$ are indices with the upper bounds $G, I, J, N, \ldots$ e.g. $i = 1, 2, \ldots, I$ and $i_n = 1, 2, \ldots, I_n$ and so on.
- $[\cdot]_{\text{index}}$ addresses elements, e.g. $[\mathscr{A}]_{i_1, i_2, \ldots i_N} = a_{i_1, i_2, \ldots i_N}$ of $\mathscr{A} \in \mathbb{R}^{I^N}$. $[\mathscr{A}]_{i_1, i_2, \ldots i_N} \in \mathbb{R}^{J^M}$ is a tensor element of tensor $\mathscr{A} \in \mathbb{R}^{I^N \times J^M}$.
- $\omega \subset \mathbb{R}$ defines an interval as $\omega = \left[\omega^{\min}, \omega^{\max}\right]$.
- $\Omega \subset \mathbb{R}^N$ is a hyperspace as $\Omega = \omega_1 \times \omega_2 \times \ldots \times \omega_N$.
- $p \in \mathbb{R}$ denotes a parameter such that $p \in \omega$.
- $\mathbf{p} \in \mathbb{R}^N$ denotes parameter vector such that $\mathbf{p} = \begin{bmatrix} p_1 & p_2 & \ldots & p_N \end{bmatrix} \in \Omega$.
- $\mathbf{S}(\mathbf{p})$ denotes a continuous bounded matrix function of parameter $\mathbf{p}(t)$ and the output of the function is characterized as $\mathbf{S}(\mathbf{p}) \in \mathbb{R}^{J_1 \times J_2}$.
- $w(p) \in \mathbb{R}$ denotes the weighting functions, and holds $\forall p : w(p) \in [0, 1]$.
- $\mathbf{w}(p) \in \mathbb{R}^I$ termed as the weighting vector and contains the weighting functions as $\mathbf{w}(p) = \begin{bmatrix} w_1(p) & w_2(p) & \ldots & w_I(p) \end{bmatrix}$. Further, it guarantees Convexity condition meaning that

$$\forall p : \mathbf{w}(p)\mathbf{1} = \mathbf{1}. \tag{1}$$

- $d \in \mathbb{R}$ denotes the location of one point in ω.
- $\mathbf{d} \in \mathbb{R}^G$ denotes the grid vector that defines the location of G number of points as $\mathbf{d} = \begin{bmatrix} d_1 & d_2 & \ldots & d_G \end{bmatrix}$, where $d_g < d_{g+1}$ for all g, and the grid covers ω such that $d_1 = \omega^{min}$ and $d_G = \omega^{max}$.
- $\mathscr{D} \in \mathbb{R}^{G^N \times N}$ denotes the grid tensor constructed from the grid vectors $\mathbf{d}_n = \begin{bmatrix} d_{n,1} & d_{n,2} & \ldots & d_{n,G} \end{bmatrix}$. It defines a rectangular grid that covers Ω. The vector elements $[\mathscr{D}]_{g_1, g_2 \cdots g_N}$ of $\mathscr{D}$ define the coordinates of the grid as:

$$[\mathscr{D}]_{g_1, g_2 \cdots g_N} = \begin{bmatrix} d_{1,g_1} & d_{2,g_2} & \ldots & d_{N,g_N} \end{bmatrix}. \tag{2}$$

- $\mathbf{i}(p) \in \mathbb{R}^G$ denotes the identity function system over ω, see Figure 1. Vector $\mathbf{i}(p) = \begin{bmatrix} i_1(p) & i_2(p) & \ldots & i_G(p) \end{bmatrix}$ contains piece-wise linear triangular shaped functions $i_g(p)$ defined by the grid vector $\mathbf{d}$ as:

$$i(p) = \lambda[\mathbf{I}]_g + (1-\lambda)[\mathbf{I}]_{g+1}; \quad \lambda = \frac{d_{g+1}-p}{d_{g+1}-d_g}, \tag{3}$$

where $d_g \leq p \leq d_{g+1}$.

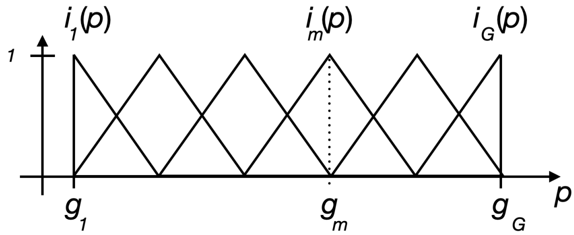

Figure 1
Triangular shaped identity weighting function system

- $\mathbf{U}$ denotes an orthonormal matrix;
- $\mathbf{W}$ denotes the matrix that satisfies:

$$\mathbf{1} = \mathbf{W}\mathbf{1} \quad \text{and} \quad \forall i,j : 0 \leq [\mathbf{W}]_{i,j}. \tag{4}$$

- **UW** denotes the Convex Transformations that transform the orthonormal matrix $\mathbf{U}$ to $\mathbf{W}$. Such transformations in the TP model transformation (also known as TS Fuzzy model transformation) related literature are Sum Normalisation - Non-Negativenes (SNNN), Normalised (NO), Close to Normalised (CNO), Relaxed Normalised (RNO), Inverse Normalised (INO), Inverse Relaxed Normalised (IRNO) transformations [9, 12–14]. These transformations guarantee further characteristics of $\mathbf{W}$ that are advantageous in polytopic model-based design as discussed in the Introduction.
- $(\cdot)^+$ denotes pseudo-inverse of a matrix, i.e. $\mathbf{A}^+$ is the pseudo inverse of matrix $\mathbf{A}$.
- $\boxtimes_{n=1}^{N}$ denotes the tensor product such that $\mathscr{F} = \mathscr{S} \boxtimes_{n=1}^{N} \mathbf{V}_n$, where $\mathscr{F} \in \mathbb{R}^{G^N \times J^M}$, $\mathscr{S} \in \mathbb{R}^{I^N \times J^M}$ and $\mathbf{V} \in \mathbb{R}^{G_n \times I_n}$. It is equivalent to

$$[\mathscr{F}]_{g_1,g_2,\dots g_N} = \sum_{i_1=1}^{I_1} \cdots \sum_{i_N=1}^{I_N} \prod_{n=1}^{N} [\mathbf{V}]_{g_n,i_n} [\mathscr{S}]_{i_1,i_2,\dots i_N}, \tag{5}$$

where $[\mathscr{S}]_{i_1,i_2,\dots i_N} \in \mathbb{R}^{J^M}$.

- CHOVD refers to Compact Higher Order Singular Value Decomposition, also known as truncated HOSVD in the literature [15]. It refers to HOSVD of a given tensor, where the zero singular values and the associated singular vectors are discarded.

3 Virtual Patient Model

3.1 Extended Identifiable Virtual Patient Model

The extended Identifiable Virtual Patient Model (IVP) [16] is a compromise between the well-known minimal model of Bergman [17] and the Hovorka model on structural complexity and precision in terms of the description of the glucose-insulin household. The model equations are the following:

$$\dot{G}(t) = -(GEZI + I_{EFF}(t)) \cdot G(t) + EGP + R_A(t), \tag{6}$$

$$\dot{I}_{EFF}(t) = -k_2 \cdot I_{EFF}(t) + k_2 \cdot S_I \cdot I_P(t), \tag{7}$$

$$\dot{I}_P(t) = -\frac{1}{\tau_2} I_P(t) + \frac{1}{\tau_2} I_{SC}(t), \tag{8}$$

$$\dot{I}_{SC}(t) = -\frac{1}{\tau_1} I_{SC}(t) + \frac{1}{\tau_1 C_I} u(t), \tag{9}$$

where $G(t)$ mg/dl is the blood glucose level at the instant of time t, $I_{EFF}(t)$ min^{-1} is the effect of insulin on glucose, $I_{SC}(t)$ and $I_P(t)$ $\mu U/mL$ represent the subcutaneous and plasma insulin concentrations, respectively. Instantaneous flow of injected insulin $u(t)$ $\mu U/min$ serves as a control input. Parameters of the insulin submodel: τ_1 min and τ_2 min are time constants of the subcutaneous and plasma insulin compartments, k_2 min^{-1} is the kinetic rate of insulin action, C_I mL/min is insulin clearance and S_I $mL/\mu U/min$ is the insulin sensitivity. Endogenous glucose production EGP $mg/dl/min$ is modeled by a constant term and $GEZI$ is the glucose effectiveness at the zero insulin level.

To have a realistic measured output of the model (6)-(9) a CGM error submodel can be used [18]. The model takes into account the delay [19] – since the measurement site is the interstitium –, sensor drift, additive sensor noise, and calibration error. To limit the number of factors that affect the CHO estimation, the sensor drift has been omitted. Sensor delay was modeled as an additional interstitial glucose compartment IG, while additive noise was an autoregressive process of order two with a white noise term $w \sim \mathcal{N}(0, \sigma^2)$ as follows:

$$\dot{IG}(t) = -\frac{1}{\tau_{IG}} IG(t) + \frac{1}{\tau_{IG}} G(t), \tag{10}$$

$$v(t) = \alpha_1 v(t - T_s) + \alpha_2 v(t - 2T_s) + w(t), \tag{11}$$

$$CGM(t) = IG(t) + v(t), \tag{12}$$

where τ_{IG} is a time constant characterizing the transfer between the blood and the interstitial glucose, while α_1 and α_2 define the autoregressive process.

CHO intake submodel – that is not included into the model directly – is summarized by $R_A(t)$ $mg/dl/min$ in the model described by the following equation:

$$R_A(t) = \sum_i^m \frac{A_g \cdot d_i}{V_G \cdot \tau_{D_i}^2} t_i \cdot e^{-\frac{t_i}{\tau_{D_i}}}, \tag{13}$$

where V_G dl is the volume of glucose distribution of the patient and A_g – is the constant utilization of carbohydrate. A given meal is the disturbance signal d_i g (carbohydrate content of the meal). The CHO absorption submodel is encapsulated in the term $R_A(t)$ $mg/dl/min$, which is the sum of the impulse responses characterized by two first-order processes in series with τ_D min time constants.

The values of the constant parameters are the following ones: $C_I = 1.11 \cdot 10^3$ mL/min, $k_2 = 2.85 \cdot 10^{-2}$ $1/min$, $V_g = 190$ dL, $\tau_1 = 52.71$ min, $\tau_2 = 45.41$ min.

In this investigation based on our previous findings [3], the most determining parameters from the model variability point of view are EGP, $GEZI$, and S_I and thus will be applied as varying parameters. The lower and upper bounds of them are $EGP = [0.062, 2.32]$, $GEZI = [5 \cdot 10^{-4}, 6 \cdot 10^{-3}]$, and $S_I = [2.35 \cdot 10^{-5}, 2.11 \cdot 10^{-3}]$.

3.2 qLPV state-space model representation

Linear parameter varying (LPV) modeling technique is an approach to handling the nonlinearities of the given system. If one of the parameters is not a free signal such as an inner state variable, then it is called quasi-LPV or qLPV. The continuous time qLPV state-space representation (qLPV-SS) is defined as follows [20]:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \mathbf{y}(t) \end{bmatrix} = \mathbf{S}(\mathbf{p}(t)) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}. \tag{14}$$

Here vector $\mathbf{x} \in \mathbb{R}^s$ represents the state vector, $\mathbf{y} \in \mathbb{R}^o$ represents the output vector and $\mathbf{u} \in \mathbb{R}^i$ represents the input vector. Matrix $\mathbf{S}(\mathbf{p}(t)) \in \mathbb{R}^{(s+o)\times(s+i)}$ represents the parameter-dependent system matrix having partitions as:

$$\mathbf{S}(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{A}(\mathbf{p}(t))_{s\times s} & \mathbf{B}(\mathbf{p}(t))_{s\times i} \\ \mathbf{C}(\mathbf{p}(t))_{o\times s} & \mathbf{D}(\mathbf{p}(t))_{o\times i} \end{bmatrix}. \tag{15}$$

The extended IVP model (6)-(10) can be transformed into a qLPV-SS representation form. From the model equation (6) the EGP as a varying parameter can be attached to the $G(t)$ by $\frac{EGP \cdot G(t)}{G(t)}$.

Selecting $\mathbf{p}(t) = [G(t), EGP(t), GEZI(t), S_I(t)]^T$ as scheduling parameters leads to the following qLPV representation:

$$\mathbf{A}(\mathbf{p}(t)) = \begin{bmatrix} a_{1,1} & -p_1(t) & 0 & 0 \\ 0 & -k_2 & k_2 p_4(t) & 0 \\ 0 & 0 & -\frac{1}{\tau_2} & \frac{1}{\tau_2} \\ 0 & 0 & 0 & -\frac{1}{\tau_1} \end{bmatrix}, \tag{16}$$

$$a_{1,1} = -p_3(t) + \frac{p_2(t)}{p_1(t)}, \tag{17}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\tau_1 C_I} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \tag{18}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{19}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}. \tag{20}$$

4 TP model-type Polytopic representation and convex hull manipulation by TP model transformation

The TP model transformation (also known as TS Fuzzy model transformation) is developed to transform given models to TP model type polytopic representation with various beneficial characteristics to enhance further design outcomes. The TP model-type polytopic model and TP model transformation were introduced in publications [9–11, 13, 21–27] and will be briefly summarized here within the context of this paper . The key properties of the TP model transformation:

- **Property 1.** It provides the minimal complexity, such as the minimal number of vertices in the polytopic model, or provides an approximation accuracy vs complexity trade-off [27–35].

- **Property 2.** It can vary the convex hull of the resulting polytopic model [9–11, 13, 21–23, 36–38].

4.1 TP model transformation

In the case of state-space modeling and controller design the main objective of the TP model transformation is to transform a given qLPV model

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \mathbf{y}(t) \end{bmatrix} = \mathbf{S}(\mathbf{p}(t)) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \tag{21}$$

to a TP model-type polytopic (polytopic for short) representation

$$\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \mathbf{y}(t)\end{bmatrix} = \mathscr{S} \mathop{\boxtimes}_{n=1}^{N} \mathbf{w}_n(p_n(t)) \begin{bmatrix}\mathbf{x}(\mathbf{t})\\ \mathbf{u}(t)\end{bmatrix}. \tag{22}$$

Method 1. TP model transformation

Step 1. Define discretization grid vector $\mathbf{d}_n$ for each dimension and the discretization tensor $\mathscr{D} \in \mathbb{R}^{G^N \times B}$.

Step 2. Discretise the given system matrix $\mathbf{S}(\mathbf{p}(t))$ over $\mathscr{D}$:

$$[\mathscr{F}]_{g_1,g_2,\cdots g_M} = \mathbf{S}([\mathscr{D}]_{g_1,g_2,\cdots g_M}). \tag{23}$$

Step 3. Extract the minimal number of weighting functions - determined by the columns of matrix $\mathbf{U}_n$ - via CHOSVD as

$$\mathscr{F} = \mathscr{T} \mathop{\boxtimes}_{n=1}^{N} \mathbf{U}_n. \tag{24}$$

Here, CHOSVD stands for Compact Higher-Order Singular Value Decomposition, indicating that all zero singular values are discarded. Consequently, the number of columns in $\mathbf{U}_n$ corresponds to the number of non-zero singular values for each dimension, effectively representing the rank along each dimension. Since each column numerically reconstructs the weighting functions (as will be introduced in Step 5), the number of weighting functions is minimized. This remains true even if the convex transformation in the next step adds one more column to $\mathbf{U}_n$, as it only introduces an additional column when necessary. For a more comprehensive understanding, please refer to [13].

Step 4. Define convex hull via Convex Transformation $\mathbf{U}_n\mathbf{W}_n$ that leads to

$$\mathscr{F} = \mathscr{S} \mathop{\boxtimes}_{n=1}^{N} \mathbf{W}_n, \qquad \text{where} \qquad \mathscr{S} = \mathscr{F} \mathop{\boxtimes}_{n=1}^{N} \mathbf{W}_n^{+}. \tag{25}$$

Step 5. Define the piece-wise linear weighting functions as:

$$\mathbf{w}_n(p_n(t)) = \mathbf{i}_n(p(t))\mathbf{W}_n \tag{26}$$

that leads to

$$\widehat{\mathbf{S}}(\mathbf{p}(t)) = \mathscr{S} \mathop{\boxtimes}_{n=1}^{N} \mathbf{w}_n(p_n(t)). \tag{27}$$

Remark 1. Note that, when the grid density approaches infinity as $\forall n : G_n \to \infty$ then $\widehat{\mathbf{S}}(\mathbf{p}(t)) \to \mathbf{S}(\mathbf{p}(t))$.

To facilitate the utilization of further LMI design tools one may transform the resulting tensor product form to

$$\mathbf{S}(\mathbf{p}(t)) = \sum_{l}^{L} w'_l(\mathbf{p}(t))\mathbf{S}_l, \tag{28}$$

where l is the linear index equivalent of array index $i_1, i_2, \ldots i_N$ as

$$l = ordering(i_1, i_2, \ldots i_N), \qquad \text{where} \qquad L = \prod_{n}^{N} I_n, \tag{29}$$

and, hence the vertex system matrices and the weighting functions are:

$$\mathscr{S}_l = [\mathscr{S}]_{i_1, i_2 \ldots i_N} \qquad \text{and} \qquad w'_l(\mathbf{p}) = \prod_{n=1}^{N} w_{n,i_n}(p_n(t)). \tag{30}$$

Remark 2. Note that functions $w'_n(\mathbf{p})$ hold Convexity condition, see Equ. (1).

4.2 Interpolation of the polytopic models

In order to determine the convex hull between the tight and the loose convex hulls we can apply the interpolation technique proposed in [11]. Consider two polytopic representations of $\mathbf{S}(\mathbf{p}(t))$ derived by TP model transformation executed over the same grid. One is with a tight convex hull. In the present case let the tight convex hull be a CNO type:

$$\mathbf{S}(\mathbf{p}(t)) = \mathscr{S}^{CNO} \boxtimes_{n=1}^{N} \mathbf{w}_n^{CNO}(p_n), \tag{31}$$

where the piece-wise weighting functions are determined as

$$\mathbf{w}_n^{CNO}(p_n(t)) = \mathbf{i}_n(p(t))\mathbf{W}_n^{CNO} \tag{32}$$

in **Step 5** of the TP model transformation. Let the other polytopic representation have a loose convex hull. In the present case let the loose convex hull be SNNN type:

$$\mathbf{S}(\mathbf{p}(t)) = \mathscr{S}^{SNNN} \boxtimes_{n=1}^{N} \mathbf{w}_n^{SNNN} p(n), \tag{33}$$

where the piece-wise weighting functions are determined as

$$\mathbf{w}_n^{SNNN}(p_n(t)) = \mathbf{i}_n(p(t))\mathbf{W}_n^{SNNN} \tag{34}$$

in **Step 5** of the TP model transformation.

Let us set $\lambda \in [0,1]$ the interpolation parameter. The goal of the interpolation is to derive

$$\mathbf{S}(\mathbf{p}(t)) = \mathscr{S}^{\lambda} \boxtimes_{n=1}^{N} \mathbf{w}_n^{\lambda}(p_n), \tag{35}$$

where

$$\forall n : \mathbf{w}_n^{\lambda}(\mathbf{p}) = \lambda \mathbf{w}_n^{CNO}(\mathbf{p}) + (1-\lambda)\mathbf{w}_n^{SNNN} p(n). \tag{36}$$

Method 2. Interpolation between polytopic models

- **Step 1: Determine the interpolated weighting matrices**

  Let

  $$\forall n : \mathbf{W}_n^{\lambda} = \lambda \mathbf{W}_n^{CNO} + (1-\lambda)\mathbf{W}_n^{SNNN}. \tag{37}$$

- **Determination of the vertices**

  The core tensor, which stores the vertices, is calculated by the pseudo inverse as:

  $$\mathscr{S}^{\lambda} = \mathscr{F} \boxtimes_{n=1}^{N} \left(\mathbf{W}_n^{\lambda}\right)^{+}. \tag{38}$$

5 Polytopic representation of the Diabetes Mellitus model

5.1 Tight and loose polytopic models

Let us execute the TP model transformation with CNO-type convex transformation on the model given in Equ. (16). Based on the above model parameters $\Omega = [70,400] \times [0.062,2.32] \times [0.0005,0.006] \times [0.0000235,0.00211]$. Since $p_1(t)$ is in a nonlinear term of the system matrix (see Eq. (17)), let the grid density be defined as $G_1 = 10000$. However, $p_2(t)$, $p_3(t)$, and $p_4(t)$ have a linear effect on the elements of the system matrix, which means the weighting functions to be derived will be linear in those dimensions. Therefore, it is enough to set $G_2 = G_3 = G_4 = 3$. When we execute the TP model transformation it turns out that the rank of the discretized tensor $\mathscr{F} \in \mathbb{R}^{10000\times3\times3\times3\times6\times6}$ is 3 on the first dimension and 2 on the 2,3,4-th dimensions. Therefore, the resulting TP model takes the form of

$$\mathbf{S}(\mathbf{p}(t)) = \mathscr{S} \boxtimes_{n=1}^{4} \mathbf{w}_n(p_n(t)), \tag{39}$$

where $\mathbf{w}_1(p_1(t)) \in \mathbb{R}^3$, $\mathbf{w}_2(p_2(t)) \in \mathbb{R}^2$, $\mathbf{w}_3(p_3(t)) \in \mathbb{R}^2$ and $\mathbf{w}_4(p_4(t)) \in \mathbb{R}^2$. Thus, the number of vertices $\mathbf{S}_{i_1,i_2,i_3,i_4} \in \mathbb{R}^{6\times6}$ stored in tensor $\mathscr{S} \in \mathbb{R}^{3\times2\times2\times2\times2\times6\times6}$

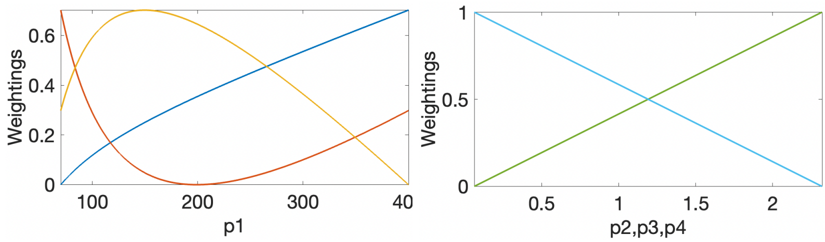

Figure 2
CNO type weighting functions

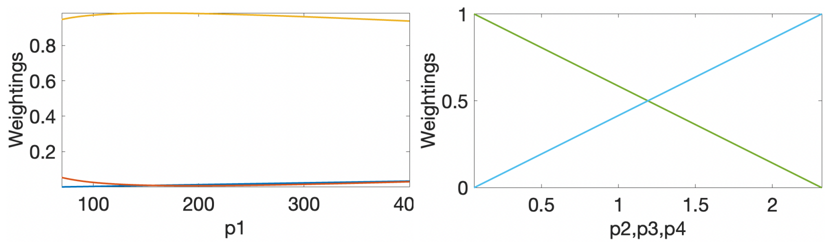

Figure 3
SNNN type weighting functions

is $24 = 3 \times 2 \times 2 \times 2$. Since CNO type weighting functions are derived, the resulting convex hull of the vertices is a tight hull [9, 10, 13]. The weighting functions are depicted in Figure 2. The relative L_2 norm approximation error over the grid is 3.0237×10^{-10} which is caused by the numerical computation of the HOSVD. The L_2 norm relative error caused by the piece-vise approximation of the weighting functions is around 5×10^{-9}. Thus the resulting TP model is exact in numerical sense. If we execute the SNNN convex transformation then we arrive at weighting functions depicted in Figure 3 that yields a loose convex hull. The interpolated weighting functions are depicted in Figure 4. Since the convex hull is 24 dimensional it is not possible to illustrate on a figure. However, Figure 5 illustrates how the elements of the vertices change and gradually converge to a tight convex hull. Since these elements are in a very different range, Figure 5 illustrates only a few elements. It shows how the $[\mathbf{S}_{1,1}]_{1,2}$, $[\mathbf{S}_{2,1}]_{1,2}$ and $[\mathbf{S}_{3,1}]_{1,2}$ element are converging. All other elements of the vertices converge in a similar way from a loose convex hull to a tight hull.

6 Conclusions

The paper combines the TP model transformation and the convex hull interpolation developed for TP model-type polytopic models. Then the paper applied this combination to demonstrate how to derive systematically an infinite number of

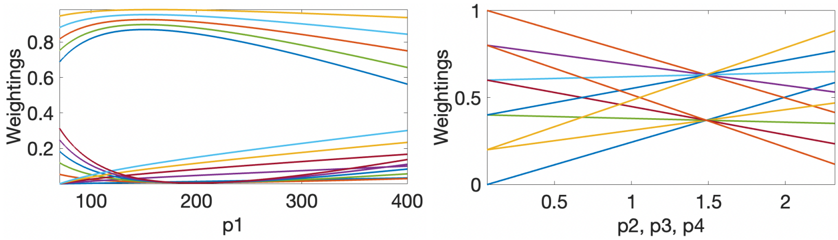

Figure 4
Interpolated weighting functions

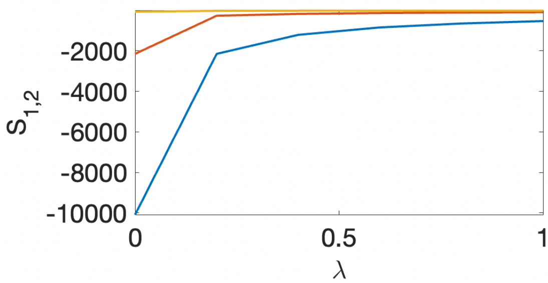

Figure 5
Convergence of $[\mathbf{S}_{1,1}]_{1,2}$, $[\mathbf{S}_{2,1}]_{1,2}$ and $[\mathbf{S}_{3,1}]_{1,2}$ vertex elements

TP model-type polytopic models with different convex hulls. The core message of this paper is that the TP model transformation and transition can be effectively applied to the IPV model. The study demonstrates that the IPV model can be precisely represented using a $3 \times 2 \times 2 \times 2$ weighting function system, resulting in an associated convex hull with 24 vertices—representing the minimum complexity. The paper further reveals that the SNNN-type TP model defines a loose convex hull, whereas the CNO-type TP model defines a tight convex hull. Additionally, it is shown that interpolating the weighting functions results in a smooth transition between the tight and loose convex hulls in the case of the IPV model. As future work, we intend to explore how these different TP model-type polytopic representations can be leveraged to select an optimal LMI structure for controller design.

Acknowledgment

This project has been supported by the National Research, Development, and Innovation Fund of Hungary, financed under the TKP2021-NKTA-36 funding scheme. The work of Gy. Eigner was partially supported by the Researcher Program of Obuda University.

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